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On the existence of infinitely many solutions for a class of semilinear elliptic equations in $\mathbb{R}^N$


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<http://www.bdim.eu/item?id=RLIN_1998_9_9_3_157_0>
Analisi matematica. — On the existence of infinitely many solutions for a class of semilinear elliptic equations in $\mathbb{R}^N$. Nota di Francesca Alessio, Paolo Caldiroli e Piero Montecchiari, presentata (*) dal Corrisp. A. Ambrosetti.

Abstract. — We show, by variational methods, that there exists a set $A$ open and dense in $\{a \in L^\infty(\mathbb{R}^N) : a \geq 0\}$ such that if $a \in A$ then the problem $-\Delta u + u = a(x)|u|^{p-1}u$, $u \in H^1(\mathbb{R}^N)$, with $p$ subcritical (or more general nonlinearities), admits infinitely many solutions.

Key words: Semilinear elliptic equations; Locally compact case; Minimax arguments; Multiplicity of solutions; Genericity.

Riassunto. — Sull'esistenza di infinite soluzioni per una classe di equazioni ellittiche semilineari su $\mathbb{R}^N$. Usando metodi variazionali, si dimostra che esiste un insieme $A$ aperto e denso in $\{a \in L^\infty(\mathbb{R}^N) : a \geq 0\}$ tale che per ogni $a \in A$ il problema $-\Delta u + u = a(x)|u|^{p-1}u$, $u \in H^1(\mathbb{R}^N)$, con $p$ sottocritico (o con nonlinearità più generali), ammette infinite soluzioni.

1. Statement of the result

In this Note we state a result concerning the existence of infinitely many solutions for a class of semilinear elliptic problems of the form

$$(Pa) \quad -\Delta u + u = a(x)f(u), \quad u \in H^1(\mathbb{R}^N)$$

where $a \in L^\infty(\mathbb{R}^N)$, with ess inf $a > 0$, and $f : \mathbb{R} \to \mathbb{R}$ satisfies:

(f1) $f \in C^1(\mathbb{R})$,

(f2) there exists $C > 0$ such that $|f(t)| \leq C(1 + |t|^p)$ for any $t \in \mathbb{R}$, where $p \in (1, (N + 2)/(N - 2))$ if $N \geq 3$ and $p > 1$ if $N = 1, 2$,

(f3) there exists $\theta > 2$ such that $0 < \theta F(t) \leq f(t)t$ for any $t \neq 0$, where $F(t) = \int_0^t f(s) \, ds$,

(f4) $f(t)/t < f'(t)$ for any $t \neq 0$.

Note that $f(t) = |t|^{p-1}t$ verifies (f1)–(f4) whenever $p \in (1, (N + 2)/(N - 2))$ if $N \geq 3$ or $p > 1$ if $N = 1, 2$.

Such kind of problem has been widely studied with variational methods and its main feature is given by a lack of global compactness due to the unboundedness of the domain. Indeed the imbedding of $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ is not compact and the Palais Smale condition fails.

The existence of nontrivial solutions of $(Pa)$ strongly depends on the behaviour of $a$. We refer to [6-9, 15, 18, 27, 28] for existence results in the case in which $a$ is a positive constant or $a(x) \to a_\infty > 0$ as $|x| \to \infty$.

When \(a\) is periodic, the invariance under translations permits to prove existence, [24], and also multiplicity results, as in [1, 5, 13, 22], where, applying a technique developed in [26], infinitely many solutions (distinct up to translations) are found.

Multiplicity results have been obtained also without periodicity or asymptotic assumptions on \(a\), in some “perturbative” settings, where concentration phenomena occur and a localization procedure can be used to get some compactness in the problem. We mention for instance [3, 4, 10-12, 14, 17, 19, 20, 23, 25].

Although some non existence examples are known (see [16]) we show that the existence of infinitely many solutions for the problem (\(P_a\)) is a generic property with respect to \(a \in L^\infty(\mathbb{R}^N)\) with \(a \geq 0\) a.e. in \(\mathbb{R}^N\). Precisely we prove

**Theorem 1.1.** Let \(f : \mathbb{R} \to \mathbb{R}\) satisfy (f1)-(f4). Then there exists a set \(A\) open and dense in \(\{a \in L^\infty(\mathbb{R}^N) : a(x) \geq 0\ a.e.\ in \mathbb{R}^N\}\) such that for every \(a \in A\) the problem (\(P_a\)) admits infinitely many solutions.

