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Variational construction of homoclinics and chaos in presence of a saddle-saddle equilibrium

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Analisi matematica. — *Variational construction of homoclinics and chaos in presence of a saddle-saddle equilibrium.* Nota di Massimiliano Berti e Philippe Bolle, presentata (*) dal Corrisp. A. Ambrosetti.

Abstract. — We consider autonomous Lagrangian systems possessing two homoclinic orbits to an hyperbolic equilibrium of saddle-saddle type with two different characteristic exponents. Under a nondegeneracy assumption on the homoclinics and under suitable conditions on the geometric behaviour of these homoclinics near the equilibrium we show, by variational methods, that they give rise to an infinite family of multibump homoclinic solutions. We relax the nondegeneracy assumption when the two characteristic exponents are close one to the other.

Key words: Homoclinic orbits; Chaos; Saddle-saddle equilibrium; Variational methods.

Riassunto. — *Costruzione variazionale di orbite omocline e di una dinamica caotica in presenza di un equilibrio di tipo saddle-saddle.* Consideriamo sistemi Lagrangiani autonomi aventi due orbite omocline ad un equilibrio iperbolico di tipo *saddle-saddle* con due differenti esponenti caratteristici. Con una ipotesi di nondegenerazione per le omocline e sotto opportune condizioni sul comportamento geometrico di queste omocline vicino all'equilibrio proviamo, con metodi variazionali, che esse danno luogo ad una famiglia infinita di soluzioni omocline di tipo *multibump*. Quando gli esponenti caratteristici sono vicini tra loro rilassiamo la condizione di nondegenerazione per le omocline.

1. Introduction and main result

We outline in this *Note* some recent results obtained in [2] where we refer for complete proofs and details. We consider autonomous Lagrangian systems of the form

$$-\ddot{q} + \psi(q)\mathcal{J}\dot{q} + Aq = \nabla W(q)\,, \tag{1}$$

where $q = (q_1\,, q_2) \in \mathbb{R}^2$, $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$. System (1) can be obtained by the following Lagrangian

$$\mathcal{L}(q\,, \dot{q}) = (1/2)|\dot{q}|^2 + (1/2)Aq \cdot q - \dot{q} \cdot v(q) - W(q)\,,$$

where $v = (v_1\,, v_2)$ satisfies

$$\psi(q) = \partial_{q_1} v_2(q) - \partial_{q_2} v_1(q). \tag{2}$$

System (1) admits the energy $\mathcal{E}(q\,, \dot{q}) = (1/2)|\dot{q}|^2 - (1/2)Aq \cdot q + W(q)$ as an invariant of the motion. We shall assume

- (W1) $W \in C^2(\mathbb{R}^2\,, \mathbb{R})$, $W(0) = 0\,, \nabla W(0) = 0\,, D^2 W(0) = 0$; for some $0 < \rho_0 < \rho_1$ (ρ_1 is specified later after hypothesis $(S2)$) $D^2 W$ is L_1-Lipschitz continuous on the

(*) Nella seduta del 24 aprile 1998.

ball $B_0 := B_0(0, \rho_0)$ of center 0 and radius ρ_0 and $\overline{L}_1$-Lipschitz continuous on $B_1 := B_1(0, \rho_1)$ of center 0 and radius ρ_1;

- $(P1)$ $\psi \in C^1(\mathbb{R}^2, \mathbb{R})$ is L_2-Lipschitz continuous (resp. $\overline{L}_2$-Lipschitz continuous) on B_0 (resp. B_1), $\psi(0) = 0$; $\nabla\psi$ is $\overline{L}_3$-Lipschitz continuous on B_1.

By $(P1)$, we may assume (2), with

- $(v1)$ $v \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $v(0) = 0$, $Dv(0) = 0$.

Under these assumptions 0 is a hyperbolic equilibrium of (1) and the characteristic exponents are two couples of opposite real numbers $\pm\lambda_1, \pm\lambda_2$. In this case the equilibrium is called of saddle-saddle type. We shall assume in the sequel that

$$(S1) \qquad \lambda_1 > \lambda_2 > 0.$$

We are interested in a chaotic behaviour of the dynamics of (1) on the zero energy level.

