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Unconditional nonlinear exponential stability in the Bénard problem for a mixture: necessary and sufficient conditions

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Meccanica dei fluidi. — *Unconditional nonlinear exponential stability in the Bénard problem for a mixture: necessary and sufficient conditions.* Nota di Giuseppe Mulone e Salvatore Rionero, presentata (*) dal Corrisp. S. Rionero.

Abstract. — The Lyapunov direct method is applied to study nonlinear exponential stability of a basic motionless state to imposed linear temperature and concentration fields of a binary fluid mixture heated and salted from below, in the Oberbeck-Boussinesq scheme. Stress-free and rigid surfaces are considered and absence of Hopf bifurcation is assumed. We prove the coincidence of the linear and (unconditional) nonlinear critical stability limits, when the ratio between the Schmidt and the Prandtl numbers is less or equal to 1. Precisely, we obtain necessary and sufficient conditions of unconditional nonlinear exponential stability of the basic motionless state.

Key words: Fluid mixture heated and salted; Lyapunov unconditional nonlinear stability; Natural convection.

Riassunto. — *Stabilità non lineare esponenziale incondizionata nel problema di Bénard per una miscela: condizioni necessarie e sufficienti.* Si applica il metodo diretto di Lyapunov allo studio della stabilità non lineare esponenziale della soluzione di conduzione-diffusione di una miscela fluida binaria riscaldata e salata da sotto, nello schema di Oberbeck-Boussinesq. Si considerano superfici rigide e *stress-free* ; si suppone che non ci sia biforcazione di Hopf. Supposto che il rapporto fra i numeri di Schmidt e di Prandtl è minore o uguale a 1, proviamo la coincidenza fra i parametri critici della stabilità lineare e non lineare. Si ottengono condizioni necessarie e sufficienti di stabilità non lineare esponenziale del moto base.

1. Introduction

The study of stability and instability of motions of a binary fluid mixture heated and salted from below is relevant in many geophysical applications, in particular in the context of the «salt pond» (or «solar pond»), ([1-3], see also [4 and the references therein]).

Here we study the asymptotic stability of a motionless state, to imposed linear temperature and concentration fields (*conduction - diffusion solution*), of a fluid layer of a binary mixture heated and salted from below, bounded by two horizontal parallel planes. We assume that $P_C \leq P_T$, in this case Hopf bifurcation's cannot occur, and consider the following possibilities:

(*a*) both the bounding surfaces are stress-free,

(*b*) both the bounding surfaces are rigid,

(*c*) one of the bounding surfaces is rigid and the other bounding surface is stress-free.

Moreover we discuss the stability problem of the conduction-diffusion solution in the case when the domain is arbitrary and its boundary can be composed of both stress-free and rigid elements.

(*) Nella seduta del 13 marzo 1998.

This problem, known also as *the Bénard problem for a mixture*, has been studied in the linear context in [5-9].

The linear results show that the gradient of the solute stabilizes the onset of convection and the motionless state is linearly unstable when the Rayleigh number of the temperature $\mathcal{R}^2$ (a measure of the gradient of temperature) is related to the Rayleigh number of the concentration $\mathcal{C}^2$ (a measure of the gradient of the concentration of the solute) by the inequality

$$\mathcal{R}^2 > \mathcal{C}^2 + \mathcal{R}_B^2 , \tag{1.1}$$

where

$$\mathcal{R}_B^{-1} = \max_{\mathcal{S}} \frac{2(w, \vartheta)}{\mathcal{D}[\mathbf{u}, \vartheta]} \tag{1.2}$$

is the smallest critical value for the Bénard problem for a homogeneous fluid (see [4, Chap. IX]). In (1.2) $\mathbf{u}$ and ϑ are the perturbations to the velocity and temperature fields, $w = \mathbf{u} \cdot \mathbf{k}$, $\mathcal{D}[\mathbf{u}, \vartheta] = 2\|\mathbf{D}(\mathbf{u})\|^2 + \|\nabla\vartheta\|^2$, $\mathbf{D}(\mathbf{u})$ is the symmetric part of $\nabla\mathbf{u}$, $\|f\|$ is the norm of f in $L_2(\Omega)$ (Ω being a suitable cell of periodicity) and $\mathcal{S}$ is the space of the admissible fields. This result also holds when the domain is arbitrary, and the boundary can be composed of both stress-free and rigid elements (provided that the values assumed by the linear distributions of temperature and concentration of the basic motion are in accord with preassigned boundary values), [4, 9].

In the nonlinear context, this problem has been studied, in [9], with the classical energy

$$E_0(t) = 1/2\left(\|\mathbf{u}\|^2 + P_T\|\vartheta\|^2 + P_C\|\gamma\|^2\right) , \tag{1.3}$$

where γ is the perturbation to the concentration field, and global exponential stability has been obtained for

$$\mathcal{R}^2 < \mathcal{R}_B^2. \tag{1.4}$$

This nonlinear stability region is the same as in the case of the simple Bénard problem for a homogeneous fluid, [4]. In fact, in the simple Bénard problem, the linear operator associated to the system of the disturbance to the basic motion is symmetric in the inner product associated to the energy norm, while in our case (binary fluid mixture) the linear operator is divided into two parts: one is symmetric and the other is skewsymmetric (this second part is connected to $\mathcal{C}^2$) and then, the gradient of solute has only a not-destabilizing effect in this norm (like in other convection problems, *e.g.*, the Bénard problems with rotation and a magnetic field, see [10-15]). Other nonlinear stability results are those of [16-18]. In [17] a perturbation method has been used and in [18] a method of modal truncation and a computer has been adopted. Joseph in [16] introduces a generalized energy method to show the stabilizing effect of the gradient of solute on thermal convection, for an arbitrary bounded domain and a layer, and he proves a nonlinear stability theorem which gives global stability in the sense that it holds for any initial value bounded in L_2. But the stability so guaranteed *has not been shown to be exponential*, [4, 16], and the possibility that a very large, stable disturbance, could be extraordinarily persistent is not excluded (at least when a part of bounding surface is rigid).