In fact, given any \(a \in L^\infty(\mathbb{R}^N)\), with \(\text{ess inf} a > 0\), for all \(\bar{\alpha} > 0\) we are able to construct a family of functions \(\{\alpha_\omega \in C(\mathbb{R}^N) : \omega \in (0, \hat{\omega})\}\) with \(0 < \alpha_\omega(x) < \bar{\alpha}\) in \(\mathbb{R}^N\) for which the problem (\(P_{a + \alpha_\omega}\)) admits infinitely many solutions. Then we show that this class of solutions is stable with respect to small \(L^\infty\)-perturbations of the functions \(a + \alpha_\omega\).

Let us note that the condition \(\text{ess inf} a \geq 0\) can be weakened by requiring just \(\lim\inf_{|x| \to \infty} a(x) \geq 0\). We refer to [2] for the complete proof of the result.

2. Outline of the proof of Theorem 1.1

Let us fix \(\bar{\alpha} > 0\) and \(a \in L^\infty(\mathbb{R}^N)\) with \(\text{ess inf} a > 0\) and let us denote \(\mathcal{F} = \{b \in L^\infty(\mathbb{R}^N) : a_0 \leq b(x) \leq a_1 \ a.e.\ in \mathbb{R}^N\}\) where \(a_0 = \frac{1}{2}\text{ess inf} a\) and \(a_1 = 2(\|a\|_{L^\infty} + \bar{\alpha})\).

Let \(X = H^1(\mathbb{R}^N)\) be endowed with its standard norm \(\|u\| = (\int_{\mathbb{R}^N}(\|\nabla u\|^2 + u^2) dx)^{1/2}\) and, for every \(b \in \mathcal{F}\) let us introduce the functional

\[
\varphi_b(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} b(x) F(u(x)) \, dx.
\]

By (f2) and (f3), \(\varphi_b \in C^1(X, \mathbb{R})\) for all \(b \in \mathcal{F}\) and \(\varphi'_b(u)v = \langle u, v \rangle - \int_{\mathbb{R}^N} b(x) f(u(x))v(x) \, dx\) where \(\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv) \, dx\). The critical points of \(\varphi_b\) are solutions of the problem (\(P_b\)) and we set \(K_b = \{u \in X : \varphi'_b(u) = 0, \ u \neq 0\}\).

Moreover let us denote \(\langle u, v \rangle_\Omega = \int_{\Omega} (\nabla u \cdot \nabla v + uv) \, dx\) and \(\|u\|_\Omega = \langle u, u \rangle_\Omega^{1/2}\) for all \(u, v \in X\) and \(\Omega\) measurable subset of \(\mathbb{R}^N\).

We start by describing the behavior of any functional \(\varphi_b\) near the origin.

**Lemma 2.1.** \(\varphi_b(u) = \|u\|^2/2 + o(\|u\|^2)\) and \(\varphi'_b(u) = \langle u, \cdot \rangle + o(\|u\|)\) as \(u \to 0\), uniformly with respect to \(b \in \mathcal{F}\).

Moreover there exists \(\bar{\rho} \in (0, 1)\) such that if \(\Omega\) is a regular open subset of \(\mathbb{R}^N\) satisfying the uniform cone property with respect to the cone \(\{x = (x_1, \ldots, x_N) \in B_1(0) : x_1 > |x|/2\}\) and
if \( \sup_{y \in \Omega} \|u\|_{B_1(y)} \leq 2\bar{p} \) then

\[
\int_{\Omega} b(x) F(u) \, dx \leq \frac{1}{4} \|u\|_{\Omega}^2 \quad \text{and} \quad |\int_{\Omega} b(x) f(u) v \, dx| \leq \frac{1}{2} \|u\|_{\Omega} \|v\|_{\Omega}
\]

for every \( b \in \mathcal{F} \) and for every \( u, v \in X \).

