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Analisi matematica. — *Linear elliptic equations with BMO coefficients.* Nota di Menita Carozza, Gioconda Moscariello e Antonia Passarelli di Napoli, presentata (*) dal Socio E. Magenes.

Abstract. — We prove an existence and uniqueness theorem for the Dirichlet problem for the equation $\operatorname{div}(a(x)\nabla u) = \operatorname{div} f$ in an open cube $\Omega \subset \mathbb{R}^N$, when f belongs to some $L^p(\Omega)$, with p close to 2. Here we assume that the coefficient a belongs to the space $BMO(\Omega)$ of functions of bounded mean oscillation and verifies the condition $a(x) \geq \lambda_o > 0$ for a.e. $x \in \Omega$.

Key words: Dirichlet problem; Existence and regularity; *BMO*-space.

Riassunto. — *Equazioni lineari ellittiche a coefficienti BMO.* Si prova un teorema di esistenza ed unicità per il problema di Dirichlet per l'equazione $\operatorname{div}(a(x)\nabla u) = \operatorname{div} f$ in un cubo aperto $\Omega \subset \mathbb{R}^N$, dove f appartiene a $L^p(\Omega)$, con p vicino a 2. Si assume che il coefficiente a appartenga allo spazio $BMO(\Omega)$ delle funzioni ad oscillazione media limitata e verifichi la condizione $a(x) \geq \lambda_o > 0$ per q.o. $x \in \Omega$.

1. Introduction

The main objective of this paper is to investigate the existence and uniqueness of solutions $u \in W_o^{1,p}(\Omega)$ of the Dirichlet problem

$$\begin{cases} \operatorname{div}(a(x)\nabla u) = \operatorname{div} f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where Ω is an open cube of $\mathbb{R}^N$, $(N \geq 2)$, f belongs to $L^p(\Omega; \mathbb{R}^N)$, with p close to 2, and a is a function verifying the following assumptions

$$(i) \qquad a(x) \geq \lambda_o > 0 \qquad \text{for a.e. } x \in \Omega$$

$$(ii) \qquad a(x) \in BMO(\Omega).$$

$BMO(\Omega)$ denotes the space of John and Nirenberg of functions of bounded mean oscillation (for the definition see Section 2 below).

We say that $u \in W_o^{1,p}(\Omega)$ is a solution of problem at (1.1) if u verifies

$$\int_\Omega a(x)\nabla u \nabla\varphi = \int_\Omega f\nabla\varphi \qquad \forall\, \varphi \in C_o^\infty(\Omega)\,.$$

We establish the following

Theorem A. *Under the assumptions* (i), (ii), *there exists* $\varepsilon_o > 0$, *depending on* λ_o *and the BMO-norm of* a, *such that for every* $f \in L^{2-\varepsilon}(\Omega; \mathbb{R}^N)$, $|\varepsilon| < \varepsilon_o$, *problem* (1.1) *admits a*

(*) Nella seduta del 13 novembre 1998.

unique solution $u \in W_o^{1,2-\varepsilon}(\Omega)$ *and the following estimate*

$$||u||_{W_o^{1,2-\varepsilon}} \leq c||f||_{L^{2-\varepsilon}} \tag{1.2}$$

with $c = c(\lambda_0, ||a||_{BMO}, N)$, *holds.*

The main tool in the proof of this theorem is an a priori estimate obtained by using the Hodge decomposition in conjunction with a local version of a regularity result of *div-curl* quantities on the Hardy space $\mathcal{H}^1$ (see [2] and Proposition 2.5 below).

Let us remark that Theorem A is an extension of a well known result due to Meyers who considered the case when $a(x)$ is bounded (see [6]).

It is worth pointing out that for $a(x)$ bounded, the result of Meyers gives ε which gets small as $||a||_\infty$ tends to ∞. From this point of view Theorem A is somewhat surprising.