In recent papers [19-23], using the Lyapunov direct method, the nonlinear conditional stability of the motionless state and parallel shear flows and of plane parallel convective flows of a homogeneous fluid and a binary fluid mixture has been examined. In particular, in [23], a nonlinear stability theorem, which gives exponential stability and shows the stabilizing effect of gradient of solute on thermal convection, has been proved for a fluid layer. When the ratio p of the Schmidt and the Prandtl number is less than 1, *a region of coincidence of linear and nonlinear exponential stability* has been found, but this coincidence has been given *only* for $\mathcal{C}^2 < (3 + p)/(1 - p)\mathcal{R}_B^2$ and for stress-free boundary conditions.

The aim of this paper is to show that for *any* $\mathcal{C}^2$, and *any* of the boundary conditions (a)-(c), the critical linear instability Rayleigh numbers $\mathcal{R}_c^2$ *coincide* with the critical unconditional exponential nonlinear stability Rayleigh numbers $\mathcal{R}_E^2$. Therefore, we obtain *necessary and sufficient conditions of unconditional nonlinear exponential* stability of the basic motion. The main idea is to introduce (like in [16]) two new fields ϕ and ψ which are in a one-to-one correspondence with ϑ and γ and obtain a system for the perturbations $\mathbf{u}\,,\phi\,,\psi$ which is equivalent to the system for the perturbations $\mathbf{u}\,,\vartheta\,,\gamma$. Then, we give a Lyapunov function $E(t)$, depending on ϕ and ψ which is equivalent to the energy norm $E_0(t)$, and prove that, by choosing properly the Lyapunov parameters, the critical nonlinear exponential stability Rayleigh numbers (obtained with this norm) coincide with the linear ones (obtained via the *normal modes* analysis).

The plan of the paper is as follows: in Section 2 we give the basic motion m_0 and write the system for the perturbations $\mathbf{u}\,,\vartheta\,,\gamma\,,p_1$ to m_0 (p_1 is the perturbation to the pressure field). Then, we recall some linear instability and nonlinear stability results. In Section 3 we introduce the fields ϕ and ψ and get the system for $\mathbf{u}\,,\phi\,,\psi\,,p_1$. Then, we define a Lyapunov function $E(t)$, equivalent to $E_0(t)$, and write the time evolution of $E(t)$. We obtain a nonlinear stability condition and solve the corresponding maximum problem, in the case of a horizontal layer bounded by two stress-free surfaces. Then, by properly choosing the Lyapunov parameters, our main result is proved: the critical linear instability Rayleigh number $\mathcal{R}_c^2$ coincides with the critical nonlinear stability Rayleigh number $\mathcal{R}_E^2$. In Section 4, by a suitable change of fields, we prove that the aforesaid coincidence holds also when the boundary conditions are of type rigid-rigid or rigid-free and for any arbitrary domain. Finally, we make some remarks for the case when the basic motion is a laminar flow and for $p > 1$.

2. Basic equations and recall of some linear and non linear stability results

Let us consider a layer of a binary fluid mixture heated and salted from below, in the Oberbeck-Boussinesq scheme, bounded by two horizontal parallel planes. Let $d > 0$, $\Omega_d = \mathbb{R}^2 \times (-d/2\,, d/2)$ and $Oxyz$ be a cartesian frame of reference with unit vectors $\mathbf{i}\,,\ \mathbf{j}\,,\ \mathbf{k}$ respectively. Let us assume that the layer is parallel to the plane $z = 0$.

The basic motion $m_0 = (\mathbf{U}, T, C, p_1)$ is given by the following equations (see [4, 21]):

$$(2.1)\qquad \begin{cases} \mathbf{U} = 0, \quad T = -\dfrac{T_1 - T_2}{d} z + T_0, \quad C = -\dfrac{C_1 - C_2}{d} z + C_0 \\ \dfrac{p_1}{\rho_0} = -gz - \dfrac{\alpha_T g}{2d}(T_1 - T_2) z^2 + \dfrac{\alpha_C g}{2d}(C_1 - C_2) z^2 + p_0 \end{cases}$$

where $\mathbf{U}$, T, C and p_1 are the velocity, temperature, concentration and pressure fields; ρ_0, T_0, C_0 are reference density, temperature and concentration, respectively; α_T and α_C are volume expansion coefficients, $\mathbf{g} = -g\mathbf{k}$ is the acceleration of gravity, p_0 is a real number, $T_0 = (T_1 + T_2)/2$, $C_0 = (C_1 + C_2)/2$ and T_1, T_2, C_1, C_2 are real numbers, with $T_1 > T_2$ and $C_1 > C_2$.

The non-dimensional equations which govern the evolution disturbance to the velocity, temperature, concentration and pressure fields $(\mathbf{u}, \vartheta, \gamma, p_2)$ to m_0 are [4, 15]:

$$(2.2)\qquad \begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p_2 + (\mathcal{R}\vartheta - \mathcal{C}\gamma)\mathbf{k} + \Delta\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0 \\ P_T\big(\vartheta_t + \mathbf{u}\cdot\nabla\vartheta\big) = \mathcal{R}w + \Delta\vartheta \\ P_C\big(\gamma_t + \mathbf{u}\cdot\nabla\gamma\big) = \mathcal{C}w + \Delta\gamma, \end{cases}$$

in $\Omega_1 \times (0, \infty)$, where $\Omega_1 = \mathbb{R}^2 \times (-1/2, 1/2)$, with initial condition

$$(2.3)\qquad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \vartheta(\mathbf{x}, 0) = \vartheta_0(\mathbf{x}), \quad \gamma(\mathbf{x}, 0) = \gamma_0(\mathbf{x}),$$

$\mathbf{x} = (x, y, z) \in \Omega_1$, and boundary conditions

$$(2.4)\qquad \begin{cases} \vartheta(x, y, \pm 1/2, t) = \gamma(x, y, \pm 1/2, t) = 0, \quad t > 0, \qquad \text{and} \\ w(\mathbf{x}, t) = 0, \ u_z(\mathbf{x}, t) = v_z(\mathbf{x}, t) = 0, \quad \text{on a stress-free surface,} \\ \mathbf{u}(\mathbf{x}, t) = 0, \qquad \text{on a rigid surface.} \end{cases}$$

The subscripts z and t denote partial derivatives, $\mathbf{u}_0$, ϑ_0, γ_0 are assigned regular fields with $\nabla\cdot\mathbf{u}_0(\mathbf{x}) = 0$, $\mathbf{u} = (u, v, w)$, ∇ is the «nabla» operator and Δ is the laplacian. The stability parameters in (2.2) are the Rayleigh numbers for heat and solute and are given by

$$\mathcal{R}^2 = g\beta_1\alpha_T d^4/\nu k_T, \quad \mathcal{C}^2 = g\beta_2\alpha_C d^4/\nu k_C \quad \text{(Rayleigh number for heat and solute),}$$

where β_1 and β_2 are the constant gradients of temperature and concentration, respectively, k_T and k_C are the thermal and solute diffusivity coefficients, ν is the kinematic viscosity. Moreover

$$P_T = \nu/k_T \quad \text{and} \quad P_C = \nu/k_C$$

are the Prandtl and Schmidt numbers.