The existence of chaos in presence of a saddle-saddle equilibrium on small energy levels $\{\mathcal{E} = h\}$ has been studied by Turaev and Shil'nikov [7] and more recently by Bolotin and Rabinowitz [3]. However, the chaotic trajectories which are obtained in [7] as well as in [3] are not preserved when the energy vanishes. The existence of a Bernoulli shift at energy level $\{\mathcal{E} = 0\}$ was studied by Holmes in [6] (see also [8]). He assumed the existence of two nondegenerate homoclinics and introduced some conditions on the way these homoclinics approach 0, which ensure, when $(S1)$ is satisfied, the existence of a horseshoe at the zero energy level. However Holmes' conditions are not very specific and it is not obvious that they can be fulfilled in the systems (3) and (S_ϵ) given below.

First we prove our results assuming that

- $(S2)$ System (1) has 2 nondegenerate homoclinics $\overline{q}, \widetilde{q}$. «Nondegenerate» means that the unique solutions, that tend to 0 as $t \to \pm\infty$, of the linearized equation at (for instance) $\overline{q}$
$$-\ddot{h} + Ah + \psi(\overline{q})\mathcal{J}\dot{h} + \nabla\psi(\overline{q}) \cdot h\mathcal{J}\dot{\overline{q}} - D^2W(\overline{q})h = 0$$
are $c\dot{\overline{q}}$, $c \in \mathbb{R}$. It means that the homoclinic is «transversal» on the 0 energy level.

We can now specify the constant ρ_1 in $(W1)$: $\rho_1 > \max\{|\overline{q}|_\infty, |\widetilde{q}|_\infty\} + \rho_0$.

We remark that there exist systems with several nondegenerate homoclinic orbits which do not have a chaotic behaviour. Consider for example system (1) and assume that $W(q) = q_1^4 + q_2^4$ and $\psi(q) = 0$. Then (1) possesses 4 nondegenerate homoclinic trajectories but it is an integrable system. Thus additional assumptions are needed for chaotic behaviour: some hypotheses of geometrical nature on $\overline{q}$ and $\widetilde{q}$, similar to the ones given in [6], are required.

We need some notations. For $r \in (0, \rho_0/2)$ we define $\overline{T} > 0$ by $|\overline{q}(\pm\overline{T})| = r$ and $|\overline{q}(t)| < r$ for $|t| > \overline{T}$. We define in the same way $\widetilde{T}$ and we set $T = \min\{\overline{T}, \widetilde{T}\}$. Call $(\overline{\alpha}_1, \overline{\alpha}_2) = (\overline{q}_1(-\overline{T}), \overline{q}_2(-\overline{T}))$, $(\overline{\beta}_1, \overline{\beta}_2) = (\overline{q}_1(\overline{T}), \overline{q}_2(\overline{T}))$ the extremal intersection points of $\overline{q}(\mathbb{R})$ with the circle in $\mathbb{R}^2$ of radius r; similarly we introduce $(\tilde\alpha_1, \tilde\alpha_2) = (\tilde q_1(-\tilde T), \tilde q_2(-\tilde T))$, $(\tilde\beta_1, \tilde\beta_2) = (\tilde q_1(\tilde T), \tilde q_2(\tilde T))$. Let $\overline\omega_u$, $\overline\omega_s$ be defined by $(\overline\alpha_1, \overline\alpha_2) = (r\cos\overline\omega_u, r\sin\overline\omega_u)$, $(\overline\beta_1, \overline\beta_2) = (r\cos\overline\omega_s, r\sin\overline\omega_s)$; $\tilde\omega_u$, $\tilde\omega_s$ are defined in the same way. We set $\Lambda = (L_1/\lambda_2^2) + (3L_2\lambda_1/\lambda_2^2)$, $\overline\Lambda = (\overline L_1/\lambda_2^2) + (3\overline L_2\lambda_1/\lambda_2^2) + \max\left\{|\dot{\overline q}|_\infty, |\dot{\tilde q}|_\infty\right\}(\overline L_3/\lambda_2^2)$. Note that Λ, $\overline\Lambda$ do not change if the equation is modified by a time rescaling $q(t)\to q(\alpha t)$.

