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On the regularity of abstract Cauchy problems and boundary value problems


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0. Introduction

The aim of this paper is to compare several notions of regularity associated with the abstract inhomogeneous Cauchy-problem of order one:

\[
(P_{1,T}) \quad \begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T], \quad 0 < T < \infty, \\ u(0) = 0 \end{cases}
\]

and the abstract boundary value problems of order two:

\[
(P_{2,DD}) \quad \begin{cases} -u''(t) + A^2 u(t) = f(t), & t \in [0, T], \quad 0 < T < \infty, \\ u(0) = u(T) = 0 \end{cases}
\]

and

\[
(P_{2,NN}) \quad \begin{cases} -u''(t) + A^2 u(t) = f(t), & t \in [0, T], \quad 0 < T < \infty, \\ u'(0) = u'(T) = 0. \end{cases}
\]

The function \( u : [0, T] \to X \) takes its values in a complex Banach space \((X, |\cdot|)\), \( u' \) (resp. \( u'' \)) denotes the first derivative (resp. the second derivative) of \( u \) with respect to \( t \), the operator \( A \) (in general unbounded and not necessarily densely defined) is supposed to be the negative generator of an exponentially stable analytic semigroup on \( X \), denoted by \((e^{-tA})_{t \geq 0}\), see [11]. Notice that an analytic semigroup is exponentially
stable if and only if 0 is in the resolvent set of the generator $A$, see Remark 7 below. The function $f$ is a given function in $L^p([0, T]; X)$ for some $p \in (1, \infty)$.

The problem $(P_{1,T})$ (resp. $(P_{2,DD}), (P_{2,NN})$) is called $p$-regular if for every $f \in L^p([0, T]; X)$, it has one and only one solution $u \in W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A))$ (resp. $u \in W^{2,p}([0, T]; X) \cap L^p([0, T]; D(A^2))$) where $D(A)$ (resp. $D(A^2)$) denotes the domain of $A$ (resp. $A^2$) equipped with the graph norm, see Definition 1.1 below.

It will appear (see Lemma 1.0 below) that if $(P_{1,T})$ (resp. $(P_{2,DD})$ or $(P_{2,NN})$) is $p$-regular, then (0.1) holds (resp. (0.2)):

\[
\begin{cases}
\text{There exists } M \geq 1 \text{ such that } \\
\quad \|u\|_{L^p([0, T]; X)} + \|Au\|_{L^p([0, T]; X)} \leq M\|u\|_{L^p([0, T]; X)} + \|Au\|_{L^p([0, T]; X)}, \\
\quad \text{for every } u \in W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A)) \\
\text{satisfying } u(0) = 0,
\end{cases}
\]

respectively:

\[
\begin{cases}
\text{There exists } M \geq 1 \text{ such that } \\
\quad \|u\|_{L^p([0, T]; X)} + \|A^2 u\|_{L^p([0, T]; X)} \leq M\|u\|_{L^p([0, T]; X)} + \|A^2 u\|_{L^p([0, T]; X)}, \\
\quad \text{for every } u \in W^{2,p}([0, T]; X) \cap L^p([0, T]; D(A^2)) \\
\text{satisfying } u(0) = u(T) = 0 \text{ (resp. } u'(0) = u'(T) = 0). 
\end{cases}
\]

Recall that if $(P_{1,T})$ is $p$-regular for some $p \in (1, \infty)$ and some $T > 0$, it is also $p$-regular for all $p \in (1, \infty)$ and all $T > 0$, [26, 8, 5], but of course the constant $M$ may depend on $p$ and $T$.

We shall call problem $(P_{1,T})$ (resp. $(P_{2,DD}), (P_{2,NN})$) $\lambda$-p-regular if problem

\[
(P_{1,T})_{\lambda}\begin{cases}
\quad u'(t) + \lambda Au(t) = f(t), \quad t \in [0, T], \quad 0 < T < \infty, \\
\quad u(0) = 0,
\end{cases}
\]

(resp. $(P_{2,DD})_{\lambda}, (P_{2,NN})_{\lambda}$) is $p$-regular for all $\lambda > 0$ and inequality (0.1) (resp. (0.2)) holds where $A$ (resp. $A^2$) is replaced by $\lambda A$ (resp. $\lambda^2 A^2$) with $\lambda > 0$ and $M$ is independent of $\lambda$, see Definition 1.2 below. We will denote by (0.1)$_{\lambda}$ and (0.2)$_{\lambda}$ the corresponding inequalities.

In §1, we first give precise definitions of regularity and $\lambda$-regularity in a general setting. These notions will be used in an essential way in §2 and §3. Then, we recall without proofs some results of Da Prato-Grisvard [9] and Sobolevskii [25] concerning the sum of commuting positive operators in Banach spaces, which are used in this paper. Finally, we recall some results concerning the $\lambda$-p-regularity of the Cauchy problem of order one.

In §2, under hypothesis $(H_A)$ defined in §1, we compare the $p$-regularity (resp. the $\lambda$-p-regularity) of problems of order one and two in a general setting.
In §3, under the same assumption, we consider the $p$-regularity (resp. the $\lambda$-$p$-regularity) of periodic problems $(P_{1,p},T)$ and $(P_{2,p},T)$ of order one and two, that is:

$$(P_{1,p},T) \begin{cases}
  u'(t) + Au(t) = f(t), & t \in [0, T], \\
  u(0) = u(T),
\end{cases}$$

and

$$(P_{2,p},T) \begin{cases}
  -u''(t) + A^2 u(t) = f(t), & t \in [0, T], \\
  u(0) = u(T), & u'(0) = u'(T).
\end{cases}$$

We study the different notions of $p$-regularity of these problems and compare them with the $p$-regularity of $(P_{1,1},T)$ for the first order and with the $p$-regularity of $(P_{2,DD})$ and $(P_{2,NN})$ for the second order.

We deduce from this that, if $(P_{1,1},T)$ is $p$-regular, then both $(P_{2,DD})$ and $(P_{2,NN})$ are $\lambda$-$p$-regular. Moreover if $X$ has the UMD-property, then the $p$-regularity of $(P_{2,DD})$ or $(P_{2,NN})$ implies the $\lambda$-$p$-regularity of $(P_{1,1},T)$.

As a consequence of these results, it appears that if $p \in (1, \infty)$, the space $X$ has the UMD property and the operator $A$ satisfies $(H_2)$, then all problems defined above are simultaneously $p$-regular and $\lambda$-$p$-regular.

1. Preliminaries

Let $E$ be a complex Banach space with norm $\| \cdot \|$ and let $A$ and $B$ be two closed operators in $E$ (not necessarily densely defined). Consider the problem:

$$(P) \quad Au + Bu = f \text{ in } E.$$  

**Definition 1.1.** Problem $(P)$ is called regular in $E$, or equivalently, the pair $(A, B)$ is called regular, if for all $f \in E$, there exists a unique $u \in D(A) \cap D(B)$ such that $(P)$ holds.

If $(P)$ is regular, it follows from Banach theorem that inequality (1.0) holds:

$$\|u\| + \|Au\| + \|Bu\| \leq M\|Au + Bu\|$$

for some $M \geq 1$ and for all $u \in D(A) \cap D(B)$.

It is easy to verify the following lemma:

**Lemma 1.0.** Let $A$ and $B$ be two closed operators in $E$ (not necessarily densely defined). Then $(P)$ is regular if and only if:

1) (1.0) holds
2) $R(A+B)$ is dense in $E$.

If moreover $0 \in \rho(A)$ or $\rho(B)$ (where $\rho(\cdot)$ denotes the resolvent set of an operator), (1.0) is equivalent to (1.1):

$$\|Au\| + \|Bu\| \leq M\|Au + Bu\|$$

for some $M \geq 1$ and for all $u \in D(A) \cap D(B)$. 
Remark 0. The operator \( A + B \) is closed if and only if
\[
\|u\| + \|Au\| + \|Bu\| \leq M (\|Au + Bu\| + \|u\|)
\]
for some \( M \geq 1 \) and for all \( u \in D(A) \cap D(B) \). In particular, if \((P)\) is regular, \( A + B \) has to be closed.

