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On the order of transitive permutation groups with cyclic point-stabilizer

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Teoria dei gruppi. — *On the order of transitive permutation groups with cyclic point-stabilizer.* Nota (*) di ANDREA LUCCHINI, presentata dal Socio G. Zacher.

ABSTRACT. — If G is a transitive permutation group of degree n with cyclic point-stabilizer, then the order of G is at most $n^2 - n$.

KEY WORDS: Permutation groups; Transitivity; Normal-core.

RIASSUNTO. — *Sui gruppi di permutazione transitivi con stabilizzante ciclico.* Se G è un gruppo di permutazioni transitivo in cui lo stabilizzante di un punto è ciclico, allora l'ordine di G è al più $n^2 - n$.

In this *Note* we prove a conjecture, due to Babai, Goodman and Pyber [1], concerning the order of a transitive permutation group with cyclic point-stabilizer subgroup. Precisely we prove:

THEOREM. *Let G be a transitive permutation group of degree $n > 1$ whose point-stabilizer subgroup is cyclic. Then $|G| \leq n^2 - n$.*

The bound $n^2 - n$ is the best possible as shown by the groups $AGL(1, p^r) = \{x \to ax + b \mid a, b \in GF(p^r), a \neq 0\}$ of degree $n = p^r$ and order $n(n-1)$.

As usual, $\mathrm{Core}_G(H)$ denotes the largest normal subgroup of G contained in the subgroup H. Of course our theorem can be equivalentely written as follows:

PROPOSITION. *If a group G contains a proper cyclic subgroup H of index n, then the index in G of $\mathrm{Core}_G(H)$ is at most $n^2 - n$.*

The proof is elementary. The crucial step (Lemma 1.1) makes use of the methods introduced in [2] by Chermak and Delgado to prove that if S is a finite nonabelian simple group and A is an abelian subgroup then $|A|^2 < |S|$.

1. A PRELIMINARY RESULT

To prove our theorem we need the following:

LEMMA 1.1. *Let G be a finite group with the property that all its nonidentity normal subgroups are nonabelian. Then*
(a) $|G| \geq |H||C_G(H)|$ *for each subgroup H of G.*
(b) $|G| > |H||C_G(H)|$ *for each nonidentity soluble subgroup H of G.*

The proof of this lemma relies on the methods introduced by Chermak and Delgado in [2]. We recall some results from that paper. Let G be a finite group and let $\mathcal{S}(G)$

(*) Pervenuta in forma definitiva all'Accademia il 16 luglio 1998.

denote the set of all nonidentity subgroups of G. For any subgroup H define $f(H) = |H||C_G(H)|$ and let $m = \sup\{f(H) \mid H \in \mathcal{S}(G)\}$, $\mathcal{M} = \{H \in \mathcal{S}(G) \mid f(H) = m\}$, and let $\mathcal{M}_*$ be the set of minimal members of $\mathcal{M}$, under inclusion. In [2, Section 1], the following results are proved:

1.2. *If* $H_1, H_2 \in \mathcal{M}$ *then* $H_1H_2 = H_2H_1$ *and* $H_1H_2 \in \mathcal{M}$. *In particular if* $H \in \mathcal{M}$ *then* $H^G = \langle H^g \mid g \in G\rangle \in \mathcal{M}$.

1.3. *If* $|\mathcal{M}_*| > 1$ *and* H_1, H_2 *are distinct members of* $\mathcal{M}_*$ *then* $[H_1, H_2] = 1$.

Proof of Lemma 1.1. Let $N_1, \dots, N_t$ be the minimal normal subgroups of G. By 1.2 the set $\mathcal{M}$ contains a normal subgroup of G, say K. Let $X = \{1, \dots, t\}$, $X_1 = \{j \in X | N_j \le K\}$, $X_2 = \{j \in X | N_j \not\le K\}$ and consider $U = \prod_{i\in X_1} N_i$, $V = \prod_{i \in X_2} N_i$. Since $[K, V] \le K \cap V = 1$, we have $K \le C_G(V)$. On the other hand $U \le K$ implies $C_G(K) \le C_G(U)$. So we have $m = f(K) = |K||C_G(K)| \le |C_G(V)||C_G(U)| \le |G| = f(G)$, where the last inequality follows from $C_G(V) \cap C_G(U) = C_G(\mathrm{Soc}\, G) = 1$. So we conclude $m = f(G)$ which means $|H||C_G(H)| \le |G|$ for each $H \le G$. This proves (*a*).