In the next conditions ω_u stands for $\overline\omega_u$ or $\tilde\omega_u$ and ω_s for $\overline\omega_s$ or $\tilde\omega_s$.

- $(H1)$ $\omega_u, \omega_s \neq n\pi/2$, $n\in\mathbb{Z}$, $\tan\omega_u\tan\omega_s<0$ and $(\cos\overline\omega_u\cos\tilde\omega_u<0$ or $\cos\overline\omega_s\cos\tilde\omega_s<0)$. (the above inequalities are satisfied for example if $\overline\omega_u\in(0, \pi/2)$, $\overline\omega_s\in(3\pi/2, 2\pi)$, $\tilde\omega_u\in(\pi, 3\pi/2)$ and $\tilde\omega_s\in(\pi/2, \pi)$);
- $(H2)$
$$\frac{\lambda_2^2}{\lambda_1^2}\frac{|\alpha_2||\beta_2| + (15\lambda_1/4\lambda_2)\Lambda r^3}{|\alpha_1||\beta_1|} \le$$
$$\le l\left(\frac{\lambda_1}{\lambda_2}\right)\min\left(e^{-2\frac{\lambda_1-\lambda_2}{\lambda_2}}, \left(\frac{C_1 e^{\lambda_2(T-T_{C_1})}}{18}\right)^{2\frac{\lambda_1-\lambda_2}{\lambda_2}}, \left(\frac{C_1^2}{40\overline\Lambda r}\right)^{\frac{\lambda_1-\lambda_2}{\lambda_2}}\right)$$
where $l(\nu) = \max_{s\in(0,1/8)}[(1-s)^2(1-s/5)/(1+s)^3]s^{\nu-1}$ and C_1 is a constant defined by (6), which measures the transversality of the homoclinics: smaller is C_1 weaker is the transversality. $T_{C_1} = \max(\overline T_{C_1}, \tilde T_{C_1})$ where for example $\overline T_{C_1}$ is defined as the smallest positive time such that
$$\forall t\in\mathbb{R}\backslash[-\overline T_{C_1}, \overline T_{C_1}] \quad |\overline q(t)|\le\rho_0 \quad\text{and}\quad 8\Lambda\max\left(|\overline q(t)|, |\dot{\overline q}(t)|/\lambda_2\right)\le C_1.$$
T_{C_1} depends only on C_1 and ρ_0.
- $(H3)$
$$\frac{\lambda_2^2}{\lambda_1^2}\frac{|\alpha_2||\beta_2| + (15\lambda_1/4\lambda_2)\Lambda r^3}{|\alpha_1||\beta_1|} \le l\left(\frac{\lambda_1}{\lambda_2}\right)\left(\frac{C_1\mathcal{M}}{36S_2 + 28\Lambda r^2}\right)^{(\lambda_1/\lambda_2)-1}$$
where $\mathcal{M} = \min_j\{|\overline\alpha_j|, |\overline\beta_j|, |\tilde\alpha_j|, |\tilde\beta_j|\}$ and $S_2 = \max\{|\overline\alpha_2|, |\overline\beta_2|, |\tilde\alpha_2|, |\tilde\beta_2|\}$.
- $(H4)$ $\min(|\sin\overline\omega_{u,s}|, |\sin\tilde\omega_{u,s}|) \ge \sqrt{(\lambda_1/\lambda_2)20\Lambda r}$, $(12\lambda_1/\lambda_2)\Lambda r\le C_1$.

Roughly speaking the first geometric assumption $(H1)$ means that the homoclinics $\overline q$, $\tilde q$ enter and leave the origin from different «quadrants». $(H2$-$H3)$ quantify how small $|\tan\omega_u\tan\omega_s|$ and r must be. Note that if the system is linear (that is $W=0$, $\psi=0$) in the ball $B(0, \rho_0)$ then condition $(H4)$ disappears and conditions $(H2$-$H3)$ are simplified (in $(H2$-$H3)$, $\Lambda=0$). Moreover if $\lambda_1/\lambda_2\to 1$ then $l(\lambda_1/\lambda_2)\to 1$ and the second members in inequalities $(H2$-$H3)$ tend to 1.