2. Definitions and preliminary results

Here and subsequently, Ω is a cube in $\mathbb{R}^N$. In order to introduce the Hardy space $\mathcal{H}^1(\Omega)$, we consider a function $\varphi \in C_0^\infty(\Omega)$ such that $\operatorname{supp}\varphi \subset B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$ and $\int \varphi dx = 1$. For every $t > 0$ we consider the mollifying function $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$ and given any function $h \in L^1_{\text{loc}}(\Omega)$ we set

$$h_t(x) = h \star \varphi_t(x) \qquad \text{for} \quad a.e. \quad x \in \Omega\,,$$

whenever $0 \leq t < \operatorname{dist}(x, \partial\Omega)$. Then we define the *radial maximal function* of f by

$$\mathcal{M}h(x) = \mathcal{M}_\Omega h(x) = \sup\{|h_t(x)|;\, 0 < t < \operatorname{dist}(x, \partial\Omega)\}$$

for all $x \in \Omega$. Now, we are in position to give the following

Definition 2.1. The Hardy space $\mathcal{H}^1(\Omega)$ consists of the functions $h \in L^1_{\text{loc}}(\Omega)$ such that

$$||h||_{\mathcal{H}^1(\Omega)} = ||\mathcal{M}_\Omega h(x)||_{L^1(\Omega)} < \infty \tag{2.1}$$

The functional $||\;||_{\mathcal{H}^1}$ is a norm which makes $\mathcal{H}^1(\Omega)$ a Banach space (see [7, 5]).

Let us remark that the Definition 2.1 does not depend on the choice of the function φ up to equivalence of norms and that if $\Omega = \mathbb{R}^N$ it is the classical definition of $\mathcal{H}^1(\mathbb{R}^N)$.

It is well known that the dual space of $\mathcal{H}^1(\mathbb{R}^N)$ is $BMO(\mathbb{R}^N)$, where

$$BMO(\mathbb{R}^N) = \left\{ g \in L^1_{\text{loc}}(\mathbb{R}^n): \quad ||g|| = \sup_Q \fint_Q |g - g_Q| < \infty \right\}$$

is the space of all functions of bounded mean oscillation.

Definition 2.2. The space $BMO(\Omega)$ is the space of functions $g \in L^1(\Omega)$ such that

$$||g|| = \sup_{Q \subseteq \Omega} \fint_Q |g - g_Q| + \fint_\Omega |g| < \infty. \tag{2.2}$$

In order to establish our result we need the following

THEOREM 2.3. *For every* $g \in BMO(\Omega)$ *there exists a unique* $T \in (\mathcal{H}^1(\Omega))'$ *such that*

$$T(h) = \int_\Omega h(x)g(x)dx \qquad \text{for all} \quad h \in \mathcal{H}^1(\Omega). \tag{2.3}$$

Conversely, for every $T \in (\mathcal{H}^1(\Omega))'$ *there exists a unique* $g \in BMO(\Omega)$ *which satisfies* (2.3). *The correspondence* $T \to g$ *determined by* (2.3) *is a Banach space isomorphism between* $(\mathcal{H}^1(\Omega))'$ *and* $BMO(\Omega)$.

PROOF. See Theorem 2 in [7, p. 223]. □

It is well known that the space $BMO(\Omega)$ is related to the $L^p(\Omega)$ space, for $0 < p < \infty$, by the following continuous embeddings

$$L^\infty(\Omega) \hookrightarrow BMO(\Omega) \hookrightarrow L^p(\Omega)\,.$$

Let us consider two vector fields $E = (E^1, \dots, E^n)$ and $B = (B^1, \dots, B^n)$ such that

$$\mathrm{curl} E = \left(\frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i}\right)_{i,j=1,\dots,n} = 0$$

and

$$\mathrm{div} B = \sum_{i=1}^n \frac{\partial B_i}{\partial x_i} = 0\,.$$

The scalar product of such vector fields enjoy a higher degree of regularity than generic products of arbitrary vector fields. The first result in such direction goes back to the div-curl Lemma due to Murat and Tartar (see [8, 9]) and the subsequent theory of compensated compactness. More recently in [2] the following celebrated result was proved