According to Lemma 2.1, 0 is a strict local minimum for \( \varphi_b \). Moreover, by \((f3)\), for any \( u \in X \setminus \{0\} \) there exists \( s(u) > 0 \) such that \( \varphi_b(s(u)u) < 0 \) for every \( b \in \mathcal{F} \). Hence, any functional \( \varphi_b \) has the mountain pass geometry with mountain pass level

\[
c(b) = \inf_{\gamma \in \Gamma} \sup_{r \in [0,1]} \varphi_b(\gamma(r))
\]

where \( \Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \varphi_b(\gamma(1)) < 0 \ \forall \ b \in \mathcal{F} \} \).

Note that \( c(b_1) \geq c(b_2) \) if \( b_1, b_2 \in \mathcal{F} \) with \( b_1(x) \leq b_2(x) \) a.e. in \( \mathbb{R}^N \). In particular \( 0 < c(a_t) \leq c(b) \leq c(a_0) \) for every \( b \in \mathcal{F} \).

**Remark 2.1.** By \((f4)\) for every \( u \in X \setminus \{0\} \) there exists a unique \( s_u > 0 \) such that \( \frac{d}{ds} \varphi_b(su)|_{s=s_u} = 0 \) and hence \( c(b) = \inf_{\|u\| = 1} \sup_{r \geq 0} \varphi_b(su) \) and \( \varphi_b \geq c(b) \) for any \( b \in \mathcal{F} \).

Now we state some properties of sequences \( (u_n) \subset X \) such that \( \varphi_{b_n}(u_n) \to l \) and \( \varphi'_{b_n}(u_n) \to 0 \) for some sequence \( (b_n) \subset \mathcal{F} \) (generalized Palais Smale sequences for the class \( \mathcal{F} \)).

**Remark 2.2.** Letting \( \overline{\lambda} = (1 - \frac{2}{\beta})\bar{p}^2 \), by Lemma 2.1 if \( (u_n) \subset X \) is a generalized Palais Smale sequence for the class \( \mathcal{F} \), then

(i) \((u_n)\) is bounded and \( \lim \varphi_{b_n}(u_n) \geq 0 \);
(ii) if \( \lim \varphi_{b_n}(u_n) \in [0, \overline{\lambda}] \) then \( u_n \to 0 \);
(iii) if \( \lim \varphi_{b_n}(u_n) \geq \overline{\lambda} \) then there exists a sequence \( (y_n) \subset \mathbb{R}^N \) such that \( \lim \inf \|u_n\|_{B_1(y_n)} \geq \overline{p} \).

Let us note that \((i)\) follows by the fact that, thanks to \((f3)\), for every \( b \in \mathcal{F} \)

\[
(\frac{1}{\beta} - \frac{1}{\beta})\|u\|^2 \leq \varphi_b(u) + \frac{1}{\beta}\|\varphi'_b(u)\|\|u\| \quad \forall u \in X.
\]

Now, the following characterization holds for the generalized Palais Smale sequences for the class \( \mathcal{F} \).

**Lemma 2.2.** Let \( (b_n) \subset \mathcal{F} \), \( (u_n) \subset X \) and \( (y_n) \subset \mathbb{R}^N \) be such that \( \varphi_{b_n}(u_n) \to l \), \( \varphi'_{b_n}(u_n) \to 0 \) and \( \lim \inf \|u_n\|_{B_1(y_n)} \geq \overline{p} \). Then there exists \( u \in X \) with \( \|u\|_{B_1(0)} \geq \overline{p} \) such that, up to a subsequence,

(i) \( u_n(\cdot + y_n) \to u \) weakly in \( X \), \( \varphi_b(u) \leq l \) and \( \varphi'_b(u) = 0 \), where \( b = \lim b_n(\cdot + y_n) \) in the \( w^{1,\infty} \)-topology,
(ii) \( \varphi_{b_n}(u_n - u(\cdot - y_n)) \to l - \varphi_b(u) \) and \( \varphi'_{b_n}(u_n - u(\cdot - y_n)) \to 0 \).
According to the above result, it is convenient to introduce some definitions concerning the problems «at infinity» associated to any functional \( \varphi_b \). Given \( b \in \mathcal{F} \), let us denote

\[
H_\infty(b) = \{ b \in L^\infty(\mathbb{R}^N) : \exists (y_n) \subset \mathbb{R}^N \text{ s.t. } |y_n| \to \infty, b(\cdot + y_n) \to h \text{ w}^*-L^\infty \}
\]

and \( c_\infty(b) = \inf_{b \in H_\infty(b)} c(b) \).