Before stating our first result we introduce some other notations. For $j=(j_1, \dots, j_k)\in\{0, 1\}^k$ and for $\Theta = (\theta_1, \dots, \theta_k)\in\mathbb{R}^k$ we define $T_i = \overline T$ if $j_i = 0$ and $T_i = \tilde T$ if $j_i = 1$; $d_i = (\theta_{i+1} - T_{i+1}) - (\theta_i + T_i)$ and $\overline d = \min_{1\le i\le k-1} d_i$.

Theorem 1. *Assume* $(W1)$, $(P1)$, $(v1)$, $(S1\text{-}S2)$ *and* $(H1\text{-}H4)$. *Then there exist* $0 < D < J$ *such that for every* $k \in \mathbb{N}$, *for every sequence* $j = (j_1, \dots, j_k) \in \{0, 1\}^k$ *there is* $\Theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ *with* $d_i \in (D, J)$ *for all* $i = 1, \dots, k-1$ *and a homoclinic solution of* (1) x_j *such that*

- *if* $j_i = 0$ *then on the interval* $[\theta_i - \overline{T}, \theta_i + \overline{T}]$
$$|x_j(t) - \overline{q}(t-\theta_i)| \le (r/8) \min \Big(|\cos \overline{\omega}_{u,s}|, |\cos \widetilde{\omega}_{u,s}|, |\sin \overline{\omega}_{u,s}|, |\sin \widetilde{\omega}_{u,s}| \Big),$$
- *if* $j_i = 1$ *then on the interval* $[\theta_i - \widetilde{T}, \theta_i + \widetilde{T}]$
$$|x_j(t) - \widetilde{q}(t-\theta_i)| \le (r/8) \min \Big(|\cos \overline{\omega}_{u,s}|, |\cos \widetilde{\omega}_{u,s}|, |\sin \overline{\omega}_{u,s}|, |\sin \widetilde{\omega}_{u,s}| \Big).$$
- *Outside* $(\cup_{j_i=0}[\theta_i - \overline{T}, \theta_i + \overline{T}]) \cup (\cup_{j_i=1}[\theta_i - \widetilde{T}, \theta_i + \widetilde{T}])$, $|x_i(t)| \le 2r$.

Remark 1. (*i*) Since the distance d_i between two consecutive bumps is bounded by the constant J which is independent of the number of bumps k, by the Ascoli-Arzela theorem there follows easily the existence of solutions with infinitely many bumps. In particular there is a lower bound for the topological entropy of the dynamical system on the zero energy level given by $h^0_{top} > \log 2/(2\max\{\overline{T}, \widetilde{T}\} + J)$.

(*ii*) The fact that $\lambda_1 > \lambda_2$ is crucial to be able to construct multibump homoclinics.

(*iii*) Smaller are the quantities $\Lambda r/|\cos\omega_u \cos\omega_s| + |\tan\omega_u \tan\omega_s|$, $|\lambda_1 - \lambda_2|/\lambda_2$, greater is the distance between the bumps.

(*iv*) We do not prove the existence of multibump homoclinics in an arbitrary small neighborhood of $\overline{q}, \widetilde{q}$. Indeed in [7] it is proved that there is a neighborhood V of $\overline{q}(\mathbb{R}) \cup \widetilde{q}(\mathbb{R})$ such that the only homoclinic solutions contained in V are $\overline{q}$ and $\widetilde{q}$.