A pair of operators \((A, B)\) such that problem \((P)\) is regular is called coercive in [25]. Also, the stronger notion of coercively positive pair is introduced in [25], which motivates our Definition 1.2:

Definition 1.2. Problem \((P)\) is called \( \lambda \)-regular in \( E \), or equivalently, the pair \((A, B)\) is called \( \lambda \)-regular, if problem
\[
(P)_{\lambda} \quad \lambda Au + Bu = f \text{ in } E
\]
is regular for all \( \lambda > 0 \) and moreover, inequality \((1.1)_{\lambda}\) holds, for all \( \lambda > 0 \):
\[
\|\lambda Au\| + \|Bu\| \leq M \|\lambda Au + Bu\|
\]
for some \( M \geq 1 \), independent of \( \lambda \) and for all \( u \in D(A) \cap D(B) \).

Remark 1. 1) In [2], an example of two operators \( A \) and \( B \), with \( B \) bounded, on a Hilbert space \( H \) such that \((A, B)\) is regular but not \( \lambda \)-regular is given.
2) Clearly if \((1.1)_{\lambda}\) holds, then the inequality
\[
\lambda \|Au\| + \mu \|Bu\| \leq M \|\lambda Au + \mu Bu\|
\]
holds for some \( M \geq 1 \), for all \( \lambda, \mu > 0 \) and \( u \in D(A) \cap D(B) \), which shows that the definition of \( \lambda \)-regularity is symmetric in \( A \) and \( B \). It is also clear that this inequality is equivalent to the following ones:
\[
\|Au\| \leq M \|Au + \lambda Bu\|
\]
for some \( M \geq 1 \) and all \( \lambda > 0 \) and \( u \in D(A) \cap D(B) \), and
\[
\lambda \|Bu\| \leq M \|Au + \lambda Bu\|
\]
for some \( M \geq 1 \) and all \( \lambda > 0 \) and \( u \in D(A) \cap D(B) \).

The following analogue for the \( \lambda \)-regularity of Lemma 1.0 is easily verified:

Lemma 1.0.\( \lambda \). Let \( A \) and \( B \) be two closed operators in \( E \) (not necessarily densely defined). If \( 0 \in \rho(A) \), then \((P)\) is \( \lambda \)-regular if and only if:
1) \((1.1)_{\lambda}\) holds for all \( \lambda > 0 \),
2) There exists \( \lambda_0 > 0 \) such that \( R(\lambda_0 A + B) \) is dense in \( E \).

Let us recall classical definitions on closed operators:

A closed linear operator \( A : D(A) \subseteq E \to E \) (not necessarily densely defined) is called positive in \((E, \| \cdot \|)\), [28], if there exists \( C > 0 \) such that \((1.2)\) holds:
\[
\|u\| \leq C \|u + \lambda Au\|, \text{ for every } \lambda > 0 \text{ and } u \in D(A),
\]
and if \( R(I + \lambda A) = E \) for some \( \lambda > 0 \), equivalently for all \( \lambda > 0 \).
Remark 2. In [28], an operator $A$ is called positive if it is positive and satisfies the additional assumption that $0 \in \rho(A)$. In this paper, it is convenient to relax this extra condition.

If $A$ is positive, injective and densely defined, it is easy to prove that $A^{-1}$ is also positive.

If $X$ is reflexive and $A$ is positive, then $A$ is densely defined [20].

Let $\Sigma_{\omega} := \{ \lambda \in \mathbb{C} \setminus \{0\}; | \arg \lambda | \leq \sigma \} \cup \{0\}$, for $\sigma \in [0, \pi)$. If $A$ is positive, there exists $\theta \in [0, \pi)$ such that (1.3) holds, [20, p. 288]:

$$
\begin{align*}
(1.3) & \\
& \begin{cases}
  i) \quad \sigma(A) \subseteq \Sigma_{\theta} \\
  ii) \quad \text{for each } \theta' \in (\theta, \pi], \text{ there exists } \\
                     M(\theta') \geq 1 \text{ such that } \|\lambda(\lambda I - A)^{-1}\| \leq M(\theta'), \\
                     \text{for every } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } |\arg \lambda| \geq \theta',
\end{cases}
\end{align*}
$$

where $\sigma(A)$ denotes the spectrum of $A$.

The number $\omega_{A} := \inf\{\theta \in [0, \pi); (1.3) \text{ holds}\}$ is called the spectral angle of the operator $A$.

Clearly $\omega_{A} \in [0, \pi)$.

An operator $A$ is said to be of type $(\omega, M)$ [27], if $A$ is positive, $\omega$ is the spectral angle of $A$ and $M := \min\{C \geq 0; (1.2) \text{ holds}\}$.

Note that $M$ is also the smallest constant in (1.3) ii) for $\theta' = \pi$.

In general, we have $M \geq 1$.

If $M = 1$, the operator $A$ is called $m$-accretive, and then $\omega \leq \pi/2$ [27].

If $A$ is of type $(\omega, M)$ with $\omega < \pi/2$, then $A^{2}$ (with $D(A^{2}) := \{ u \in D(A); Au \in \in D(A)\}$) is of type $(2\omega, M')$ for some $M' \geq 1$.

Let $A$ be an operator of type $(\omega, M)$ and densely defined, then the fractional powers of $A$, $A^{\alpha}$ with $\alpha \in (0, 1)$, [20], are well-defined and are positive operators of type $(\alpha \omega, M')$. Moreover the domain of $A$ is dense in the domain of $A^{\alpha}$ equipped with the graph norm [27]. If $A$ is an operator of type $(\omega, M)$ with $\omega < \pi/2$ and if $0 \in \rho(A)$, then $(A^{2})^{1/2} = A$.

Two positive operators $A$ and $B$ in $E$ are said to be (resolvent) commuting if the bounded operators $(I + \lambda A)^{-1}$ and $(I + \mu B)^{-1}$ commute for some $\lambda, \mu > 0$, equivalently for all $\lambda, \mu > 0$.

If $A$ and $B$ are commuting positive operators then $A + B$ (with domain $D(A) \cap D(B)$) is closable [9].

The following theorem, which is a consequence of a theorem of Da Prato-Grisvard [9], will be essential in the sequel:

**Theorem 1.1.** Let $A$ and $B$ be two commuting positive operators in $E$ satisfying:

1. $D(A) + D(B)$ is dense in $E$,

2. $\omega_{A} + \omega_{B} < \pi$. 
Then the closure of \( A + B \) is of type \(( \omega, M) \) with \( \omega \leq \max(\omega_A, \omega_B) \).

If moreover

iii) \( 0 \in \rho(A) \) (resolvent set of \( A \)), then

\[
\| u \| \leq M \| Au + Bu \| , \quad \text{for all } u \in D(A) \cap D(B) ,
\]

and \( 0 \in \rho(A+B) \).

b) \( R(A + B) \supseteq D(A) + D(B) \),

c) \( A + B \) is closed if and only if \( R(A + B) = E \) if and only if (1.1) holds.

**Remark 3.** 1) Under hypotheses i)-iii) of Theorem 1.1, assumption 2) of Lemma 1.0 is always satisfied. Therefore, in order to prove the regularity of problem \((P)\), it is sufficient to verify inequality (1.1).

2) Similarly, under hypotheses i)-iii) of Theorem 1.1, assumption 2) of Lemma 1.0. \( \lambda \) is always satisfied. Therefore, in order to prove the \( \lambda \)-regularity of problem \((P)\), it is sufficient to verify inequality (1.1)\( \lambda \).

In this paper, we shall always be in the situation of i)-iii) of Theorem 1.1.

**Remark 4.** Under assumptions i) - iii) of Theorem 1.1, the pair \((A, B)\) is \( \lambda \)-regular if and only if the closed operator \( BA^{-1} \) is positive.

A useful consequence of \( \lambda \)-regularity is the following result of Sobolevskii [25]:

**Theorem 1.2.** Let \( A \) and \( B \) be two commuting positive operators in \( E \) satisfying:

i') \( D(A) \) and \( D(B) \) are dense in \( E \),

ii') \( \omega_A + \omega_B < \pi \),

iii') \( 0 \in \rho(A) \cap \rho(B) \),

iv') \( (A, B) \) satisfies (1.1)\( \lambda \).