Using the fact that \( H_\infty(b) \) is sequentially closed with respect to the \( w^*-L^\infty \) topology, it is possible to prove that the value \( c_\infty(b) \) is attained. In fact we have:

**Lemma 2.3.** For every \( b \in \mathcal{F} \) there exist \( b_\infty \in H_\infty(b) \) and \( u_\infty \in X \setminus \{0\} \) such that \( \varphi_{b_\infty}(u_\infty) = c(b_\infty) = c_\infty(b) \) and \( \varphi_{b_\infty}(u_\infty) = 0 \).

In particular we are interested in applying the above result with \( b = a + \alpha \) as follows.

By Lemma 2.3, since \( H_\infty(a + \alpha) = H_\infty(a) + \alpha \), there exist \( a_\infty \in L^\infty(\mathbb{R}^N) \) and a sequence \( (x_j) \subset \mathbb{R}^N \) such that \( a(\cdot + x_j) \to a_\infty \text{ w}^*-L^\infty \) \(|x_{j+1}| - |x_j| \uparrow \infty \) and \( c_\infty(a + \alpha) = c(a_\infty + \alpha) \). Then, for \( \omega \in (0,1) \) we define \( j(\omega) = \inf\{ j \in \mathbb{N} : |x_j| - |x_{j-1}| \geq \frac{\omega}{\alpha} \} \) and

\[
\alpha_\omega(x) = \begin{cases} 
\alpha(1 - \omega^2|x - x_j|^2/4) & \text{for } |x - x_j| \leq 2/\omega, j \geq j(\omega) \\
0 & \text{otherwise.}
\end{cases}
\]

Note that \( \max_{x \in \mathbb{R}^N} \alpha_\omega(x) = \alpha = \alpha(x_j) \) for all \( j \geq j(\omega) \) and \( \alpha_\omega(x) \leq \frac{12}{16} \alpha \) for every \( x \in \mathbb{R}^N \setminus \bigcup_{j \in \mathbb{N}} B_{1/4}(x_j) \).

To simplify the notation, for \( \omega \in (0,1) \) we set \( \varphi_\omega = \varphi_{a_\infty + \alpha_\omega} \), \( \mathcal{K}_\omega = \mathcal{K}_{a_\infty + \alpha_\omega} \). In addition we denote \( \varphi_\infty = \varphi_{a_\infty + \alpha} \) and \( c_\infty = c_\infty(a + \alpha) \).

**Remark 2.3.** By definition of \( c_\infty \), if \( b \in H_\infty(a) \) and \( \beta \in L^\infty(\mathbb{R}^N) \) with \( 0 \leq \beta \leq \alpha \) a.e. in \( \mathbb{R}^N \), then \( c(b + \beta) \geq c(b + \alpha) \geq c_\infty \). Moreover, if \( \beta \in (0, \bar{\alpha}) \), then \( c_\infty(a + \beta) > c_\infty(a + \alpha) = c_\infty \). This is proved using suitable estimates on the critical points of the functionals \( \varphi_{b_\infty} \), being \( b \in \mathcal{F} \).

In the following lemmas we state some properties concerning the sequences \( (u_n) \subset X \) such that \( \varphi'_{\omega_n}(u_n) \to 0 \) and that «carry mass» at infinity, i.e., for which \( \|u_n\|_{B_1(y_n)} \geq \bar{\rho} \) for some sequence \( |y_n| \to \infty \).