As it is usual, we assume that the perturbations $\mathbf{u}, \vartheta, \gamma, p_2$ are periodic functions of x and y of periods $2\pi/a_x$, $2\pi/a_y$, respectively, $(a_x > 0,\ a_y > 0)$ and denote by Ω the periodicity cell $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [-1/2, 1/2]$ and by $a = (a_x^2 + a_y^2)^{1/2}$

the wave number. Moreover, taking into account the fact that the stability of m_0 makes sense only in a class of solutions of (2.2)-(2.4) in which m_0 is unique, in the case when both the bounding planes are stress-free, we exclude any other rigid solution requiring the «average velocity condition»

$$\int_\Omega u\, d\Omega = \int_\Omega v\, d\Omega = 0. \tag{2.5}$$

For a bounded domain the stress-free boundary conditions correspond to the conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0\,, \qquad [\mathbf{D}(\mathbf{u})\cdot\mathbf{n} - \mathbf{n}\cdot\mathbf{D}(\mathbf{u})\cdot\mathbf{nn}]|_{\partial\Omega} = 0\,,$$

where $\mathbf{n}$ is the unit vector of the outward normal to $\partial\Omega$.

The stability of the conduction - diffusion solution of a mixture heated and salted from below, has been studied in the linear case in [5-8].

For a horizontal layer bounded by both stress-free surfaces on which ϑ and γ are required to vanish, instability is guaranteed when

$$\begin{aligned}
&\mathcal{R}^2 > \mathcal{R}_B{}^2 + \mathcal{C}^2 \qquad (p < 1)\,,\\
&\mathcal{R}^2 > \mathcal{R}_B{}^2 + \mathcal{C}^2 \qquad (p > 1\,,\ \mathcal{C}/\mathcal{C}_0 < 1)\\
&\mathcal{R}^2 > \frac{\mathcal{R}_B{}^2(p+1)(1+P_Tp)}{P_Tp^2} + \frac{(1+P_Tp)\mathcal{C}^2}{(1+P_T)p^2} \quad (p > 1\,,\ \mathcal{C}/\mathcal{C}_0 > 1)\,,
\end{aligned} \tag{2.6}$$

where

$$\mathcal{R}_B{}^2 = 657.511\,, \qquad \mathcal{C}_0{}^2 = (\mathcal{R}_B{}^2/P_T)\big((P_T+1)/(p-1)\big)\,, \tag{2.7}$$

and

$$p = P_C/P_T. \tag{2.8}$$

For any arbitrary domain whose boundary can be composed of both stress-free and rigid elements (on which $\vartheta = \gamma = 0$) instability is guaranteed (see [4, 9]) when

$$\mathcal{R}^2 > \mathcal{R}_B{}^2 + \mathcal{C}^2\,, \tag{2.9}$$

with $\mathcal{R}_B{}^2$ given by (1.2).

In the nonlinear case the stability problem has been studied in [9, 16-18].

In [9], the classical energy

$$E_0(t) = 1/2\big(\|\mathbf{u}\|^2 + P_T\|\vartheta\|^2 + P_C\|\gamma\|^2\big)\,, \tag{2.10}$$

has been used, and global exponential stability has been obtained for

$$\mathcal{R}^2 < \mathcal{R}_B^2. \tag{2.11}$$

Thus the nonlinear stability region, obtained with this norm, is the same as in the case of the simple Bénard problem for a homogeneous fluid [4], and then, the stabilizing effect of the gradient of concentration on the onset of convection is not achieved. In order to obtain this effect, Joseph, [16, 4], uses a generalized energy method to show the stabilizing effect of the gradient of solute on thermal convection. He proves a nonlinear stability theorem, [16, Theorem 1], which gives global stability, in the sense that it holds for any initial value bounded in L_2. But the stability so guaranteed *has not been shown* to be exponential. In [23], a theorem of nonlinear exponential stability has been shown and the coincidence between the *critical linear instability Rayleigh number*

$$\mathcal{R}_c^2 = \mathcal{R}_B{}^2 + \mathcal{C}^2 \tag{2.12}$$

and the nonlinear one has been obtained for a layer, with stress-free boundary conditions, when

$$\mathcal{C}^2 < (3 + p)/(1 - p)\mathcal{R}_B{}^2 .$$

3. Unconditional nonlinear exponential stability and the maximum problem. Coincidence of the critical linear and nonlinear limits in the case of stress-free surfaces

Here we prove a nonlinear exponential stability theorem which shows, for $p < 1$, the *coincidence* between the linear critical Rayleigh number and the nonlinear one for *any* $\mathcal{C}^2$. For this, now we introduce two fields ϕ and ψ, which are in a one-to-one correspondence with ϑ and γ, and give the system for the new perturbations $\mathbf{u}, \phi, \psi, p_2$.

Let us define the fields

$$\phi = \mathcal{R}\vartheta - \mathcal{C}\gamma \tag{3.1}$$

$$\psi = \mathcal{R}\vartheta - p\delta\mathcal{C}\gamma \tag{3.2}$$

with p defined by (2.8) and $\delta \in \mathbb{R}$ is a constant to be chosen. We assume that

$$p\delta \neq 1. \tag{3.3}$$

Let us observe that the first field ϕ comes naturally from $(2.2)_1$, while the second is related to the first as in the Joseph's coupling parameters method, [16].