THEOREM 2.4. *Let* $B \in L^p(\mathbb{R}^N, \mathbb{R}^N)$ *and* $E \in L^q(\mathbb{R}^N, \mathbb{R}^N)$ *where* p, q *are Hölder conjugate exponents, be two vector fields such that* $\mathrm{div} B = 0$ *and* $\mathrm{curl} E = 0$. *Then the scalar product* $E \cdot B$ *belongs to the Hardy space* $\mathcal{H}^1(\mathbb{R}^N)$ *and*

$$||E \cdot B||_{\mathcal{H}^1} \le c||E||_{L^q}||B||_{L^p}$$

where c *is a constant depending only on the dimension* N.

From now on it is essential to consider Ω an open cube of $\mathbb{R}^N$ in order to apply a local version of the previous theorem (see [5]), namely

PROPOSITION 2.5. *Let* $B \in L^p(\Omega, \mathbb{R}^N)$ *and* $E \in L^q(\Omega, \mathbb{R}^N)$ *be two vector fields such that* $\mathrm{div} B = 0$ *and* $\mathrm{curl} E = 0$, *where* $1/p + 1/q = 1$. *Then their scalar product* $E \cdot B$ *belongs to the Hardy space* $\mathcal{H}^1(\Omega)$ *and*

$$||E \cdot B||_{\mathcal{H}^1(\Omega)} \le c||E||_{L^q(\Omega)}||B||_{L^p(\Omega)}$$

where c *is a constant depending on the dimension* N.

Finally we recall the following

THEOREM 2.6 (Hodge decomposition). *Let* $w \in W_0^{1,r}(\Omega)$, $r > 1$ *and let* $1 - r < \varepsilon < 1$. *Then there exist* $\psi \in W_0^{1,\frac{r}{1-\varepsilon}}(\Omega)$ *and a divergence free vector field* $H \in L^{\frac{r}{1-\varepsilon}}(\Omega;\mathbb{R}^N)$ *such that*

$$|\nabla w|^{-\varepsilon}\nabla w = \nabla\psi + H$$

and

$$||H||_{\frac{r}{1-\varepsilon}} \leq c_1|\varepsilon|||\nabla w||_r^{1-\varepsilon},$$
$$||\nabla\psi||_{\frac{r}{1-\varepsilon}} \leq c_2||\nabla w||_r^{1-\varepsilon}$$

where $c_i = c_i(N)$.

PROOF. See Theorem 3 in [4]. □

3. PROOF OF THEOREM A

Before proving the existence and uniqueness result stated in Theorem A, we have to establish a Lemma which is a refined version of a result contained in [3].

LEMMA 3.1. *Let us consider the Dirichlet problem*

$$\begin{cases} \operatorname{div}(\alpha(x)\nabla u) = \operatorname{div} f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

where α *is a function verifying the following assumptions*

$$\begin{aligned} &0 < \lambda_o \leq \alpha(x) && \text{for a.e. } x \in \Omega \\ &\alpha \in L^\infty(\Omega). \end{aligned}$$

Then for every $|\varepsilon| < \varepsilon_o = \dfrac{\lambda_o}{\tilde{c}||\alpha||_{BMO}}$ $\tilde{c}$ *positive number, and for* $f \in L^{2-\varepsilon}(\Omega;\mathbb{R}^N)$ *the problem* (3.1) *admits a unique solution u and the following estimate*

$$||u||_{W_o^{1,2-\varepsilon}} \leq c||f||_{L^{2-\varepsilon}}$$

with $c = c(\lambda_0, ||\alpha||_{BMO}, N)$, *holds.*

PROOF. Let $f_j \in L^2(\Omega;\mathbb{R}^N)$ converge to f in $L^{2-\varepsilon}(\Omega;\mathbb{R}^N)$ and u_j be the solution to the problem

$$\begin{cases} \operatorname{div}(\alpha(x)\nabla u_j) = \operatorname{div} f_j & \text{in} \quad \Omega \\ u_j = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