Then for every \( \alpha \in (0, 1) \), \( D(A) \cap D(B) \subset D(A^\alpha B^{1-\alpha}) \) and there is \( M_\alpha > 0 \) such that (1.5) holds:

\[
\| A^\alpha B^{1-\alpha} u \| \leq M_\alpha \| Au + Bu \| ,
\]

for all 

\[
u \in D(A) \cap D(B).
\]

**Remark 5.** It is also shown in [25] that under the assumptions of Theorem 1.2, the following interpolation inequality holds:

\[
\| A^\alpha B^{1-\alpha} u \| \leq M'_\alpha \| Au \| ^\alpha \| Bu \| ^{1-\alpha} ,
\]

for some \( M'_\alpha > 0 \) and for all \( u \in D(A) \cap D(B) \).

We mention in Remark 6, some sufficient conditions on \( A, B \) and \( E \) of Theorem 1.1 which guarantee the regularity of the pair \((A, B)\):

**Remark 6.** 1) If \( E \) has the UMD property, \( A \) and \( B \) are positive, injective, resolvent commuting and satisfy \( \| A^s \| \leq M_\epsilon^\alpha |s| \) and \( \| B^s \| \leq M_\epsilon^\beta |s| \) for some \( M \geq 1, \alpha \geq 0 \) and \( \beta \geq 0 \) such that \( \alpha + \beta < \pi \) and for all \( s \in \mathbb{R} \), then:
a) in [15], if $0 \in \rho(A) \cap \rho(B)$, then $(A, B)$ is a regular pair.

b) in [16] and later in [24], inequality (1.1) is proved without using the condition $0 \in \rho(A) \cap \rho(B)$.

c) It follows from the proof of [24] that not only (1.1) holds but also stronger inequality (1.1)$_\lambda$ holds. This implies in particular that if $0 \in \rho(A)$, $(A, B)$ forms a $\lambda$-regular pair.

2) If $E$ is a Hilbert space, $A$ and $B$ are positive with $\omega_A + \omega_B < \pi$, injective, resolvent commuting and satisfying $\|A^s\| \leq M$ for some $M \geq 1$ and for all $s \in [-1, +1]$, then:

a) in [15], if $0 \in \rho(A) \cap \rho(B)$, then $(A, B)$ is a regular pair.

b) in [2], it is shown that under the weaker condition $0 \in \rho(A)$ only, then $(A, B)$ is a $\lambda$-regular pair.

In this paper we use Theorem 1.1 in the following particular case:

Let $(X, |\cdot|)$ be a complex Banach space and $T > 0$.

For $p \in (1, \infty)$, we shall denote by $E_p$, the Banach space $L^p([0, T]; X)$, equipped with the usual norm, which will be denoted by $\|\cdot\|$.

Let $A$ be a positive operator on $X$. Define the operator $A$ on $E_p$ by:

$$D(A) := L^p([0, T]; D(A))$$

where $D(A)$ is equipped with the graph norm and

(1.7) $$(Au)(t) := Au(t), \ a.e. \ in \ [0, T].$$

The operator $A$ is also a positive operator on $E_p$ with spectral angle $\omega_A = \omega_A$. Moreover the following holds:

$$((I + A)^{-1}u)(t) = (I + A)^{-1}(u(t)), \ a.e. \ in \ [0, T], \ and \ all \ u \in E_p.$$ Next, we define the operator $B_T$ acting in $E_p$ as follows:

$$D(B_T) := \{u \in W^{1,p}([0, T]; X); u(0) = 0\}$$

and

$$(B_T u)(t) := u'(t), \ a.e. \ in \ [0, T].$$

We recall that the operator $B_T$ (see [9]) is $m$-accretive and densely defined on $E_p$ and commute in the sense of the resolvent with the operator $A$ defined in (1.9).

Since the spectral angle of $B_T$ is equal to $\pi/2$, Theorem 1.1 applies to the pair $(A, B_T)$, provided that the operator $A$ satisfies assumption $(H_A)$:

$$(H_A)_{1) \omega_A < \pi/2,}$$

$$(H_A)_{2) 0 \in \rho(A).}$$
Remark 7. 1) We recall that if $D(A)$ is dense in $X$, the assumption $(H_A)_1$ is equivalent to saying that $(-A)$ is the generator of a bounded analytic semi-group $(e^{-tA})_{t>0}$ and moreover that if $(H_A)_1$ holds, then $(H_A)_2$ is equivalent to saying that the semi-group $(e^{-tA})_{t>0}$ is exponentially stable, which means that

$$|e^{-tA}| \leq M_A e^{-\omega t}$$

for some $M \geq 1$, $\omega > 0$ and for all $t > 0$.

2) More generally, if $D(A)$ is not dense in $X$ and $(H_A)_1$ holds, then it is shown in [11] that $(-A)$ generates in an appropriate sense a bounded analytic semi-group $(e^{-tA})_{t>0}$ on $X$ which is strongly continuous only on $\overline{D(A)}$ and satisfying (1.8) when $(H_A)_2$ is fulfilled:

$$(1.8) \quad |e^{-tA}| \leq M_A e^{-\omega t}, \quad |tA e^{-tA}| \leq M_A e^{-\omega t}, \quad |t^2 A^2 e^{-tA}| \leq M_A e^{-\omega t}$$

for some $M_A \geq 1$, $\omega > 0$ and for all $t > 0$.

Moreover, if $(H_A)$ is satisfied, then

$$(1.9) \quad 1 \in \rho(e^{-tA})$$

for all $t > 0$.

It is clear from the definitions that the pair $(A, B_T)$ is regular in $E_p$ (resp. $\lambda$-regular) if and only if $(P_{1, T})$ is $p$-regular (resp. $\lambda$-$p$-regular), see §0 for definitions of Cauchy problem $(P_{1, T})$.

The analogy is the same between general inequality (1.1) and inequalities (0.1) for Cauchy problems of §0 (resp. (1.1)$_{\lambda}$ and inequalities (0.1)$_{\lambda}$).

In Proposition 1.3, we show that, under the hypothesis $(H_A)$, the $p$-regularity and the $\lambda$-$p$-regularity of problem $(P_{1, T})$, is equivalent; the main step is a result of Dore-Kato [14, Theorem 2.4] which proves that if the pair $(A, B_T)$ is regular in $E_p$ for some $T > 0$, then the natural extension of this pair to $[0, \infty)$ is regular in $L_p([0, \infty); X)$.

Recently, Le Merdy, [21], gave an interesting example of a bounded operator $A$ in $X$ for which the Cauchy problem is not regular in $L_p([0, \infty); X)$, hence probleme $(P_{1, T})$ is not $\lambda$-$p$-regular.

**Proposition 1.3.** Let $p \in (1, \infty)$ and $A$ satisfy $(H_A)$, then the following statements are equivalent:

1. Problem $(P_{1, T})$ is $p$-regular
2. Problem $(P_{1, T})$ is $\lambda$-$p$-regular.

**Proof.** 1$\Rightarrow$2

If $(P_{1, T})$ is $p$-regular and $(H_A)$ holds, it follows from [14] that the problem on $[0, \infty)$ is also $p$-regular (in an obvious sense).

Next, by the change of variables $t \rightarrow \lambda t$, the problem on $[0, \infty)$ is clearly $\lambda$-$p$-regular.
Let \( u \in W^{1,p}([0,T];X) \cap (L^p([0,T]; D(A))) \) such that \( u(0) = 0 \) be given and set
\[
 f(t) = u'(t) + \lambda Au(t) \quad \text{for } t \in [0,T]
 = 0 \quad \text{for } t > T.
\]
Clearly, \( f \in L^p([0,\infty);X) \). From the \( \lambda\)-\( p \)-regularity of the problem on \([0,\infty)\), there exists one and only one \( \tilde{u} \in W^{1,p}((0,\infty);X) \cap (L^p((0,\infty); D(A))) \) with \( \tilde{u}(0) = 0 \), satisfying \( \tilde{u}'(t) + \lambda A\tilde{u}(t) = f \) on \([0,\infty)\). Moreover there exists \( C > 0 \) with, for all \( \lambda > 0 \):
\[
\|\tilde{u}'\| + \lambda \|A\tilde{u}\| \leq C\|f\|.
\]
Since \( \tilde{u}(t) = u(t) \) on \([0,T]\), by uniqueness of the solution of \((P_{1,T})\), we get by restriction, since \( \|\cdot\|_{L^p([0,T];X)} \leq \|\cdot\|_{L^p([0,\infty);X)} \):
\[
\|u'\| + \lambda \|Au\| \leq C\|f\|
\]
and thus \((P_{1,T})\) is \( \lambda \)-\( p \)-regular, which proves 2.