First, we give an estimate from below of the level of such sequences:

**Lemma 2.4.** Let \( (\omega_n) \subset (0,1) \), \( (u_n) \subset X \) and \( (y_n) \subset \mathbb{R}^N \) be such that \( \varphi'_{\omega_n}(u_n) \to 0 \), \( |y_n| \to \infty \) and \( \|u_n\|_{B_1(y_n)} \geq \bar{\rho} \) for every \( n \in \mathbb{N} \). Then \( c_\infty \leq \lim \inf_{n} \varphi'_{\omega_n}(u_n) \).

Secondly, a compactness result holds for those sequences \( (u_n) \subset X \) at a level close to \( c_\infty \) and such that \( \varphi'_{\omega_n}(u_n) \to 0 \) and every \( u_n \) has a «mass» located in \( \overline{B}_{1/\omega_n}(x_{j_n}) \).

**Lemma 2.5.** There exist \( h_0 > 0 \) and \( \omega_0 \in (0,1) \) such that if \( (\omega_n) \subset (0, \omega_0) \), \( (u_n) \subset X \) and \( (y_n) \subset \mathbb{R}^N \) satisfy \( \varphi'_{\omega_n}(u_n) \to 0 \), \( \|u_n\|_{B_1(y_n)} \geq \bar{\rho} \), \( y_n \in \overline{B}_{1/\omega_n}(x_{j_n}) \) with \( j_n \geq j(\omega_n) \), and \( \lim \sup \varphi_{\omega_n}(u_n) \leq c_\infty + h_0 \), then \( (u_n(\cdot + y_n)) \) is precompact in \( X \).
The above Lemma suggests to introduce the following sets

\[ A_j(\omega, b, \nu) = \{ u \in X : \varphi_\omega(u) \leq c_\infty + b, \| \varphi'_\omega(u) \| \leq \nu \text{ and } \sup_{y \in B_{\frac{\rho_0}{2}}(x_j)} \| u \|_{B_{1}(y)} \geq \overline{p} \} \]

defined for every \( \omega \in (0, 1) \), \( b > 0 \), \( \nu > 0 \) and \( j \geq j(\omega) \). Let us note that, by Lemma 2.5, for \( \omega \in (0, \omega_0) \) the functional \( \varphi_\omega \) satisfies the Palais Smale condition in each set \( A_j(\omega, b, \nu) \) with \( j \geq j(\omega) \) and \( 0 < h \leq h_0 \).

Hence, the next goal will be to construct a pseudogradient flow which leaves invariant suitable localized minimax classes, in order to get the existence of Palais Smale sequences for \( \varphi_\omega \) in each set \( A_j(\omega, b, \nu) \).

To this extent, we need suitable estimates in neighborhoods of the sets \( A_j(\omega, b, \nu) \).

In fact the following holds:

**Lemma 2.6.** There exist \( \omega \in (0, \omega_0) \), \( \overline{h} \in (0, h_0) \) and \( \forall > 0 \) such that:

(i) if \( u \in B_{\rho_0^2}(A_j(\omega)) \) for some \( \omega \in (0, \overline{\omega}) \) and \( j \geq j(\omega) \), then \( \| u \|_{\| \|_{N - B_{\frac{1}{2}}(x_j)}} \leq 6\rho_0 \);

(ii) if \( u \in (B_{\rho_0^2}(A_j(\omega)) \setminus A_j(\omega)) \cap \{ \varphi_\omega \leq c_\infty + \overline{h} \} \) for some \( \omega \in (0, \overline{\omega}) \) and \( j \geq j(\omega) \), then \( \| u \|_{\| \|_{N - B_{\frac{1}{2}}(x_j)}} < \rho_0 \) and \( \| \varphi'_\omega(u) \| > \forall \),

where \( A_j(\omega) = A_j(\omega, \overline{h}, \forall) \) and \( \rho_0 = \overline{p}/8 \).