By the stability of the Hodge decomposition stated in Theorem 2.6, for every j, k there exist $\psi \in W_o^{1,\frac{2-\varepsilon}{1-\varepsilon}}(\Omega)$ and a divergence free vector field $H \in L^{\frac{2-\varepsilon}{1-\varepsilon}}(\Omega, \mathbb{R}^N)$ such that

$$|\nabla u_j - \nabla u_k|^{-\varepsilon}(\nabla u_j - \nabla u_k) = \nabla\psi + H$$

and

(3.2)
$$||H||_{\frac{2-\varepsilon}{1-\varepsilon}} \leq c_1|\varepsilon|||\nabla u_j - \nabla u_k||_{2-\varepsilon}^{1-\varepsilon}$$
$$||\nabla\psi||_{\frac{2-\varepsilon}{1-\varepsilon}} \leq c_2||\nabla u_j - \nabla u_k||_{2-\varepsilon}^{1-\varepsilon}$$

where c_1 , c_2 are constants depending only on N.

From our assumptions it follows that for every j, k

$$\begin{aligned}\lambda_o \int_\Omega |\nabla u_j - \nabla u_k|^{2-\varepsilon} &\leq \int_\Omega \langle \alpha(x)(\nabla u_j - \nabla u_k) , |\nabla u_j - \nabla u_k|^{-\varepsilon}(\nabla u_j - \nabla u_k)\rangle = \\ &= \int_\Omega \langle \alpha(x)(\nabla u_j - \nabla u_k) , \nabla\psi + H\rangle = \\ &= \int_\Omega (f_j - f_k)\nabla\psi + \int_\Omega \alpha(x)\langle \nabla u_j - \nabla u_k , H\rangle = I + II.\end{aligned}$$

To estimate I, it suffices to apply Hölder inequality and (3.2) to get

$$\text{I} \leq ||f_j - f_k||_{2-\varepsilon}||\nabla\psi||_{\frac{2-\varepsilon}{1-\varepsilon}} \leq c_2||f_j - f_k||_{2-\varepsilon}||\nabla u_j - \nabla u_k||_{2-\varepsilon}^{1-\varepsilon}. \tag{3.3}$$

To estimate II we observe that since $\mathrm{div}H = 0$ and $\mathrm{curl}(\nabla u_j - \nabla u_k) = 0$, by Proposition 2.5 their scalar product belongs to the Hardy space $\mathcal{H}^1(\Omega)$ and moreover

$$||\langle H, \nabla u_j - \nabla u_k\rangle||_{\mathcal{H}^1} \leq c_3||H||_{\frac{2-\varepsilon}{1-\varepsilon}}||\nabla u_j - \nabla u_k||_{2-\varepsilon.}$$

Since $\alpha \in L^\infty(\Omega) \subset BMO(\Omega)$ by Theorem 2.3, thanks to (3.2) we obtain

$$\begin{aligned}\text{II} &\leq c_4||\alpha||_{BMO}||\langle H, \nabla u_j - \nabla u_k\rangle||_{\mathcal{H}^1} \leq c_3c_4||\alpha||_{BMO}||H||_{\frac{2-\varepsilon}{1-\varepsilon}}||\nabla u_j - \nabla u_k||_{2-\varepsilon} \leq \\ &\leq c_1c_3c_4|\varepsilon|\,||\alpha||_{BMO}||\nabla u_j - \nabla u_k||_{2-\varepsilon}^{2-\varepsilon}.\end{aligned} \tag{3.4}$$

From estimates (3.3) and (3.4) we have

$$\lambda_o \int_\Omega |\nabla u_j - \nabla u_k|^{2-\varepsilon} \leq c_2||f_j - f_k||_{2-\varepsilon}||\nabla u_j - \nabla u_k||_{2-\varepsilon}^{1-\varepsilon} + c_1c_3c_4|\varepsilon|||\alpha||_{BMO}||\nabla u_j - \nabla u_k||_{2-\varepsilon}^{2-\varepsilon}$$

and then

$$||\nabla u_j - \nabla u_k||_{2-\varepsilon} \leq c||f_j - f_k||_{2-\varepsilon}.$$

Then for every ε such that

$$|\varepsilon| < \frac{\lambda_o}{c_1c_3c_4||\alpha||_{BMO}}$$

we have that

$$||u_j - u_k||_{W_o^{1,2-\varepsilon}} \leq c||f_j - f_k||_{L^{2-\varepsilon}}. \tag{3.5}$$

Formula (3.5) implies that u_j is a Cauchy sequence and then it converges to a function u in $W_o^{1,2-\varepsilon}(\Omega)$. One can easily check that u solves problem (3.1). □

Now, we are in position to give the proof of Theorem A.