2 \( \Rightarrow \) 1 is obvious. \(\square\)

**Remark 8.** For the sake of completeness, we recall without proof that if \((P_{1,T})\) is \( p \)-regular, the solution \( u \) is given by:
\[
u(t) = \int_0^t e^{-(t-s)A}f(s)\,ds = \int_0^t e^{-tA}f(t-s)\,ds
\]
for \( t \geq 0 \).

### 2. First order versus second order problems

In this Section we are concerned with the regularity of pairs \((A, B)\) and \((A^2, -B^2)\), in a general setting, as it was done in Theorem 1.1. This will give us a technical result for comparing problems of order one and of order two.

**Proposition 2.1.** Let \( E \) be a Banach space and \( A \) be a positive operator in \( E \) satisfying:

\( i \) \( \omega_A < \pi/2 \),

\( ii \) \( 0 \in \rho(A) \).

Let \( B \) be the generator of a bounded strongly continuous group on \( E \). Suppose that \( A \) and \( B \) as well as \( A \) and \( -B \) are resolvent commuting. Then, the two following assumptions are equivalent:

\( a \) The pairs \((A, B)\) and \((A, -B)\) are regular (resp. \( \lambda \)-regular).

\( b \) The pair \((A^2, -B^2)\) is regular (resp. \( \lambda \)-regular) and satisfies (2.1) (resp. (2.1)\( _\lambda \)):

\[
(2.1) \quad D(A^2) \cap D(B^2) \subset D(BA) \quad \|BAu\| \leq M\|A^2u - B^2u\|
\]
for some \( M > 0 \) and for all \( u \in D(A^2) \cap D(B^2) \),

\[
(2.1)_\lambda \quad D(A^2) \cap D(B^2) \subset D(BA) \quad \lambda\|BAu\| \leq M\|\lambda^2A^2u - B^2u\|
\]
for some \( M > 0 \), independent of \( \lambda \), for all \( u \in D(A^2) \cap D(B^2) \) and all \( \lambda > 0 \).
We shall need the following:

**Lemma 2.2.** Let $A$ and $B$ be two positive operators in a Banach space $E$, resolvent commuting, with $0 \in \rho(A)$. Then:

\[(2.2) \quad D(BA) \subset D(AB) \text{ and } BAu = ABu\]

for all $u \in D(BA)$.

**Proof of Lemma 2.2.** We shall identify operators in $E$ with their graphs in $E \times E$. From $A^{-1}(I + B)^{-1} = (I + B)^{-1}A^{-1}$, we obtain $(I + B)A^{-1} = (A(I + B))^{-1}$, hence $(I + B)A = A(I + B)$. Then for any $x \in D(BA)$, [that is $x \in D(A)$ with $Ax \in D(B)$], we have $x \in D(B)$ together with $x + Bx \in D(A)$ and $Ax + BAx = Ax + ABx$. This implies (2.2). □

**Proof of Proposition 2.1.** \(a) \Rightarrow b):\)

In order to prove the regularity of the pair $(A^2, -B^2)$, we shall use Theorem 1.1 with $A$ replaced by $A^2$ and $B$ replaced by $-B^2$.

Observe that the operator $A^2$ is positive with $\omega_{A^2} = 2\omega_A < \pi$ and $0 \in \rho(A^2)$.

It is known (see e.g. [22]) that under the assumption on $B$, the operator $-B^2$ is positive with $\omega_{-B^2} = 0$.

Moreover, $A^2$ and $-B^2$ are resolvent commuting: indeed, it is enough to check the equality: $(A^2)^{-1}(I - B^2)^{-1} = (I - B^2)^{-1}(A^2)^{-1}$ which follows from the assumption that $A$ and $B$ as well as $A$ and $-B$ are resolvent commuting and the observation that $(A^2)^{-1}(I - B^2)^{-1} = A^{-1}A^{-1}(I - B)^{-1}(I + B)^{-1}$.

Moreover, since $B$ is in particular the generator of a strongly continuous semigroup on $E$, $D(B^2)$ is dense in $E$, see [23].

In view of Theorem 1.1, the regularity of the pair $(A^2, -B^2)$ is a consequence of the surjectivity of $A^2 - B^2$. In order to prove it, we need the following regularity property of the solution to the equations $Ax + Bx = y \in E$ and $Ax - Bx = y \in E$. Without loss of generality, we only consider the first equation. We claim that if $y \in D(A) \cap D(B)$, then the unique solution $x$ satisfies $x \in D(A^2) \cap D(BA) \cap D(B^2)$. Indeed if $y \in D(B)$ and $z = (I + B)y$, let $\hat{x} \in D(A) \cap D(B)$ be the unique solution to $Ax + Bx = z$. We obtain:

\[y = (I + B)^{-1}z = (I + B)^{-1}A\hat{x} + (I + B)^{-1}B\hat{x} = A(I + B)^{-1}\hat{x} + B(I + B)^{-1}\hat{x}\]

using the resolvent commutativity of $A$ and $B$.

By uniqueness, we get $x = (I + B)^{-1}\hat{x}$, hence $x \in D(AB) \cap D(B^2)$. Similarly, if $y \in D(A)$, we have $x \in D(A^2) \cap D(BA)$, which establishes the claim.

We are in a position to prove the surjectivity of $A^2 - B^2$: let $f \in E$ and let $v \in D(A) \cap D(B)$ be such that $Av + Bv = f$. Let $u \in D(A) \cap D(B)$ satisfy $Au - Bu = v$. We have $u \in D(A^2) \cap D(AB) \cap D(BA) \cap D(B^2)$, hence by Lemma 2.2,

\[f = (A + B)v = (A + B)(A - B)u = A^2u - ABu + BAu - B^2u = A^2u - B^2u \]

which shows that $A^2 - B^2$ is surjective.
Next we prove (2.1) and (2.1)$_\lambda$. We only consider the case of (2.1)$_\lambda$. Let $u \in D(A^2) \cap D(B^2)$ and let $f := A^2 u - B^2 u$. From what precedes and the injectivity of $A^2 - B^2$, we have $u \in D(B,A)$. For any $\lambda > 0$,
\[
\lambda \|B A u\| = \|(B)(\lambda A u)\| \leq M_1 \|\lambda A + B\| \lambda A u\|
\]
in view of the $\lambda$-regularity of the pair $(A, B)$.

By using (2.2), we have $(\lambda A + B)\lambda A u = \lambda A(\lambda A + B)u$ and by the $\lambda$-regularity of the pair $(A, -B)$, we obtain
\[
\|\lambda A + B\| \lambda A u\| = \|\lambda A(\lambda A + B)u\| \leq M_2 \|\lambda A - B\| (\lambda A + B) u\| = M_1 M_2 \|\lambda^2 A^2 u - B^2 u\|.
\]

$b) \Rightarrow a)$:

Since the pairs $(A, B)$ and $(A, -B)$ satisfy the hypotheses $i)$-$iii)$ of Theorem 1.1, it suffices to prove (1.1) (resp. (1.1)$_\lambda$) for these two pairs.

Let $u \in D(A) \cap D(B)$ and set $f := (A + B) u$. By hypothesis, there exists $w \in D(A^2) \cap D(B^2)$ such that
\[
f = A^2 w - B^2 w.
\]

By using (2.1) and (2.2), we write:
\[
f = (A + B)(A - B) w.
\]

By the injectivity of $(A + B)$, we get $u = (A - B) w$.

By the regularity of the pair $(A^2, -B^2)$, Lemma 2.2 and (2.1), we get $M, M' > 0$ such that:
\[
\|A u\| \leq \|A^2 w\| + \|A B w\| \leq (M + M') \|f\|,
\]
which proves the regularity of the pair $(A, B)$. We proceed similarly for the regularity of the pair $(A, -B)$ and the $\lambda$-regularity. \hfill \Box

3. The periodic case and the conclusion

In this Section, we are concerned with the regularity and the $p$-regularity of problems $(P_{1,T})$ on one side and $(P_{2,DD})$ and $(P_{2,NN})$ on the other side.