Then, the system (2.2) is equivalent to the system

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p_2 + \phi\mathbf{k} + \Delta\mathbf{u}\,, \qquad \nabla\cdot\mathbf{u} = 0 \\ P_T(\psi_t + \mathbf{u}\cdot\nabla\psi) = (\mathcal{R}^2 - \delta\mathcal{C}^2)w + \dfrac{\delta(p-1)}{p\delta - 1}\Delta\phi + \dfrac{\delta - 1}{p\delta - 1}\Delta\psi \\ P_C(\phi_t + \mathbf{u}\cdot\nabla\phi) = (p\mathcal{R}^2 - \mathcal{C}^2)w + \dfrac{p^2\delta - 1}{p\delta - 1}\Delta\phi - \dfrac{p-1}{p\delta - 1}\Delta\psi\,, \end{cases} \tag{3.4}$$

with boundary conditions $(2.4)_2$, $(2.4)_3$ and

$$\phi = \psi = 0 \qquad \text{on} \quad z = \pm 1/2\,, \qquad t > 0\,,$$

and initial conditions $(2.3)_1$ and

$$\phi(\mathbf{x}, 0) = \mathcal{R}\vartheta_0 - \mathcal{C}\gamma_0\,, \quad \psi(\mathbf{x}, 0) = \mathcal{R}\vartheta_0 - p\delta\mathcal{C}\gamma_0\,, \quad \mathbf{x} \in \Omega_1 .$$

In order to study the stability of the basic motion, we use the following Lyapunov function

$$E(t) = 1/2[\lambda\|\mathbf{u}\|^2 + \mu P_T\|\psi\|^2 + P_C\|\phi\|^2], \tag{3.5}$$

with λ, μ positive constants to be chosen.

The evolution equation of $E(t)$ is:

$$\begin{aligned}\dot{E}(t) &= (p\mathcal{R}^2 - \mathcal{C}^2 + \lambda)(\phi, w) + \mu(\mathcal{R}^2 - \delta\mathcal{C}^2)(w, \psi) - [2\lambda\|\mathbf{D}(\mathbf{u})\|^2 + \\ &+ \frac{p^2\delta - 1}{p\delta - 1}\|\nabla\phi\|^2 + \frac{\mu(\delta - 1)}{p\delta - 1}\|\nabla\psi\|^2 + \frac{(p-1)(\mu\delta - 1)}{p\delta - 1}(\nabla\phi, \nabla\psi)] = \mathcal{I}_0 - \mathcal{D}_0,\end{aligned} \tag{3.6}$$

with

$$\mathcal{I}_0 = (p\mathcal{R}^2 - \mathcal{C}^2 + \lambda)(\phi, w) + \mu(\mathcal{R}^2 - \delta\mathcal{C}^2)(w, \psi), \tag{3.7}$$

$$\mathcal{D}_0 = 2\lambda\|\mathbf{D}(\mathbf{u})\|^2 + \frac{p^2\delta - 1}{p\delta - 1}\|\nabla\phi\|^2 + \mu\frac{\delta - 1}{p\delta - 1}\|\nabla\psi\|^2 + \frac{(p-1)(\mu\delta - 1)}{p\delta - 1}(\nabla\phi, \nabla\psi). \tag{3.8}$$

In order to assure that the functional $\mathcal{D}_0$ is positive-definite, we require

$$\frac{p^2\delta - 1}{p\delta - 1} > 0, \quad \frac{\delta - 1}{p\delta - 1} > 0, \quad [(p-1)(\mu\delta - 1)]^2 - 4\mu(p^2\delta - 1)(\delta - 1) < 0. \tag{3.9}$$

From the assumption (3.9) it follows that there exists a positive real number α_1 such that

$$\begin{aligned}&\alpha_1(\|\nabla\phi\|^2 + \|\nabla\psi\|^2) \leq \\ &\qquad \leq \frac{p^2\delta - 1}{p\delta - 1}\|\nabla\phi\|^2 + \mu\frac{\delta - 1}{p\delta - 1}\|\nabla\psi\|^2 + \frac{(p-1)(\mu\delta - 1)}{p\delta - 1}(\nabla\phi, \nabla\psi), \\ &\alpha_1 = \frac{4\mu(\delta - 1)(p^2\delta - 1) - (p-1)^2(\mu\delta - 1)^2}{4(p\delta - 1)[p^2\delta - 1 + \mu(\delta - 1)]}.\end{aligned} \tag{3.10}$$

From this it follows that

$$\dot{E}(t) \leq (m - 1)\mathcal{D}_0, \tag{3.11}$$

where

$$m = \max_{\mathcal{H}} \frac{\mathcal{I}_0}{\mathcal{D}_0}, \tag{3.12}$$

and $\mathcal{H}$ is the space of the admissible fields:

$$\begin{aligned}\mathcal{H} = \{\ &\mathbf{u}, \psi, \phi \text{ regular fields, periodic in } x \text{ and } y \text{ of periods } 2\pi/a_x,\ 2\pi/a_y, \\ &0 < \mathcal{D}_0 < \infty, \text{ satisfying (2.4) and, for both stress-free surfaces, (2.5)}\}.\end{aligned} \tag{3.13}$$

Assuming (*stability condition*)

$$m < 1, \tag{3.14}$$

then, from (3.11), we have

$$\dot{E}(t) \leq (m - 1)(\alpha_1\|\nabla\phi\|^2 + \alpha_1\|\nabla\psi\|^2 + \lambda\|\nabla\mathbf{u}\|^2). \tag{3.15}$$

Now we use the *Poincaré inequality*

$$k^2(\Omega) \int_\Omega f^2 \, d\Omega \le \int_\Omega |\nabla f|^2 \, d\Omega \,, \tag{3.16}$$

where f is a regular field in the periodicity cell Ω, $f = 0$ on $z = \pm 1/2$ or $f = 0$ on $z = -1/2$, $f_z = 0$ on $z = 1/2$ ($k^2(\Omega) = \pi^2$ in the first case, and $k^2(\Omega) = \pi^2/4$ in the second case), and the *Wirtinger inequality*

$$\pi_0^2 \int_\Omega f^2 \, d\Omega \le \int_\Omega |\nabla f|^2 \, d\Omega \,, \tag{3.17}$$

where f is a regular field in the periodicity cell Ω, $\int_\Omega f \, d\Omega = 0$, $f_z = 0$ on $z = \pm 1/2$, and $\pi_0^2 = \min(\pi^2 \,, a_x^2 \,, a_y^2)$ (W. von Wahl, private communication, July 1997), to obtain

$$\dot{E}(t) \le 2\gamma_1 (m - 1) E(t) \,, \tag{3.18}$$

where

$$\gamma_1 = \min\left(\frac{\alpha_1 \pi^2}{P_C} \,, \frac{\alpha_1 \pi^2}{\mu P_T} \,, \alpha_2 \tau^2\right) , \tag{3.19}$$

and $\tau^2 = \pi^2$, $\tau^2 = \frac{\pi^2}{4}$, $\tau^2 = \pi_0^2$, in the rigid-rigid, rigid-free and free-free boundary cases, respectively, α_2 is the constant (depending on Ω) in the Korn inequality $\alpha_2 \|\nabla \mathbf{u}\|^2 \le 2 \|\mathbf{D}(\mathbf{u})\|^2$.