By the above listed properties of the sets \( A_j(\omega) \), we can state the existence of a pseudogradient vector field acting in \( A_j(\omega) \). Precisely:

**Lemma 2.7.** There exist \( \varepsilon > 0 \) and \( \overline{\forall} > 0 \) such that for any \( \varepsilon \in (0, \varepsilon) \) there is \( \omega_\varepsilon \in (0, \overline{\omega}) \) for which if \( A_j(\omega) \cap K_\omega = \emptyset \) for some \( \omega \in (0, \omega_\varepsilon) \) and \( j \geq j(\omega) \), then there exist \( \mu_{j\omega} > 0 \) and a locally Lipschitz continuous function \( V_{j\omega} : X \to X \) verifying:

(i) \( \| V_{j\omega}(u) \| \leq 1, \varphi_\omega(u) V_{j\omega}(u) \geq 0 \) for all \( u \in X \) and \( V_{j\omega}(u) = 0 \) for all \( u \in X \setminus B_{\rho_0}(A_j(\omega)) \),

(ii) \( \varphi_\omega(u) V_{j\omega}(u) \geq \mu_{j\omega} \) if \( u \in B_{\rho_0}(A_j(\omega)) \cap \{ \varphi_\omega \leq c_\infty + \overline{h}/2 \} \),

(iii) \( \varphi_\omega(u) V_{j\omega}(u) \geq \overline{\forall} \) if \( u \in (B_{\rho_0}(A_j(\omega)) \setminus B_{\rho_0}(A_j(\omega))) \cap \{ \varphi_\omega \leq c_\infty + \overline{h}/2 \} \),

(iv) \( \langle u, V_{j\omega}(u) \rangle_{\| \|_{N - B_{\frac{1}{2}}(x_j)}} \geq 0 \) if \( \| u \|_{\| \|_{N - B_{\frac{1}{2}}(x_j)}} \geq \varepsilon \).

Now we construct infinitely many minimax classes of mountain pass type for any functional \( \varphi_\omega \) with \( \omega > 0 \) sufficiently small.

First, we point out that, by Lemma 2.3, there exists \( u_\infty \in X \) such that \( \varphi_\infty(u_\infty) = c_\infty \) and \( \varphi'_\infty(u_\infty) = 0 \). Moreover, by Remark 2.1, there exists \( \gamma_\infty \in \Gamma \), with range \( \gamma_\infty \subset \{ s u_\infty : s \geq 0 \} \), satisfying:

(i) \( \max_{s \in [0, 1]} \varphi_\infty(\gamma_\infty(s)) = \varphi_\infty(u_\infty) \),

(ii) for every \( r > 0 \) there is \( h_r > 0 \) such that \( \varphi_\infty(u) \leq c_\infty - h_r \) for any \( u \in \text{range } \gamma_\infty \) with \( \| u - u_\infty \| \geq r \).

Let us fix \( M > 0 \) such that \( \sup_{u \in B_{\rho_0}(A_j(\omega))} \| u \| \leq M \) for all \( \omega \in (0, \overline{\omega}), j \geq j(\omega) \) and \( \max_{s \in [0, 1]} \| \gamma_\infty(s) \| \leq M \). This is possible because of (2.1).
Then, fixing $\hat{\varepsilon} > 0$ small enough (precisely $\hat{\varepsilon} < (1/8) \min\{\varepsilon, h_{\rho_0}, \varpi\}$ where $h_{\rho_0}$ is defined in the above property (ii) and $\varpi$ and $\varepsilon$ in Lemma 2.7), let us define
\[
\Gamma_j(\omega) = \{\gamma \in \Gamma : \|\gamma(s)\| \leq M \text{ and } \|\gamma(s)\|_{\mathbb{R}^N \setminus B_{\rho_j}(s_j)} \leq \hat{\varepsilon} \ \forall \ s \in [0, 1]\}.
\]
The classes of mountain pass paths $\Gamma_j(\omega)$ satisfy the following properties:

**Lemma 2.8.** There exists $\hat{\omega} \in (0, \hat{\omega}_e)$ such that for all $\omega \in (0, \hat{\omega})$ and $j \geq j(\omega)$, setting $\gamma_j(s) = \gamma_{\omega}(s) - x_j$ for all $s \in [0, 1]$, there results:

(i) $\gamma_j \in \Gamma_j(\omega)$,

(ii) $\max_{s \in [0, 1]} \varphi_\omega(\gamma_j(s)) \leq c_\omega + \hat{\varepsilon}$,

(iii) if $\gamma_j(s) \notin B_{\rho_0}(A_j(\omega))$ then $\varphi_\omega(\gamma_j(s)) \leq c_\omega - h_{\rho_0}/2$.