In order to prove their equivalence, we introduce periodic problems of order one and two, $(P_{1,p,T})$ and $(P_{2,p,T})$, which are defined in the Introduction.

As in §0, we define the $p$-regularity of $(P_{1,p,T})$ and $(P_{2,p,T})$ as follows: $(P_{1,p,T})$ (resp. $(P_{2,p,T})$) is $p$-regular if for all $f \in L^p([0, T]; X)$ there exists a unique function $u \in W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A))$ (resp. $u \in W^{2,p}([0, T]; X) \cap L^p([0, T]; D(A^2))$) with $u(0) = u(T)$ (resp. $u(0) = u(T)$ and $u'(0) = u'(T)$) such that $(P_{1,p,T})$ (resp. $(P_{2,p,T})$) is satisfied.

As above, it appears that if $(P_{1,p,T})$ (resp. $(P_{2,p,T})$) is $p$-regular, then (3.1) holds.
(resp. (3.2)):
\[
\left\{ \begin{array}{l}
\text{There exists } M \geq 1 \text{ such that } \\
\|u'\| + \|Au\| \leq M\|u' + Au\|, \\
\text{for every } u \in W^{1,p}(0, T); X) \cap L^p(0, T); D(A) \\
satisfying u(0) = u(T),
\end{array} \right.
\]

respectively:
\[
\left\{ \begin{array}{l}
\text{There exists } M \geq 1 \text{ such that } \\
\|u''\| + \|A^2u\| \leq M\|u'' + A^2u\|, \\
\text{for every } u \in W^{2,p}(0, T); X) \cap L^p(0, T); D(A^2) \\
satisfying u(0) = u(T) \text{ and } u'(0) = u'(T).
\end{array} \right.
\]

As in §0, we shall call problem \((P_{1,p,T})\) (resp. \((P_{2,p,T})\)) \(\lambda\)-\(p\)-regular if problem
\[
(P_{1,p,T})_\lambda \left\{ \begin{array}{l}
u'(t) + \lambda Au(t) = f(t), \quad t \in [0, T], \quad 0 < T < \infty, \\
u(0) = u(T),
\end{array} \right.
\]

(resp. \((P_{2,p,T})_\lambda\), is \(p\)-regular for all \(\lambda > 0\) and inequality (3.1) (resp. (3.2)) holds where \(A\) (resp. \(A^2\)) is replaced by \(\lambda A\) (resp. \(\lambda^2 A^2\)) with \(\lambda > 0\) and \(M\) is independent of \(\lambda\). We will denote by \((3.1)_{\lambda}\)-\((3.2)_{\lambda}\) the corresponding inequalities.

Let us state our results:

**Theorem 3.1.** Let \(p \in (1, \infty)\) and \(A\) satisfy \((H_\lambda)\) on \(X\). Then:

1) If \((P_{1,p,T})\) is \(p\)-regular (respectively \(\lambda\)-\(p\)-regular), then \((P_{1,p,T})\) is also \(p\)-regular (respectively \(\lambda\)-\(p\)-regular).

2) If \((P_{2,p,T})\) is \(p\)-regular, then \((P_{2,p,T})\) is \(\lambda\)-\(p\)-regular.

**Theorem 3.2.** Let \(p \in (1, \infty)\) and \(A\) satisfy \((H_\lambda)\) on \(X\). Then, if \(X\) has the UMD property:

1) \((P_{2,DD})\) is \(p\)-regular if and only if \((P_{2,NN})\) is also \(p\)-regular.

2) If \((P_{2,p,T})\) is \(p\)-regular, then (3.3) holds:
\[
\|Au'\| \leq M\|u'' + A^2u\|
\]

for some \(M > 0\) and for all \(u \in L^p(\mathbb{R}; D(A^2)) \cap W^{1,p}(\mathbb{R}; D(A)) \cap W^{2,p}(\mathbb{R}; X)\) satisfying \(u(0) = u(T), u'(0) = u'(T)\).

In order to finish to compare the regularity of these problems, we need the following proposition:

**Proposition 3.3.** Let \(p \in (1, \infty)\) and \(A\) satisfy \((H_\lambda)\) on \(X\).

1) \((P_{1,p,T})\) is \(p\)-regular if and only if \((P_{2,p,T})\) is \(p\)-regular and verifies (3.3).

2) \((P_{2,p,T})\) is \(p\)-regular (resp. \(\lambda\)-\(p\)-regular) if and only if \((P_{2,DD})\) and \((P_{2,NN})\) are \(p\)-regular (resp. \(\lambda\)-\(p\)-regular).

3) If \((P_{1,p,T})\) is \(p\)-regular, then \((P_{1,T})\) is \(p\)-regular.
All together, these results imply the following Corollary, which closes the circle of implications comparing the regularity of Cauchy problems and boundary value problems defined in §0:

**Corollary 3.4.** Let $p \in (1, \infty)$ and $A$ satisfy $(H_A)$ on $X$.

1) If $(P_{1,T})$ is $p$-regular, then $(P_{2,DD})$ and $(P_{2,NN})$ are $\lambda$-p-regular.

2) If $X$ has the UMD property, if $(P_{2,DD})$ or $(P_{2,NN})$ is $p$-regular, then $(P_{1,T})$ is $\lambda$-p-regular.

As in §1, we set $E_p = L^p([0, T]; X)$ and define the operator $A$ by (1.7). We set

$$D(B_p) := \{ u \in W^{1,p}([0, T]; X); u(0) = u(T) \}$$

and

$$B_p u(t) := u'(t), \ t \in [0, T], \ for \ u \in D(B_p).$$

The operators $B_p$ and $-B_p$ are $m$-accretive in $E$, as generators of translation groups. Obviously, $A$ and $B_p$ are resolvent commuting. So by Theorem 1.1, if $A$ satisfies $(H_A)$, $(P_{1,p,T})$ is $\lambda$-p-regular if and only if the pair $(A, B_p)$ verifies (1.1)$_\lambda$.

Similarly if $A$ satisfies $(H_A)$, since $-B_p^2$ is positive with $\omega_{-B_p^2} = 0$, $\omega_{A^2} < \pi$ and $A^2$ and $-B_p^2$ are resolvent commuting, in order to prove that $(P_{2,p,T})$ is $\lambda$-p-regular, it is sufficient to verify (1.1)$_\lambda$, with $A$ replaced by $A^2$ and $B$ by $-B_p^2$.

For the sequel, it is convenient to identify periodic functions defined on $\mathbb{R}$ and functions defined on $[0, T]$, taking the same value on 0 and $T$.

Moreover, replacing $u$ and $f$ by their periodic extensions, we get easily that $(P_{1,p,T})$ is $p$-regular if and only if the extended problem of order one on $\mathbb{R}$ is $p$-regular (in an obvious sense) and $(P_{2,p,T})$ is $p$-regular if and only if the extended problem of order two on $\mathbb{R}$ is $p$-regular.

We will use these two formulations in the proofs, when needed.

**Proof of Theorem 3.1.**

1) $(P_{1,T})$ $p$-regular (resp. $\lambda$-p-regular) $\Rightarrow$ $(P_{1,p,T})$ $p$-regular (resp. $\lambda$-p-regular).

We already know from Proposition 1.3 that problem $(P_{1,T})$ is $p$-regular if and only if it is $\lambda$-p-regular. So it is sufficient to establish: $(P_{1,T})$ $\lambda$-p-regular implies $(P_{1,p,T})$ $\lambda$-p-regular.