Integrating the last inequality, we obtain

$$E(t) \le E(0) \exp\{-2\gamma_1 (1 - m) t\} \,, \tag{3.20}$$

and the stability condition $m < 1$ assures *unconditional nonlinear exponential stability of the basic motion*. We observe that the *Friedrichs-Poincaré inequality* holds also for a bonded domain Ω with a piecewise C^1 boundary $\partial\Omega$. In fact, if Σ is a part of the boundary $\partial\Omega$ on which the unit normal vector $\mathbf{n}$ has three independent directions, then there exists a positive constant $k(\Omega)$, depending only on the domain Ω, such that $k(\Omega)\|\mathbf{u}\| \le \|\nabla \mathbf{u}\|$, $\forall \mathbf{u} \in W_2^1(\Omega)$, $\mathbf{u} \cdot \mathbf{n}|_\Sigma = 0$, [24]. Therefore (3.20) (with a suitable γ_1 depending on $k(\Omega)$) holds also in the case of any (regular) bounded domain whose boundary can be composed of both stress-free and rigid elements (see [4] for the appropriate boundary conditions. In this case $\mathcal{H}$ is the space of the admissible fields which satisfy these conditions).

In order to obtain the *critical nonlinear Rayleigh number* $\mathcal{R}_E^2$, we must solve the maximum problem (3.12). In the sequel we shall assume

$$p < 1. \tag{3.21}$$

Then, $(3.9)_1$ and $(3.9)_2$ are verified if we choose δ such that

$$\delta < 1 \qquad \text{or} \qquad \delta > 1/p^2. \tag{3.22}$$

Now we rewrite the maximum problem in the following way

$$m = \max_{\mathcal{H}} \frac{\alpha(\phi \,, w) + \beta(\psi \,, w)}{2\lambda \|\mathbf{D}(\mathbf{u})\|^2 + \Gamma \|\nabla\phi\|^2 + \zeta \|\nabla\psi\|^2 + \eta(\nabla\phi \,, \nabla\psi)} \,, \tag{3.23}$$

where

$$\text{(3.24)}\qquad \begin{cases} \alpha = p\mathcal{R}^2 - \mathcal{C}^2 + \lambda\,, & \beta = \mu(\mathcal{R}^2 - \delta\mathcal{C}^2)\,, \quad \Gamma = (p^2\delta - 1)/(p\delta - 1)\,, \\ \zeta = \mu(\delta - 1)/(p\delta - 1)\,, & \eta = (p-1)(\mu\delta - 1)/(p\delta - 1). \end{cases}$$

In the sequel of this section we shall assume that the domain is a horizontal layer with *both stress-free* bounding planes.

The Euler-Lagrange equations for the maximum problem (3.23) are the following

$$\text{(3.25)}\qquad \begin{cases} \alpha\phi\mathbf{k} + \beta\psi\mathbf{k} + 2m\lambda\Delta\mathbf{u} = \nabla p' \\ \alpha w + 2m\Gamma\Delta\phi + m\eta\Delta\psi = 0 \\ \beta w + 2m\zeta\Delta\psi + m\eta\Delta\phi = 0\ . \end{cases}$$

By taking the third component of the double *curl* of $(3.25)_1$, we have

$$\text{(3.26)}\qquad \begin{cases} \alpha\Delta_1\phi + \beta\Delta_1\psi + 2m\lambda\Delta\Delta w = 0 \\ \alpha w + 2m\Gamma\Delta\phi + m\eta\Delta\psi = 0 \\ \beta w + 2m\zeta\Delta\psi + m\eta\Delta\phi = 0\,, \end{cases}$$

where $\Delta_1 f = f_{xx} + f_{yy}$. Now, following the standard analysis of normal modes, [25], and observing that all the even derivatives of $w\,,\phi\,,\psi$ vanish on the planes $z = \pm 1/2$, it is easy to see that the appropriate solutions of (3.26) are $w = W_0 \cos n\pi z \exp i(k_1 a_1 x + k_2 a_2 y)$, $\phi = \Phi_0 \cos n\pi z \exp i(k_1 a_1 x + k_2 a_2 y)$, $\psi = \Psi_0 \cos n\pi z \exp i(k_1 a_1 x + k_2 a_2 y)$, ($n = 1\,,3\,,5\,,\ldots$, $k_1 \neq 0$ and $k_2 \neq 0$ integers), with $W_0\,,\Phi_0\,,\Psi_0$ constants. Substituting these functions in (3.26) we obtain:

$$\text{(3.27)}\qquad \begin{cases} -\alpha r^2\Phi_0 - \beta r^2\Psi_0 + 2m\lambda(n^2\pi^2 + r^2)^2 W_0 = 0 \\ -2m\Gamma(n^2\pi^2 + r^2)\Phi_0 - m\eta(n^2\pi^2 + r^2)\Psi_0 + \alpha W_0 = 0 \\ -m\eta(n^2\pi^2 + r^2)\Phi_0 - 2m\zeta(n^2\pi^2 + r^2)\Psi_0 + \beta W_0 = 0\ , \end{cases}$$

with $r^2 = k_1^2 a_1^2 + k_2^2 a_2^2$.