As an application of Theorem 1 we can prove the existence of multibump homoclinic solutions to perturbed systems like

$$\begin{aligned} -\ddot{q}_1 + \lambda_1^2 q_1 &= W_1'(q_1) + \epsilon\psi(q)\dot{q}_2 \\ -\ddot{q}_2 + \lambda_2^2 q_2 &= W_2'(q_2) - \epsilon\psi(q)\dot{q}_1, \end{aligned} \tag{3}$$

with $(q_1, q_2) \in \mathbb{R}^2$. We assume that $W_i(0) = W_i'(0) = W_i''(0) = 0$, $\psi(0) = 0$ and that $W_i(-q_i) = W_i(q_i)$ for $i = 1, 2$, $\psi(q_1, -q_2) = \psi(-q_1, q_2) = \psi(q_1, q_2)$. For $\epsilon = 0$ system (3) splits into the direct product of two 1-dimensional systems. Suppose for example that:

$$-\ddot{q}_1 + \lambda_1^2 q_1 = W_1'(q_1) \tag{4}$$

possesses an homoclinic q_0. Let $\Gamma = \int_{-\infty}^{+\infty} \psi(q_0(s), 0)\ \dot{q}_0(s)\exp(\lambda_2 s)\ ds$.

Theorem 2. *Under the above assumptions, if* $\Gamma \neq 0$ *then there is* $\epsilon_0 > 0$ *such that, for* $\epsilon \in (-\epsilon_0, 0) \cup (0, \epsilon_0)$, (3) *has a rich family of homoclinic solutions which induce chaos on the zero energy level.*

The proof consists in getting a first nondegenerate homoclinic solution q_ϵ by the implicit function theorem. Then $-q_\epsilon$ is a homoclinic solution as well and careful estimates enable to check that conditions $(H1\text{-}H4)$ are satisfied for ϵ small enough.

2. Relaxation of the nondegeneracy condition when the two eigenvalues are close

Consider the following system:

$$(S_\epsilon) \qquad -\ddot{q} + \psi(q)\mathcal{J}\dot{q} + A_\epsilon q = \nabla W(q)\,,$$

with $A_\epsilon = \begin{pmatrix} (\lambda+\epsilon)^2 & 0 \\ 0 & (\lambda-\epsilon)^2 \end{pmatrix}$ and assume that W satisfies $(W1)$, ψ satisfies $(P1)$ and that

- (A1) (S_0) has two homoclinic solutions $\overline{q}$ and $\widetilde{q}$.

It can be shown that the limits as $t \to +\infty$ and as $t \to -\infty$ of $\overline{q}(t)/|\overline{q}(t)|$ (resp. $\widetilde{q}(t)/|\widetilde{q}(t)|$) do exist. Call these limits $(\cos\overline{\omega}_s\,, \sin\overline{\omega}_s)$ and $(\cos\overline{\omega}_u\,, \sin\overline{\omega}_u)$ (resp. $(\cos\widetilde{\omega}_s\,, \sin\widetilde{\omega}_s)$ and $(\cos\widetilde{\omega}_u\,, \sin\widetilde{\omega}_u)$). Moreover we suppose that

- (A2) $\omega_u\,, \omega_s \neq n\pi/2\,,\ n\in\mathbb{Z}\,,\ -1 < \tan\omega_u \tan\omega_s < 0$ and ($\cos\overline{\omega}_u\cos\widetilde{\omega}_u < 0$ or $\cos\overline{\omega}_s\cos\widetilde{\omega}_s < 0$).

Then, as another application of Theorem 1, it can be proved that, if the two homoclinics are nondegenerate, then for $\epsilon \neq 0$ small enough, the same conclusion as in Theorem 1 holds.

We would like to relax somewhat the nondegeneracy assumption. Rather than performing this relaxation for the general case, which would require quite involved conditions, we just deal with this perturbative equation (S_ϵ).

We need some preparation. In the sequel we shall use the Banach spaces $Y = W^{1,\infty}(\mathbb{R}\,,\mathbb{R}^2)$ endowed with the norm $||y|| = \max(|y|_\infty\,, (1/\lambda_2)|\dot{y}|_\infty)$ and $E = W^{1,2}(\mathbb{R}\,,\mathbb{R}^2)$ with norm $|\cdot|_E$ associated to the scalar product $(x\,,y) = \sum_{j=1}^2 \int_{\mathbb{R}} \dot{x}_j\dot{y}_j + \lambda_j^2 x_j y_j$.

We define $S : Y \to Y$ by $S(y) = y - L_A(\nabla W(y) - \psi(y)\mathcal{J}\dot{y})$ where L_A is the linear operator which assigns to h the unique solution $z = L_A h$ of $-\ddot{z} + Az = h$ with $\lim_{|t|\to\infty} z(t) = 0$. If $S(q)=0$ and $q\in E$, then q is a homoclinic solution to system (1).