Suppose $\lambda > 0$, $u \in D(A) \cap D(B)$ and set $f_\lambda = \lambda Au + Bu$. Since $u$ is a solution to $(P_{1,p,T})$, then $u$ satisfies:

$$u(t) = e^{-\lambda t}u(0) + \int_0^t e^{-\lambda(t-s)}f_\lambda(s) \, ds$$

where $\{e^{-\lambda t}\}_{t>0}$ is the semigroup constructed in [11]. Since, $u(0) = u(T)$ and $1 \in \rho(e^{-\lambda t})$ for all $t > 0$ by Remark 7.2), property (1.9), we have:

$$u(0) = (I - e^{-\lambda T})^{-1} \int_0^T e^{-(T-s)}f_\lambda(s)ds.$$
Defining \( u_1(t) \) and \( u_2(t) \) by

\[
(3.8) \quad u_1(t) = \int_0^t e^{-\lambda(t-s)A}f_\lambda(s)\,ds,
\]

and

\[
(3.9) \quad u_2(t) = e^{-\lambda tA}(I - e^{-\lambda TA})^{-1}\int_0^T e^{-\lambda(T-s)A}f_\lambda(s)\,ds,
\]

we get by (3.6) and (3.7) that

\[
(3.10) \quad u(t) = u_1(t) + u_2(t) = \int_0^t e^{-\lambda(t-s)A}f_\lambda(s)\,ds + e^{-\lambda tA}(I - e^{-\lambda TA})^{-1}\int_0^T e^{-\lambda(T-s)A}f_\lambda(s)\,ds.
\]

From the \( \lambda\)-\( p \)-regularity of \((P_{1,r})\), we have

\[
(3.11) \quad u_1 \in W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A)),
\]

and there exists \( M \geq 1 \) independent of \( \lambda > 0 \), such that

\[
(3.12) \quad \lambda\|Au_1\| \leq M\|f_\lambda\|.
\]

Concerning \( u_2 \) observe that

\[
\int_0^T e^{-\lambda(T-s)A}f_\lambda(s)\,ds = u_1(T)
\]

lies in the trace space associated with \( W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A)) \), see [9, 11].

The same holds for \( (I - e^{-\lambda TA})^{-1}u_1(T) \) since:

\[
(3.13) \quad (I - e^{-\lambda TA})^{-1}u_1(T) = \int_0^T e^{-\lambda(T-s)A}(I - e^{-\lambda TA})^{-1}f_\lambda(s)\,ds.
\]

It follows that, since \( u_2(t) = e^{-\lambda tA}(I - e^{-\lambda TA})^{-1}u_1(T) \), then by [13, 11], \( u_2 \) belongs to \( W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A)) \).

It remains to prove (3.12) with \( u_1 \) replaced by \( u_2 \).

By definition,

\[
u_2(t) = e^{-\lambda tA} \int_0^T e^{-\lambda(T-s)A}(I - e^{-\lambda TA})^{-1}f_\lambda(s)\,ds = \int_0^T e^{-\lambda(t+s)A}(I - e^{-\lambda TA})^{-1}f_\lambda(T-s)\,ds =
\]

\[
\int_0^T e^{-\lambda(t+s)A}(I - e^{-\lambda TA})^{-1}f_\lambda(T-s)\,ds.
\]

Thus,

\[
\int_0^T |\lambda Au_2(t)|^p\,dt = \int_0^T |\lambda A e^{-\lambda(t+s)A}(I - e^{-\lambda TA})^{-1}f_\lambda(T-s)\,ds|^p\,dt \leq
\]

\[
\leq |(I - e^{-\lambda TA})^{-1}|^p \int_0^T \left[ \int_0^T M_Ae^{-\lambda\omega(t+s)} \frac{1}{t+s} |f_\lambda(T-s)|\,ds \right]^p\,dt
\]

\]
by (1.8). Thus,
\[ \int_0^T |\lambda Au_2(t)|^p \, dt \leq |(I - e^{-\lambda TA})^{-1}|^p M_A^p \int_0^T \left[ \int_0^T \frac{1}{t+s} |f_\lambda(T-s)| \, ds \right]^p \, dt. \]

Since the kernel \( \frac{1}{t+s} \) defines a bounded operator on \( L^p([0, T]; \mathbb{R}) \), [14], there exists \( M \), depending only on \( p \) and \( T \), such that:
\[ \int_0^T |\lambda Au_2(t)|^p \, dt \leq M_A^p |(I - e^{-\lambda TA})^{-1}|^p \int_0^T |f_\lambda(t)|^p \, dt. \]

Hence,
\[ ||\lambda Au_2|| \leq M_A |(I - e^{-\lambda TA})^{-1}||f_\lambda||. \]

Then we prove that
\[ (3.14) \sup_{0 < \lambda_0 < \lambda} |(I - e^{-\lambda TA})^{-1}| < \infty \text{ for every } \lambda_0 > 0. \]

Since \( |e^{-tA}| \leq M_A e^{-\omega t} \), for some \( \omega > 0 \) and \( M_A \geq 1 \) by (1.8), we have \( |e^{-\lambda TA}| \leq 1/2 \), for all \( \lambda \in [\lambda_1, \infty) \), for some \( \lambda_1 > 0 \).

It follows that \( |(I - e^{-\lambda TA})^{-1}| \leq 2 \) for \( \lambda \in [\lambda_1, \infty) \).

Observe that \( |(I - e^{-\lambda TA})^{-1}| \) is a continuous function of \( \lambda \in (0, \infty) \), since \( t \to e^{-\lambda t} \) is analytic from \( (0, \infty) \) to \( L(X) \) and \( (I - e^{-\lambda t}) \) is invertible for all \( t > 0 \) by (1.9) in Remark 7.2.

Therefore, \( |(I - e^{-\lambda TA})^{-1}| \) is bounded on intervals \([\lambda_0, \lambda_1]\), with \( 0 < \lambda_0 < \lambda_1 \), which implies (3.12) for \( u_2 \).

We have shown that given \( \lambda_0 > 0 \), there exists \( M = M(\lambda_0) \geq 1 \) such that for every \( u \in W^{1,p}(0, T; X) \cap L^p([0, T]; D(A)) \) satisfying \( u(0) = u(T) \), we have
\[ \lambda \|Au\| \leq M \|u' + \lambda Au\|, \]
for every \( \lambda \in [\lambda_0, \infty) \).

Finally we remove the restriction on \( \lambda_0 \).

For \( u \in D(A) \cap D(B_p) \) and \( n \in \mathbb{N} \), let \( \nu \) be the \( T/n \)-periodic function with values in \( X \) which coincides with \( u(t/n) \) on \([0, T/n]\). Then
\[ \nu \in D(A) \cap D(B_p) \]
and for \( \lambda \in [\lambda_0, \infty) \), we have:
\[ \lambda^\rho \|Av\|_{L^p([0, T]; X)} \leq M_A^\rho \|v' + \lambda Av\|_{L^p([0, T]; X)}. \]

Observe that \( \|Av\|_{L^p([0, T]; X)} = n\|Av\|_{L^p([0, T/n]; X)} \) as well as
\[ \|v' + \lambda Av\|_{L^p([0, T/n]; X)} = n\|v' + \lambda Av\|_{L^p([0, T/n]; X)}. \]

It follows that \( \lambda \|Av\|_{L^p([0, T/n]; X)} \leq M_1 \|v' + \lambda Av\|_{L^p([0, T/n]; X)} \), which implies by rescaling:
\[ (\lambda/n) \|Au\|_{L^p([0, T]; X)} \leq M_1 \|u' + (\lambda/n) Au\|_{L^p([0, T]; X)}. \]
It follows that (3.15) holds for all $\lambda > 0$. This completes the proof of the $\lambda$p-regularity of problem $(P_{1,p,T})$.

2) If $(P_{2,p,T})$ is $p$-regular, then $(P_{2,p,T})$ is $\lambda$-p-regular.

In this part, we will consider periodic functions extended to $\mathbb{R}$, as mentioned in the beginning of the proof.

Let us denote by $L^p_T(Y)$ the space of $T$-periodic functions defined on $\mathbb{R}$ with values in a Banach space $Y$, $p$-integrable on $[0,T]$ and by $W^{m,p}_T(Y)$ the corresponding Sobolev spaces for $m=1,2,\ldots$

Under the hypothesis that $(P_{2,p,T})$ is regular, it is sufficient to prove that for some $M>0$ independent of $\lambda$, for all $u \in W^{2,p}_T(X) \cap L^p_T(D(A))$ and for all $\lambda > 0$,

$$\lambda^2 \|A^2u\| \leq M - u'' + \lambda^2 A^2 u$$

where $\|\cdot\| := \|\cdot\|_{L^p_T(X)}$.

Choose a sequence of mollifiers $(\varphi_n)_{n \in \mathbb{N}}$, with compact supports in $\mathbb{R}$ such that, for $u \in W^{m,p}_T(Y)$:

$$\varphi_n * u \in C_\infty^\infty(T) \text{ and } \varphi_n * u \to u \text{ in } W^{m,p}_T(Y).$$

Let $u \in W^{2,p}_T(X) \cap L^p_T(D(A))$ and define $u_n := \varphi_n * u \in C_\infty^\infty(T(D(A)))$.