In order to have solutions of (3.27) which do not vanish identically, the determinant of the system (3.27) must vanish. Then, we easily obtain

$$\text{(3.28)}\qquad m^2\lambda s(4\zeta\Gamma - \eta^2) + \eta\alpha\beta - \beta^2\Gamma - \zeta\alpha^2 = 0\,,$$

where $s = (n^2\pi^2 + r^2)^3/r^2$. Because of $(3.9)_3$, we have $4\zeta\Gamma - \eta^2 > 0$; then, from (3.28), it follows that

$$\text{(3.29)}\qquad m^2 = \frac{\beta^2\Gamma + \alpha^2\zeta - \eta\alpha\beta}{\lambda s(4\zeta\Gamma - \eta^2)}.$$

By taking the maximum over k_1^2, k_2^2 and n^2 of the RHS of (3.29) we have

$$\text{(3.30)}\qquad m^2 = \frac{\beta^2\Gamma + \alpha^2\zeta - \eta\alpha\beta}{\lambda\mathcal{R}_B^2(4\zeta\Gamma - \eta^2)}\,,$$

where $\mathcal{R}_B^2$ have been introduced in Section 1.

Substituting the value of $\alpha, \beta, \Gamma, \eta, \zeta$, given by (3.24), in (3.30), we have

$$m^2 = A/B, \tag{3.31}$$

where

$$\begin{aligned} A =&\mu(p\delta-1)[\mu(\mathcal{R}^2-\delta\mathcal{C}^2)^2(p^2\delta-1)+(p\mathcal{R}^2-\mathcal{C}^2+\lambda)^2(\delta-1)- \\ &-(p-1)(\mu\delta-1)(p\mathcal{R}^2-\mathcal{C}^2+\lambda)(\mathcal{R}^2-\delta\mathcal{C}^2)], \\ B =&\lambda\mathcal{R}_B^2[4(p^2\delta-1)(\delta-1)\mu-(p-1)^2(\mu\delta-1)^2]. \end{aligned} \tag{3.32}$$

In order to obtain the best value for the nonlinear critical Rayleigh number $\mathcal{R}_E^2$, we minimize m^2 with respect to the Lyapunov parameters λ and μ. For this, by computing the partial derivatives of m^2 with respect to λ and μ, and by making them equal to zero, we obtain

$$\lambda_1 = \frac{(\mathcal{C}^2-\mathcal{R}^2)(\delta p-1)}{1-\delta}, \quad \mu_1 = \frac{(\mathcal{C}^2-\mathcal{R}^2)(p-1)}{(\delta-1)(\mathcal{C}^2\delta p-\mathcal{R}^2)},$$

$$\lambda_2 = p\mathcal{R}^2-\mathcal{C}^2, \quad \mu_2 = \frac{(1-p)(\mathcal{C}^2-p\mathcal{R}^2)}{(\delta p-1)(\mathcal{C}^2\delta p-\mathcal{R}^2)}.$$

Now we choose $\delta < 1$, in this way $(3.9)_1$ and $(3.9)_2$ are verified.

In a first moment, we also assume that, $\mathcal{C}^2 < \mathcal{R}^2$, then, in these hypotheses (*i.e.* $p<1$, $\delta < 1$, $\mathcal{C}^2 < \mathcal{R}^2$) we have

$$\lambda_1 > 0, \qquad \mu_1 > 0.$$

By choosing $\lambda = \lambda_1$, $\mu = \mu_1$ in (3.32), we obtain

$$\begin{aligned} A =&\frac{(\delta p-1)^3(1-p)(\mathcal{C}^2-\mathcal{R}^2)}{(1-\delta)^3(\mathcal{C}^2\delta p-\mathcal{R}^2)^2}\{\mathcal{C}^6\delta[\delta(3p+1)-4p]-\mathcal{C}^4\mathcal{R}^2[\delta^2(7p+1)+ \\ &+2\delta(3-5p)-4]+\mathcal{C}^2\mathcal{R}^4[4\delta^2p+2\delta(5-3p)-p-7]-\mathcal{R}^6(4\delta-p-3)\}, \end{aligned}$$

$$\begin{aligned} B =&\frac{\mathcal{R}_B^2(p-1)(\delta p-1)^3(\mathcal{C}^2-\mathcal{R}^2)}{(1-\delta)^3(\mathcal{C}^2\delta p-\mathcal{R}^2)^2}\{\mathcal{C}^4\delta[\delta(3p+1)-4p]- \\ &-2\mathcal{C}^2\mathcal{R}^2[2\delta^2p+3\delta(1-p)-2]+\mathcal{R}^4(4\delta-p-3)\}. \end{aligned}$$

Factorizing A, we have

$$\begin{aligned} A =&\frac{(\delta p-1)^3(1-p)(\mathcal{C}^2-\mathcal{R}^2)^2}{(1-\delta)^3(\mathcal{C}^2\delta p-\mathcal{R}^2)^2}K_0(\mathcal{R}^2) \\ B =&\frac{\mathcal{R}_B^2(p-1)(\delta p-1)^3(\mathcal{C}^2-\mathcal{R}^2)}{(1-\delta)^3(\mathcal{C}^2\delta p-\mathcal{R}^2)^2}K_0(\mathcal{R}^2), \end{aligned} \tag{3.33}$$

where

$$K_0(\mathcal{R}^2) = \mathcal{C}^4\delta[\delta(3p+1)-4p]-2\mathcal{C}^2\mathcal{R}^2[2\delta^2p+3\delta(1-p)-2]+\mathcal{R}^4(4\delta-p-3). \tag{3.34}$$

We observe that hypothesis $(3.9)_3$ requires

$$(p-1)^2(\mu_1\delta-1)^2-4\mu_1(p^2\delta-1)(\delta-1)<0\,,$$

and, by the definition of μ_1, this implies

$$K_0(\mathcal{R}^2)\frac{(1-p)(\delta p-1)^2}{(\delta-1)^2(\mathcal{C}^2\delta p-\mathcal{R}^2)^2}<0. \tag{3.35}$$

Since $p<1$, $\delta<1$, $\mathcal{C}^2<\mathcal{R}^2$, from last inequality, we obtain $K_0(\mathcal{R}^2)<0$.

Thus

$$K_0(\mathcal{R}^2)<0\,,\qquad \text{and}\qquad A/B<1\,, \tag{3.36}$$

imply *nonlinear stability*.