Now let H be a nonidentity soluble subgroup of G. Suppose by way of contradiction that $f(H) \ge |G|$. Since we proved that $m = f(G) = |G|$ it must be $H \in \mathcal{M}$ and there exists $J \le H$ with $J \in \mathcal{M}_*$. By hypothesis G contains no soluble normal subgroup, so $J^g \ne J$ for some $g \in G$ and $|\mathcal{M}_*| > 1$. By 1.3 $[J, J^g] = 1$ for each $g \notin N_G(J)$, hence J^G is a soluble normal subgroup of G, a contradiction. □

2. Proof of the theorem

We have to prove that if a nonidentity finite group G contains a core-free cyclic subgroup H then $|H|^2 < |G|$. We use induction on the order of G.

First suppose that G contains a minimal normal subgroup N which is abelian. Let $N \cong \mathbb{Z}_p^d$ and let $K/N = \mathrm{Core}_{G/N}(HN/N)$. By induction $|HN/K|^2 \le |G/K|$ and equality holds if and only if $G = HN = K$. By Dedekind's modular law, $K = JN$ with $J = H \cap K$. Since $Z(K) \cap N \trianglelefteq G$, one of the two following possibilities holds:

a) $Z(K) \cap N = 1$.

In that case $N \cap J = N \cap H = 1$ and $C_J(N) = Z(K) \trianglelefteq G$. Since $J_G \le \mathrm{Core}_G(H) = 1$, it must be $C_J(N) = 1$ so J is a cyclic subgroup of $\mathrm{Aut}\, N \cong GL(d, p)$. As the order of an element in $GL(d, p)$ is at most $p^d - 1$ (see, for example, [3, Theorem 3.1]), we conclude $|J| < |N|$ and

$$\frac{|H|^2}{|J|^2} = \left|\frac{HN}{K}\right|^2 \le \left|\frac{G}{K}\right| = \frac{|G|}{|N||J|} \qquad \text{so} \qquad |H|^2 \le \frac{|G||J|}{|N|} < |G|.$$

b) $N \le Z(K)$.

In that case K is abelian. If a prime q, different from p, divides $|J|$, then the Sylow q-subgroup of J is a nonidentity normal subgroup of G contained in H, in contradiction with $\mathrm{Core}_G(H) = 1$. Moreover if J contains an element x of order p^2

then, for any $g \in G$, there exist an integer i_g and an element $n_g \in N$ such that $x^g = x^{i_g} n_g$, hence $(x^p)^g = (x^g)^p = (x^{i_g} n_g)^p = x^{i_g p} = (x^p)^{i_g}$; so $\langle x^p \rangle$ is a normal subgroup of G again in contradiction with $\mathrm{Core}_G(H) = 1$. We deduce $J \leq \mathbb{Z}_p$ and, since it cannot be $J = N$, we have $|J \cap N||J| \leq |N|$. If $G = K$ then $H \trianglelefteq G$, so it must be $H = 1$ and the conclusion follows since we are assuming $G \neq 1$. If $G \neq K$, by induction

$$\frac{|H|^2}{|J|^2} = \left|\frac{HN}{K}\right|^2 < \left|\frac{G}{K}\right| = \frac{|G||J \cap N|}{|N||J|} \qquad \text{so} \qquad |H|^2 < \frac{|G||J \cap N||J|}{|N|} \leq |G|.$$

So we may assume that G contains no nonidentity abelian normal subgroup. In that case from Lemma 1.1 (*b*) we obtain $|H|^2 \leq |H||C_G(H)| < |G|$.

This concludes the proof of the theorem.

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