Proof of Theorem A. Let us consider the sequence a_j defined by

$$a_j(x) = \begin{cases} a(x) & \text{if} \quad a(x) \le j \\ j & \text{if} \quad a(x) \ge j \end{cases}$$

and the Dirichlet problems

$$\text{(3.6)} \qquad \begin{cases} \operatorname{div}(a_j(x)\nabla u_j) = \operatorname{div} f & \text{in} \quad \Omega \\ u_j = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

Since $||a_j||_{BMO} \le 2||a||_{BMO}$ for every j [1, Lemma A.2], by Lemma 3.1, if $|\varepsilon| \le \dfrac{\lambda_o}{c||a||_{BMO}}$ and $f \in L^{2-\varepsilon}(\Omega)$, each problem (3.6) admits a unique solution u_j in $W_o^{1,2-\varepsilon}(\Omega)$. As before we use the Hodge decomposition to get

$$|\nabla u_j|^{-\varepsilon}(\nabla u_j) = \nabla\psi_j + H_j$$

where $\operatorname{div} H_j = 0$ and

$$||H_j||_{\frac{2-\varepsilon}{1-\varepsilon}} \le c|\varepsilon|\,||\nabla u_j||_{2-\varepsilon}^{1-\varepsilon}$$

$$||\nabla\psi_j||_{\frac{2-\varepsilon}{1-\varepsilon}} \le c||\nabla u_j||_{2-\varepsilon}^{1-\varepsilon}$$

where c denotes a constant depending only on N.

Using that $a(x) \ge \lambda_o$ and arguing as in the previous Lemma we get

$$\lambda_o\int_\Omega |\nabla u_j|^{2-\varepsilon} \le \int_\Omega f\nabla\psi_j + \int_\Omega a_j(x)\langle\nabla u_j\,,\, H_j\rangle \le c||f||_{2-\varepsilon}||\nabla u_j||_{2-\varepsilon}^{1-\varepsilon} + c|\varepsilon|\,||a||_{BMO}||\nabla u_j||_{2-\varepsilon}^{2-\varepsilon}\,.$$

Then

$$||\nabla u_j||_{2-\varepsilon} \le c||f||_{2-\varepsilon}$$

with $c = c(\lambda_0\,,\, ||a||_{BMO}\,,\, N)$. The sequence (u_j) is bounded in $W_o^{1,2-\varepsilon}(\Omega)$, and then there exist a subsequence of (u_j), still denoted by (u_j), and a function u such that

$$u_j \to u \qquad \text{weakly} \quad \text{in} \quad W^{1,2-\varepsilon}(\Omega).$$

Since, for $\varphi \in C_o^\infty(\Omega)$, $a_j\nabla\varphi$ converges strongly to $a\nabla\varphi$ in $L^p(\Omega)$ for every $p < \infty$ and ∇u_j converges weakly to ∇u in $L^{2-\varepsilon}(\Omega)$, we obtain that u solves the problem (1.1), *i.e.* that

$$\int_\Omega a(x)\nabla u\nabla\varphi = \int_\Omega f\nabla\varphi \qquad \forall\,\varphi \in C_o^\infty(\Omega)\,.$$

Moreover, we have that

$$\text{(3.7)} \qquad ||\nabla u||_{2-\varepsilon} \le c||f||_{2-\varepsilon}$$

with $c = c(\lambda_0\,,\, ||a||_{BMO}\,,\, N)$. The uniqueness is an easy consequence of the previous estimate. □

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