Our next goal is to determine the nonlinear critical Rayleigh number. For this, now we study the system of inequalities

$$\begin{cases} K_0(\mathcal{R}^2)<0\\ \dfrac{\mathcal{R}^2-\mathcal{C}^2}{\mathcal{R}_B^2}<1. \end{cases} \tag{3.37}$$

As concerns $(3.37)_1$, by choosing $\delta\in](p+3)/4\,,1[$, we see that $(3.37)_1$ is equivalent to $\mathcal{R}_1^2<\mathcal{R}^2<\mathcal{R}_2^2$, with $\mathcal{R}_1^2=\mathcal{C}^2h_1(\delta)$, $\mathcal{R}_2^2=\mathcal{C}^2h_2(\delta)$, and

$$\begin{aligned} h_1(\delta)&=\frac{2\delta^2p+3\delta(1-p)-2-2\sqrt{(\delta-1)^3(\delta p^2-1)}}{4\delta-p-3}\,,\\ h_2(\delta)&=\frac{2\delta^2p+3\delta(1-p)-2+2\sqrt{(\delta-1)^3(\delta p^2-1)}}{4\delta-p-3}. \end{aligned} \tag{3.38}$$

When $\delta\in](p+3)/4\,,1[$, it is easy to verify that the following properties hold

$$\begin{aligned} &i)\quad h_2(\delta)>h_1(\delta)>1\,,\\ &ii)\quad \lim_{\delta\to1-}h_1(\delta)=\lim_{\delta\to1-}h_2(\delta)=1\,,\\ &iii)\quad \lim_{\delta\to((p+3)/4)+}h_2(\delta)=+\infty\,,\qquad \lim_{\delta\to((p+3)/4)+}h_1(\delta)=\frac{3(p+3)}{4(p+2)}. \end{aligned} \tag{3.39}$$

Then, the system (3.37) is satisfied for

$$\mathcal{R}_1^2<\mathcal{R}^2<\min(\mathcal{R}_2^2\,,\mathcal{C}^2+\mathcal{R}_B^2). \tag{3.40}$$

From (3.38) - (3.40), it is easy to see that, by varying δ in the interval $](p+3)/4\,,1[$, and by taking

$$\bigcup_{\delta\in\left](p+3)/4\,,1\right[}\left]\mathcal{R}_1^2\,,\mathcal{C}^2+\mathcal{R}_B^2\right[\,,$$

we shall obtain the nonlinear stability region $]\mathcal{C}^2, \mathcal{C}^2 + \mathcal{R}_B^2[$. This means that every time we choose $\hat{\mathcal{R}}^2 \in]\mathcal{C}^2, \mathcal{C}^2 + \mathcal{R}_B^2[$, it is possible to find $\hat{\delta} \in](p+3)/4, 1[$ such that $m^2 < 1$.

Now we show that also when $0 < \mathcal{R}^2 \leq \mathcal{C}^2$, as it is obvious to be expected, we have $m^2 < 1$. In fact, by choosing $\delta \in]0, 1[$, and $\mathcal{R}^2 = \delta\mathcal{C}^2$, from (3.31) and (3.33) it follows that

$$m^2 = \frac{\mu(p\delta - 1)(\delta - 1)(p\mathcal{R}^2 - \mathcal{C}^2 + \lambda)^2}{\lambda\mathcal{R}_B^2 D(\mu)}, \tag{3.41}$$

with

$$D(\mu) = 4\mu(p^2\delta - 1)(\delta - 1) - (p-1)^2(\mu\delta - 1)^2 > 0.$$

Setting

$$\mu = \overline{\mu} = \frac{2(p^2\delta - 1)(\delta - 1) + \delta(p-1)^2}{\delta^2(p-1)^2}, \qquad \lambda = (1 - p\delta)\mathcal{C}^2,$$

we have

$$D(\overline{\mu}) = \frac{4(\delta - 1)(\delta p - 1)^2(\delta p^2 - 1)}{\delta^2(p-1)^2} > 0,$$

and

$$m^2 = 0,$$

and then the stability condition is verified for $0 < \mathcal{R}^2 < \mathcal{C}^2$. Finally, when $\mathcal{R}^2 = \mathcal{C}^2$, choosing $\lambda = (1-p)\mathcal{C}^2$, $\mu = \overline{\mu}$ as before, we have

$$m^2 = \frac{\overline{\mu}^2\delta^2(1-\delta)(1-p)\mathcal{C}^2}{4\mathcal{R}_B^2(1 - \delta p)}.$$

Since for $\delta \to 1-$ it follows that $\overline{\mu} \to 1$, then we have $m^2 \to 0$. Consequently, for any $\mathcal{C}^2$, we can have $m^2 < 1$ as we take $\delta < 1$, close to 1.

Hence we conclude that whenever

$$\mathcal{R}^2 < \mathcal{C}^2 + \mathcal{R}_B^2, \tag{3.42}$$

the *nonlinear stability condition* $m < 1$ is satisfied.

The criticality, *i.e.* $m^2 = 1$, will be reached when $\mathcal{R}^2 = \mathcal{R}_E^2$, where $\mathcal{R}_E^2 = \mathcal{C}^2 + \mathcal{R}_B^2$. This last value is the *critical nonlinear stability Rayleigh number* and is coincident with the critical linear instability Rayleigh number (2.12).

From the linear instability results (2.6), the instability theorem of [26], and the previous proof, we can now formulate our main result:

THEOREM 3.1. *Assuming* $p < 1$, *the condition*

$$\mathcal{R}^2 \leq \mathcal{C}^2 + \mathcal{R}_B^2, \tag{3.43}$$

is necessary and sufficient for nonlinear stability of the basic motion m_0. *If*

$$\mathcal{R}^2 < \mathcal{C}^2 + \mathcal{R}_B^2, \tag{3.44}$$

then m_0 *is nonlinearly exponentially stable according to* (3.20). *The critical linear instability Rayleigh number* $\mathcal{R}_c^2$ *coincides with the critical nonlinear stability Rayleigh number* $\mathcal{R}_E^2$. □

REMARK 3.1. The previous method is valid also when the basic motion is different from the rest state. In particular, for Couette and Poiseuille flows of a mixture, it is possible to obtain, for any $\mathcal{C}^2$, a conditional nonlinear stability condition which does not depend on the Reynolds number and an unconditional stability result for $\mathcal{R}^2 < \mathcal{R}_E^2$, any $\mathcal{C}^2$, and Reynolds number less than $c_o(1 - m)$. These results will appear in a next paper.