Let q be a homoclinic solution of the system isolated up to time translations. Let $a = L_A(c\dot{q}\chi_{[-T,T]})$, where $c>0$ is chosen so that $|a|_E = 1$. Define $\hat{F} = a^\perp$ and $\widehat{\Pi} : Y \to \hat{F}$ the projection given by $\widehat{\Pi}(x) = x - (x\,,a)a$. Consider $G_* : \hat{F} \to \hat{F}$ defined by

$$\begin{aligned} G_*(x) &= \widehat{\Pi}S(q+x) = \\ &= \widehat{\Pi}\Big[(q+x) - L_A\Big(\nabla W(q+x) - \mathcal{J}\psi(q+x)(\dot{q}+\dot{x})\Big)\Big] = \widehat{\Pi}\Big(S(x) - K_q(x)\Big), \end{aligned}$$

where

$$\begin{aligned} &K_q(x) = \\ &= L_A\Big[\Big(\nabla W(q+x) - \nabla W(q) - \nabla W(x)\Big) - \mathcal{J}\Big((\psi(q+x)-\psi(q))\dot{q} + (\psi(q+x)-\psi(x))\dot{x}\Big)\Big]. \end{aligned}$$

We have $K_q(0) = 0$. In addition, it is easy to see that $K_q : Y \to Y$ is compact and that $K_q(Y) \subset E$.

Note also that there is $\rho > 0$ such that $\widehat{\Pi} \circ S : \hat{F} \to \hat{F}$ is a diffeomorphism from $\hat{B}(\rho)$ onto a neighborhood of 0 in $\hat{F}$ containing $\hat{B}(\rho/2)$. Let δ satisfy $\widehat{\Pi} K_q(\hat{B}(\delta)) \subset \hat{B}(\delta/2)$ and consider $\hat{G}\colon \hat{B}(\delta) \to \hat{F}$ defined by $\hat{G}(x) = x - (\widehat{\Pi} \circ S)^{-1} \widehat{\Pi} K_q(x) := x - \hat{K}_q(x)$. We have $\hat{K}_q(0) = 0$, and $\hat{K}_q(Y) \subset E$. Hence all the zeros of $\hat{G}$ must belong to E and thus be homoclinic solutions of the system. Now, q being an isolated homoclinic, 0 is an isolated zero of $\hat{G}$. Moreover $\hat{K}_q$ is a compact operator. We can then introduce the following definition

DEFINITION 1. *We shall say that* q *is a topologically nondegenerate homoclinic if there is* $0 < \nu \le \delta$ *such that* $\hat{G}$ *has no zero in* $\overline{\hat{B}(\nu)} \backslash \{0\}$ *and* $\deg(\hat{G}, \hat{B}(\nu); 0) \neq 0$.

(We could prove without difficulty that this definition is independent of the choice of a satisfying $(a\,,\,\dot{q}) \neq 0$). It is easy also to see that a nondegenerate homoclinic according to definition $(S2)$ is also «topologically nondegenerate».

We point out that in certain cases one can say that a *variationally* obtained isolated (up to time translations) homoclinic is topologically nondegenerate. For instance, an isolated local minimum for f, or, under some further assumptions, an isolated mountain-pass critical point corresponds to a topologically nondegenerate homoclinic (see [5, 4 and references therein]).

THEOREM 3. *Assume* $(W1)$, $(P1)$, $(v1)$. *Moreover assume that* (S_0) *satisfies assumptions* $(A1$-$A2)$ *and that both homoclinic solutions in* $(A1)$ *are topologically nondegenerate. Then there is* $\epsilon_1 > 0$ *such that, for* $0 < |\epsilon| < \epsilon_1$ *the same conclusion as in Theorem 1 holds. In particular* $h^0_{top} > C\epsilon$ *for a suitable positive constant* $C > 0$.

3. FINITE DIMENSIONAL REDUCTION

In this section we outline some steps of the proof of our main result.