Set $v_n := u'_n + \lambda Au_n \in C_\infty^\infty(T(D(A)))$ and $f_n := -v'_n + \lambda Av_n \in C_\infty^\infty(T(X))$.

Then, since $u_n \in C_\infty^\infty(T(D(A)))$, we have $(Au_n)' = Au'_n$ and:

$$-u'' + \lambda^2 A^2 u = f_n.$$ 

We recall a result of [11]:

For all $g \in W^{1,p}_T(X)$, if $A$ verifies $(H_A)$, there exists a solution of the problem

$$z'(t) + \lambda Az(t) = g(t), \quad t \in \mathbb{R}$$

in $W^{1,p}_T(X) \cap L^p_T(D(A))$. Let us denote by $S_{\lambda,T}g$ this solution.

We claim that

$$S_{\lambda,T}g(t) = z(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-\sigma \lambda A} g(t - \sigma) d\sigma. \quad (3.17)$$

To prove this claim, we write

$$(I - e^{-\lambda TA})z(t) = \int_0^T e^{-\sigma \lambda A} g(t - \sigma) d\sigma =$$

$$= \int_0^t e^{-\sigma \lambda A} g(t - \sigma) d\sigma + \int_t^{t+T} e^{-\sigma \lambda A} g(t - \sigma) d\sigma - \int_0^t e^{-\sigma \lambda A} g(t - \sigma) d\sigma =$$

$$= \int_0^t e^{-\sigma \lambda A} g(t - \sigma) d\sigma + \int_0^T e^{-(t+\sigma) \lambda A} g(-\sigma) d\sigma - \int_0^t e^{-(\sigma + T) \lambda A} g(t - \sigma) d\sigma =$$

$$= (I - e^{-\lambda TA}) \int_0^t e^{-\sigma \lambda A} g(t - \sigma) d\sigma + e^{-\lambda T} \int_0^T e^{-\sigma \lambda A} g(T - \sigma) d\sigma.$$ 

This last expression is the one which was already proved in (3.10).
Thus, if \( P \) denotes as before the operator on \( \mathbb{R} \) such that \((Pz)(t) = z(-t)\), we have 
\((Pz')(t) = -(Pz')(t)\) and applying this to \( v_n \), we get:

\[
(Pv_n)(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-\sigma \lambda A}(Pf_n)(t - \sigma) \, d\sigma
\]

which is the same as

\[
(3.18) \quad v_n(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-\sigma \lambda A}v_n(t - \sigma) \, d\sigma.
\]

From the definition of \( u_n \), we have:

\[
u_n(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-\sigma \lambda A}v_n(t - \sigma) \, d\sigma.
\]

Replacing \( v_n \) by its value, we get:

\[
u_n(t) = (I - e^{-\lambda TA})^{-2} \int_0^T e^{-\sigma \lambda A} \int_0^T e^{-\lambda A}f_n(s + t - \sigma) \, ds \, d\sigma.
\]

Set \( D_\lambda = (I - e^{-\lambda TA})^{-2} \).

From the proof of the first part of Theorem 3.1, \( D_\lambda \) is uniformly bounded, say by \( C \), for \( \lambda \geq 1 \). Then:

\[
\|\lambda^2 A^2 u_n\|^p = \int_0^T |\lambda^2 A^2 u_n(t)|^p \, dt = \int_0^T |\lambda^2 A^2| \int_0^T e^{-\sigma \lambda A} \int_0^T e^{-\lambda A}D_\lambda f_n(s + t - \sigma) \, ds \, d\sigma|^p \, dt.
\]

Let us make the change of variables: \( \sigma \lambda \to \sigma \), \( s \lambda \to s \), \( t \lambda \to t \):

\[
\|\lambda^2 A^2 u_n\|^p = \frac{1}{\lambda} \int_0^\lambda |A^2| \int_0^\lambda e^{-\sigma A} \int_0^\lambda e^{-\lambda A}D_\lambda f_n \left( \frac{s + t - \sigma}{\lambda} \right) \, ds \, d\sigma|^p \, dt.
\]

To complete the proof, let \( \lambda = m \in \mathbb{N}^* \) and divide the triple integral above in four parts:

\[
I_1 = \frac{1}{m} \int_0^m |A^2| \int_0^T e^{-\sigma A} \int_0^T e^{-\lambda A}D_m f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma|^p \, dt
\]

\[
I_2 = \frac{1}{m} \int_0^m |A^2| \int_T^T e^{-\sigma A} \int_T^m e^{-\lambda A}D_m f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma|^p \, dt
\]

\[
I_3 = \frac{1}{m} \int_0^m |A^2| \int_T^m e^{-\sigma A} \int_0^T e^{-\lambda A}D_m f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma|^p \, dt
\]

\[
I_4 = \frac{1}{m} \int_0^m |A^2| \int_T^T e^{-\sigma A} \int_T^m e^{-\lambda A}D_m f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma|^p \, dt.
\]
For $I_1$, by periodicity and using the $\lambda$-regularity of $(P_2, P, T)$, we get:

$$I_1 = \frac{1}{m} \sum_{k=0}^{m-1} \int_{kT}^{(k+1)T} \left| A^2 \int_0^T e^{-\sigma A} \int_0^T e^{-\lambda A} D_m f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma \right|^p \, dt =$$

$$= \frac{1}{m} \sum_{k=0}^{m-1} \int_0^T \left| A^2 \int_0^T e^{-\sigma A} \int_0^T e^{-\lambda A} D_m f_n \left( \frac{s + t + kT - \sigma}{m} \right) \, ds \, d\sigma \right|^p \, dt \leq$$

$$\leq M^p \frac{1}{m} \sum_{k=0}^{m-1} \int_0^T \left| D_m f_n \left( \frac{t + kT}{m} \right) \right|^p \, dt \leq$$

$$\leq M^p |D_m|^p \sum_{k=0}^{m-1} \int_{kT}^{(k+1)T} |f_n(t)|^p \, dt = M^p |D_m|^p \|f_n\|^p.$$

For $I_4$, using (1.8), we define a kernel on $\mathbb{R}$ by setting $K(s) = 0$ for $0 \leq s \leq T$ and $K(s) = \frac{M_A}{\omega} e^{-\omega s}$ for $s > T$ and we get by using Young's inequality for convolutions:

$$I_4 = \frac{1}{m} \int_0^{mT} \int_0^{mT} A e^{-\sigma A} \int_T^{mT} A e^{-\lambda A} D_m f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma \right|^p \, dt \leq$$

$$\leq \frac{1}{m} |D_m|^p \int_0^{mT} \left[ \int_0^{mT} K(\sigma) \int_0^{mT} K(s) f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma \right]^p \, dt =$$

$$= |D_m|^p m^2 \int_0^T \left[ \int_0^T K(m\sigma) \int_0^T K(m(s + t - \sigma)) \, ds \, d\sigma \right]^p \, dt \leq$$

$$\leq |D_m|^p \left( \int_0^T K(s) \, ds \right)^{2p} \int_0^T |f_n(t)|^p \, dt \leq \left( \frac{M_A}{\omega T} \right)^{2p} |D_m|^p \|f_n\|^p.$$

For $I_2$ (and in the same way for $I_3$) we also use (1.8):

$$I_2 = \frac{1}{m} \int_0^{mT} \int_T^{mT} A^2 e^{-(\sigma + \lambda)} A D_m f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma \right|^p \, dt \leq$$

$$\leq |D_m|^p \frac{1}{m} \int_0^{mT} \left( \int_T^{mT} \frac{M_A^2}{(s + \sigma)^2} e^{-\sigma (\sigma + \lambda)} f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma \right)^p \, dt \leq$$

$$\leq |D_m|^p \frac{1}{m} \int_0^{mT} \left[ \int_0^T \left( \frac{M_A}{T} \right)^2 e^{-\omega(t + \sigma)} f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma \right]^p \, dt =$$

$$= |D_m|^p \frac{1}{m} \int_0^{mT} \left( \int_0^T K_1(s) \int_0^T K(s) f_n \left( \frac{s + t - \sigma}{m} \right) \, ds \, d\sigma \right)^p \, dt.$$
where $K_1(\sigma) = \frac{M A}{T} e^{-\omega \sigma}$, for $\sigma \geq 0$

$$= |D_m|^p m^{2p} \int_0^T \left( \int_0^T K'(m\sigma) \int_0^T K(ms) |f_n(s + t - \sigma)| \, ds \, d\sigma \right)^p \, dt \leq$$

$$\leq |D_m|^p m^{2p} \left( \int_0^T K'(m\sigma) \, d\sigma \right)^p \left( \int_0^T K(ms) \, ds \right)^p \|f_n\|^p \leq$$

$$\leq |D_m|^p \left( \int_0^T K(s) \, ds \right)^p \left( \int_0^T K'(s) \, ds \right)^p \|f_n\|^p \leq |D_m|^p \frac{M A^{2p}}{(\omega T)^{2p}} \|f_n\|^p.$$ 

Putting together these four majorations, we see that $\|m^2 A^2 u_n\|^p$ is uniformly bounded by a constant times $\|f_n\|^p$ if $m \geq 1$.