4. THE RIGID-RIGID AND RIGID-FREE BOUNDARY CONDITIONS IN A LAYER AND THE BOUNDED DOMAIN CASE

In this section, we shall prove the coincidence between the critical linear instability Rayleigh number and the nonlinear critical one, in the case when the domain is a horizontal layer bounded by two rigid planes or by one rigid plane and the other stress-free. We shall also obtain the aforesaid coincidence for any bounded domain.

In fact, in all the previous cases, taking the boundary conditions into account, the evolution equations of the Lyapunov function $E(t)$ is given by

$$\begin{aligned}\dot{E}(t) = (p\mathcal{R}^2 - \mathcal{C}^2 + \lambda)(\phi\,, w) + \mu(\mathcal{R}^2 - \delta\mathcal{C}^2)(w\,, \psi) - [2\lambda\|\mathbf{D}(\mathbf{u})\|^2 +\\ + \frac{p^2\delta - 1}{p\delta - 1}\|\nabla\phi\|^2 + \frac{\mu(\delta - 1)}{p\delta - 1}\|\nabla\psi\|^2 + \frac{(p-1)(\mu\delta - 1)}{p\delta - 1}(\nabla\phi\,, \nabla\psi)]\,,\end{aligned} \tag{4.1}$$

and the related maximum problem is given by

$$m = \max_{\mathcal{H}} \frac{\alpha(\phi\,, w) + \beta(\psi\,, w)}{2\lambda\|\mathbf{D}(\mathbf{u})\|^2 + \Gamma\|\nabla\phi\|^2 + \zeta\|\nabla\psi\|^2 + \eta(\nabla\phi\,, \nabla\psi)}\,, \tag{4.2}$$

where α, β, Γ, ζ, η are given by (3.24) and $\mathcal{H}$ is the space of the appropriate admissible functions. We observe that, in the case of a layer with rigid-rigid or rigid-free boundary conditions and for a bounded domain with all the boundary rigid, we have $2\|\mathbf{D}(\mathbf{u})\|^2 = \|\nabla\mathbf{u}\|^2$.

Now we denote by $(\mathbf{V}\,, \chi)$ the fields which are related to $\mathbf{u}$, ϕ and ψ by the following equalities:

$$\mathbf{V} = \sqrt{\lambda}\mathbf{u}\,, \qquad \alpha\phi = k\sqrt{\sigma}\chi \qquad \beta\psi = (2 - k)\sqrt{\sigma}\chi\,, \tag{4.3}$$

with

$$\sigma = \frac{\beta^2\Gamma + \alpha^2\zeta - \eta\alpha\beta}{4\zeta\Gamma - \eta^2} > 0\,, \tag{4.4}$$

(observe that, by $(3.9)_3$, $4\zeta\Gamma - \eta^2 > 0$) and

$$k = \frac{\alpha(2\alpha\zeta - \beta\eta)}{\beta^2\Gamma + \alpha^2\zeta - \eta\alpha\beta}. \tag{4.5}$$

Since the fields $\mathbf{V}$ and χ satisfy the same boundary conditions as $\mathbf{u}$ and ϕ (or ψ), then, as it is easy to verify, the maximum (4.2) becomes

$$m = \sqrt{\frac{\sigma}{\lambda}} \max_{\mathcal{H}} \frac{2(\chi\,,\,V_3)}{2\|\mathbf{D}(\mathbf{V})\|^2 + \|\nabla\chi\|^2} = \sqrt{\frac{\sigma}{\mathcal{R}_B^2\lambda}}\,, \tag{4.6}$$

where $V_3 = \mathbf{V}\cdot\mathbf{k}$, and now $\mathcal{R}_B^2$ is the value which is appropriate to the domain with the assigned boundary conditions, for example, in the case of a layer with rigid-rigid boundary conditions $\mathcal{R}_B^2 = 1708$, in the case of a layer with rigid-free boundary conditions $\mathcal{R}_B^2 = 1100$.

From (4.6) we get the stability condition

$$\sqrt{\sigma/\mathcal{R}_B^2\lambda} < 1\,,$$

which is equivalent to

$$\sigma < \mathcal{R}_B^2\lambda.$$

Now, proceeding exactly as in Section 3, we easily obtain that, whenever

$$\mathcal{R}^2 < \mathcal{R}_E^2\,, \qquad \mathcal{R}_E^2 = \mathcal{C}^2 + \mathcal{R}_B^2\,,$$

we have $m < 1$. Thus, also in these cases (rigid-rigid, rigid-free in a layer, any bounded domain) Theorem 3.1 holds and we reach the *coincidence* between the linear and non-linear critical Rayleigh numbers.

Remark 4.1. In the case $p = 1$, we can obtain the same results as before in the limit $p \to 1$. In fact, in this case, the equations $(3.4)_{1-2}$ and $(3.4)_4$ do not contain ψ and can be solved separately. One writes the evolution equations of

$$E_1(t) = 1/2[\lambda\|\mathbf{u}\|^2 + P_C\|\phi\|^2]$$

and proves that it decays exponentially. Then it is easy to prove that also $\|\psi\|$ decays exponentially (see, *e.g.* [27]).

Remark 4.2. The case $p > 1$ will be studied in a forthcoming paper. In this case, for moderate $\mathcal{C}^2$, the principle of exchange of stabilities is still valid and we are able to find unconditional nonlinear stability up to the linear critical value. For example, for $p = 2$ and $\mathcal{C}^2 \le \mathcal{R}_B^2/3$, we obtain $\mathcal{R}_E^2 = \mathcal{R}_c^2$, while, for $\mathcal{C}^2 > \mathcal{R}_B^2/3$, we get $\mathcal{R}_E^2 < \mathcal{R}_c^2$, and this is in agreement with other nonlinear stability results of [4-6].

Remark 4.3. In the case $p < 1$, Joseph gives stability results (not exponential) only for $\mathcal{C}^2 < \mathcal{R}^2$ while here we have obtained exponential stability results for any $\mathcal{R}^2$ such that $0 < \mathcal{R}^2 < \mathcal{R}_E^2$.

Remark 4.4. The stability results, obtained in the case of a mixture heated and salted from below, apply also in the case of a mixture heated and salted from above (typical configuration at sea). To have the unconditional stability result in this case, one need only to interchange $\mathcal{R}^2$ and $\mathcal{C}^2$, P_T and P_C.

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