The multibump homoclinic solutions of (1) are obtained as critical points of the following action functional, which by $(W1)$ and $(v1)$ is well defined and of class C^2 on E:

$$f(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot{q}|^2 + \frac{1}{2} Aq \cdot q - \dot{q} \cdot v(q) - W(q).$$

The idea to prove such results goes as follows. A «pseudo-critical» manifold Z_k for the functional f is constructed by gluing together translates of the homoclinics $\overline{q}(\cdot - \theta_i)$ and $\tilde{q}(\cdot - \theta_j)$ near the equilibrium with solutions of the boundary value problem. Then we show that, when the bumps are sufficiently separate, that is when

$$\min_i (\theta_{i+1} - \theta_i) > \overline{D}\,, \tag{5}$$

a shadowing type lemma enables to construct near Z_k a k-dimensional constrained manifold M_k such that the critical points of the restriction $g(\Theta)$ of f to M_k gives rise to a k-bump homoclinic solution. The geometric properties ($H1$-$H4$) on the homoclinics $\overline{q}$ and $\widetilde{q}$ ensure the existence of critical points of $g(\Theta)$ satisfying (5). We point out that when $\max(\theta_{i+1} - \theta_i)$ is too large, θ cannot be a critical point of g; therefore we need to estimate carefully the minimal distance $\overline{D}$ for which we obtain the constrained manifold M_k.

Now we define the transversality constant C_1 quoted in assumptions ($H2$-$H4$). Let $\overline{\tau}$ be some positive real number such that $|\dot{\overline{q}}(t)| \geq 3|\dot{\overline{q}}(\overline{t})|/4$ on the interval $\overline{J} = (\overline{t} - \overline{\tau}, \overline{t} + \overline{\tau})$. Let $\overline{a}_0$ be the unique element of Y which tends to zero for $|t| \to +\infty$ such that $-\ddot{\overline{a}}_0 + A\overline{a}_0 = \dot{\overline{q}}\chi_{\overline{J}}$. We assume that for R large enough

$$\max\left(||dS(\overline{q})h - \mu\overline{a}_0||\,, R|(h\,, \overline{a}_0)|\right) \geq C_1||h|| \quad \forall\,(h\,, \mu) \in Y \times \mathbb{R}. \tag{6}$$

We shall assume that (6) holds also for $\widetilde{q}$. We prefix the following definition.

Definition 2. *A manifold $M \subset Y$ is called a natural constraint for the functional f if $\forall x \in M \quad d(f_{|M})(x) = 0$ implies that $df(x) = 0$.*

We now define the «pseudo-critical» manifold. In order to glue $\overline{q}$ and $\widetilde{q}$ we need a proposition related to the λ-lemma on existence and uniqueness of orbits connecting two points α, β in a small neighborhood B_r of 0. By a fixed point argument we can prove that:

Lemma 1. *For all $0 < r < r_1$ with $r_1 = 1/10\Lambda$, for all $\alpha, \beta \in \mathbb{R}^2$ with $|\beta| = |\alpha| = r$, for all $d > 2/\lambda_2$ there exists a unique trajectory of (1) $q_d(t)$ such that $q_d(0) = \beta$ and $q_d(d) = \alpha$.*

In the sequel we will call also $\gamma(\beta, \alpha, d) = q_d$ the connecting solution given by Lemma 1.

Consider the k parameter family of continuous functions Q_Θ defined in the following way:

$$Q_\Theta = \begin{cases} Q^1(t) & \text{if } t \in (-\infty\,, s_1]\,, \\ \gamma(Q^1(s_1)\,, Q^2(u_2)\,, d_1)(\cdot - s_1) & \text{if } t \in [s_1\,, u_2]\,, \\ \dots & \\ Q^i(t) & \text{if } t \in [u_i\,, s_i]\,, \\ \gamma(Q^i(s_i)\,, Q^{i+1}(u_{i+1})\,, d_i)(\cdot - s_i) & \text{if } t \in [s_i\,, u_{i+1}]\,, \\ \dots & \\ Q^k(t) & \text{if } t \in [u_k\,, +\infty)\,, \end{cases}$$

where $u_i = \theta_i - T_i$, $s_i = \theta_i + T_i$ and

$$Q^i(t) = \begin{cases} \overline{q}(\cdot - \theta_i) & \text{if} \quad j_i = 0 \\ \widetilde{q}(\cdot - \theta_i) & \text{if} \quad j_i = 1. \end{cases}$$