To remove this last restriction, we conclude like in the first part of the proof of Theorem 3.1 and we get inequality (1.1) for $u_n$, first for positive rational numbers of the form $m^2 / q$, $m, q \in \mathbb{N}^*$, hence for all positive real numbers $\lambda$.

Taking limits when $n \to \infty$, this inequality remains true for $u \in W_T^{2,p}(X) \cap L_T^p(D(A^2))$, which is sufficient to prove the $\lambda$-$p$-regularity of $(P_{2,p,T})$ as announced at the beginning. \(\square\)

**Proof of Theorem 3.2.**

Recall that in this part, we suppose that $X$ has the UMD property.

1. $(P_{2,DD})$ $p$-regular $\iff (P_{2,NN})$ $p$-regular

Note that, because of the boundary conditions, we can extend functions on $[0, T]$ to $[-T, T]$ by oddness if we are working with $(P_{2,DD})$ and by evenness if we are working with $(P_{2,NN})$. It is clear that like for the periodic case, we can extend these functions to $\mathbb{R}$ by periodicity.

Under the assumptions of Theorem 3.2, $(P_{2,DD})$ (resp. $(P_{2,NN})$) is $p$-regular if and only if there exists $M > 0$ such that, for all $u \in W_T^{2,p}(X) \cap L_T^p(D(A^2))$, odd (resp. even), the following inequality holds:

$$(3.19) \quad \|A^2 u\| \leq M \|A^2 u - u''\|.$$ 

It is enough to prove that if this property is true for odd functions, then it holds for even functions and conversely. Let us prove the direct part, the converse being similar. Suppose that $(P_{2,DD})$ is $p$-regular. Then inequality (3.19) is true for odd functions.

Let $u$ be an even function of $W_T^{2,p}(X) \cap L_T^p(D(A^2))$. Then if $u_0$ is the average of $u$ on $[-T, T]$, we can work with $\bar{u} = u - u_0$ and without loss of generality, we can suppose that $u_0 = 0$.

It is sufficient to show (3.19) when $u$ belongs to a dense subset of even functions belonging to $W_T^{2,p}(X) \cap L_T^p(D(A^2))$, namely the even trigonometrical polynomials with
values in $D(A^2)$:

$$P_n(t) := \sum_{k=1}^{\infty} \cos \frac{2\pi kt}{T} a_k, \text{ with } a_k \in D(A^2).$$

If $\mathcal{H}$ denotes the Hilbert transform, acting on $L^p_T(\mathbb{R}; X)$, we get that

$$(\mathcal{H}P_n)(t) := \sum_{k=1}^{\infty} \sin \frac{2\pi kt}{T} a_k.$$

Since $\mathcal{H}P_n$ is odd, (3.19) holds and we have:

$$\|A^2\mathcal{H}P_n\| \leq M \|A^2\mathcal{H}P_n - (\mathcal{H}P_n)''\|.$$

We have that $\mathcal{H}^2 = -I$, $\mathcal{H}$ commutes with $A^2$ and with the second derivative. Since $X$ has the UMD property, there exists $C >$ such that:

$$\|A^2P_n\| \leq MC\|A^2P_n - P_n''\|.$$

This proves the $p$-regularity of $(P_{2,NN})$.

2. $(P_{2,p,T})$ $p$-regular $\Rightarrow$ (3.3) holds

We want to apply Sobolevskii’s result written in Theorem 1.2. We already know that if $(P_{2,p,T})$ is $p$-regular then it is also $\lambda$-$p$-regular by Theorem 3.1 2). After substracting averages, which is possible since $0 \in \rho(A)$, it is convenient to work with spaces of periodic functions with null average. Set:

$$W^{m,p}_{T,*} := \left\{ u \in W^{m,p}_T \mid \frac{1}{T} \int_{-T/2}^{T/2} u(t) dt = 0 \right\}$$

for $m = 0, 1, 2, \ldots$. On $W^{m,p}_{T,*}(X)$, the second derivative $-B^2_{p,*}$ is invertible. Since $A^2$ is also invertible by hypothesis, we are in position to apply Theorem 1.2 to $A^2$ and $-B^2_{p,*}$, which denotes the restrictions of these operators to $L^p_{T,*}(D(A^2))$ and $W^{2,p}_{T,*}(X)$. From the Banach valued version of Marcinkiewicz’s theorem instead of Mihlin’s theorem, see [1, 31], we obtain:

$$D(B^2_{p,*}) \cap D(A^2_{p,*}) \subset D\left[ (I - B^2_{p,*})^{1/2} (A^2_{p,*})^{1/2} \right]$$

and

$$\| \left[ (I - B^2_{p,*})^{1/2} (A^2_{p,*})^{1/2} \right] u \| \leq M \| A^2_{p,*} u - u'' \|$$

for some $M > 0$ and for all $u \in W^{2,p}_{T,*}(X) \cap L^p_{T,*}(D(A^2))$.

Observe that $(A^2_{p,*})^{1/2} = A_{p,*}$ and $(I - B^2_{p,*})^{1/2} = \mathcal{H}B_{p,*}$ where $\mathcal{H}$ is the Hilbert transform, [1]. We get then that, on $W^{2,p}_{T,*}(X) \cap L^p_{T,*}(D(A^2))$

$$\| \mathcal{H}A_{p,*} u' \| \leq M \| A^2_{p,*} - u'' \|.$$

Since $X$ has the UMD property, $\mathcal{H}$ is bounded on $L^p_T(\mathbb{R}; X)$ and this inequality implies (3.3). $\Box$
Proof of Proposition 3.3.

1. \((P_{1}, P; T)\) \(p\)-regular \(\iff\) \((P_{2}, P; T)\) \(p\)-regular and \((3.3)\)

Since \(B_{P}\) is the generator of the translation group on \(L^{p}([0, T]; X)\), 1) follows from Proposition 2.1.

2. \((P_{2}, P; T)\) \(p\)-regular \(\iff\) \((P_{2, DD})\) and \((P_{2, NN})\) \(p\)-regular

It is sufficient to remark that, after extending these problems to \(\mathbb{R}\) by periodicity for \((P_{2}, P; T)\), oddness then periodicity for \((P_{2, DD})\) and evenness then periodicity for \((P_{2, NN})\), the solution of \((P_{2, DD})\) is the odd part of the solution of \((P_{2}, P; T)\) and the solution of \((P_{2, NN})\) its even part. Conversely, by decomposing functions into odd parts and even parts, the solution of \((P_{2}, P; T)\) is the sum of the solution of \((P_{2, DD})\) and the solution of \((P_{2, NN})\).

3. \((P_{1}, P; T)\) \(p\)-regular \(\implies\) \((P_{1, T})\) \(p\)-regular

It is enough to show that problem \((P_{1, T})\) has a solution for every \(f \in L^{p}([0, T]; X)\). For such function \(f\), let \(u_{1}\) be the solution to \((P_{1, T})\). Then there exists exactly one function \(u_{2} \in W^{1, p}([0, T]; X) \cap L^{p}([0, T]; D(A))\) satisfying \(u_{2}'(t) + Au_{2}(t) = 0\), \(u_{2}(0) = -u_{1}(0)\), (see [9, appendix on trace spaces] and [11]).

Thus \(u = u_{1} + u_{2}\) is a solution to \((P_{1, T})\). \(\square\)

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References


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