The k-dimensional manifold $Z_k = \{Q_\Theta\,,\ \Theta \in \mathbb{R}^k\,,\ \overline{d} > 2/\lambda_2\}$ is a k-dimensional «pseudo-critical» manifold for f. This means that $||S(Q_\Theta)|| \to 0$ as $\overline{d} \to +\infty$.

In the next «shadowing type» lemma we repeat the same arguments as in [1], based on the contraction-mapping theorem in order to build a natural constraint M_k for f, close to Z_k.

LEMMA 2. *For* $0 < r < r_1$ *and* $\overline{d} > \overline{D} = \max\left\{\frac{2}{\lambda_2}, 2(\ln(18/C_1)/\lambda_2) - 2(T - T_{C_1})), \frac{1}{\lambda_2}\ln\left(\frac{40\overline{\Lambda}r}{C_1^2}\right), \frac{1}{\lambda_2}\ln\left(\frac{40}{C_1}\right)\right\}$ *then there is a* C^1 *function* w *(with* $||w(\Theta)|| \to 0$ *as* $\overline{d} \to +\infty$*) such that* $M_k = \left\{Q_\Theta + w(\Theta) \mid \overline{d} > \overline{D}\right\}$ *is a natural constraint for* f.

By Lemma 2 we are led, in order to find k-bumps homoclinics to look for critical points of the functional f restricted to the k-dimensional manifold M_k. Since (1) is autonomous the function of $\Theta \in \mathbb{R}^k$ given by $f(Q_\Theta + w(\Theta))$ depends only on $d_1, \ldots, d_{k-1}$. Let us define $g(d_1, \ldots, d_{k-1}) = f(Q_\Theta + w(\Theta))$. A zero of the function $G : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ defined by $G(d_1, \ldots, d_{k-1}) = \left(\frac{\partial g}{\partial d_1}, \ldots, \frac{\partial g}{\partial d_{k-1}}\right)(d)$ gives rise to an homoclinic solution of (1). Careful estimates show that

LEMMA 3. *For all* $0 < r < r_1$, $\overline{d} > \overline{D}$ *there results that*

$$\left|\left(\frac{\partial}{\partial d_i} g\right)(d) - \sum_{j=1}^{2} \frac{\lambda_j^2}{(\sinh(\lambda_j d))^2}\left(\hat{\alpha}_j^{i+1}\hat{\beta}_j^i \cosh(\lambda_j d) - \frac{(\hat{\alpha}_j^{i+1})^2 + (\hat{\beta}_j^i)^2}{2}\right)\right| \leq$$

$$\leq \frac{15}{2}\lambda_1\lambda_2\Lambda r^3 e^{-\lambda_2 d},$$

where $\hat{\alpha}^{i+1} = (Q_\Theta + w(\Theta))(u_{i+1})$ *and* $\hat{\beta}^i = (Q_\Theta + w(\Theta))(s_i)$.

Theorem 1 is then proved showing that, by the previous lemma, hypothesis ($H1$-$H4$) imply $|\deg(G, U; 0)| = 1$, where $U = \prod_{i=1}^{k-1}(D, J) \subset \mathbb{R}^{k-1}$ and $J > D > \overline{D}$ are some real number which can be explicitly estimated.

The proof of Theorem 3 uses the same pattern. However, since the homoclinic solutions are no longer necessarily nondegenerate, our constrained manifold is of dimension greater than k for the k-bump solutions (roughly its dimension is kl, where l is the max of the dimensions of the solution spaces for the linearised equation at $\overline{q}$ and $\tilde{q}$ respectively). The assumption of topologically nondegeneracy on the homoclinics still allows to find critical points of the functional restricted to the constrained manifold.

The methods described above can be generalized to systems like (1) with $q \in \mathbb{R}^n$.

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