In particular $\Gamma_j(\omega) \neq \emptyset$ for all $\omega \in (0, \hat{\omega})$ and $j \geq j(\omega)$, and we can define the corresponding minimax values
\[
c_j(\omega) = \inf_{\gamma \in \Gamma_j(\omega)} \max_{s \in [0, 1]} \varphi_\omega(\gamma(s)).
\]
These mountain pass levels are close to the mountain pass level $c_\omega$ in the sense explained by the following Lemma.

**Lemma 2.9.** For all $\omega \in (0, \hat{\omega})$ there exists $j(\omega) \geq j(\omega)$ such that $|c_j(\omega) - c_\omega| \leq \hat{\varepsilon}$ for all $j \geq j(\omega)$.

Now we can prove that for $\omega > 0$ sufficiently small, the functional $\varphi_\omega$ admits infinitely many critical points. More precisely we show that:

**Lemma 2.10.** If $\omega \in (0, \hat{\omega})$ then $A_j(\omega) \cap K_\omega \neq \emptyset$ for every $j \geq j(\omega)$.

**Proof.** Arguing by contradiction, suppose that there exist $\omega \in (0, \hat{\omega})$ and $j \geq j(\omega)$ such that $A_j(\omega) \cap K_\omega = \emptyset$. Let $V_{j\omega} : X \to X$ be the pseudogradient vector field given by Lemma 2.7 and let $\eta \in C(\mathbb{R} \times X, X)$ be the associated flow, given by the solution of the Cauchy problem
\[
\begin{cases}
  \frac{d\eta(t, u)}{dt} = -V_{j\omega}(\eta(t, u)) \\
  \eta(0, u) = u.
\end{cases}
\]
Note that $\eta$ is well defined and continuous in $\mathbb{R} \times X$ because the field $V_{j\omega}$ is a bounded, locally Lipschitz continuous function. Moreover, by the properties of $V_{j\omega}$ stated in Lemma 2.7, for a fixed $\tau > 0$ large enough, the function $\eta_{j\omega}(u) = \eta(\tau, u)$ satisfies:

(i) $\eta_{j\omega}(u) = u$ for all $u \in X \setminus B_{\rho_0}(A_j(\omega))$, 

(ii) $\varphi_\omega(\eta_{j\omega}(u)) \leq \varphi_\omega(u)$ for all $u \in X$,

(iii) $\varphi_\omega(\eta_{j\omega}(u)) \leq \varphi_\omega(u) - \varpi_{\rho_0}$ if $u \in B_{\rho_0}(A_j(\omega)) \cap \{\varphi_\omega \leq c_\omega + \varpi/2\}$,

(iv) $\|\eta_{j\omega}(u)\|_{\mathbb{R}^N \setminus B_{\rho_j}(s_j)} \leq \varepsilon$ if $\|u\|_{\mathbb{R}^N \setminus B_{\rho_j}(s_j)} \leq \varepsilon$.

Let now $\hat{\gamma}_j(s) = \eta_{j\omega}(\gamma_j(s))$ for $s \in [0, 1]$, where $\gamma_j \in \Gamma_j(\omega)$ is defined as in Lemma 2.8. By the above listed properties (i) and (ii) of $\eta_{j\omega}$, the class $\Gamma_j(\omega)$ is invariant under the deformation $\eta_{j\omega}$ and then $\hat{\gamma}_j \in \Gamma_j(\omega)$. We claim that $\max_{s \in [0, 1]} \varphi_\omega(\hat{\gamma}_j(s)) \leq c_j(\omega) - \hat{\varepsilon}$
and therefore we get a contradiction with the definition of \( c_j(\omega) \). Indeed, if \( \gamma_j(s) \not\in B_{\rho_0}(A_j(\omega)) \), by the property (ii) of \( \eta_{j_\omega} \) and by Lemma 2.8 (iii), we have \( \varphi_{\omega}(\gamma_j(s)) \leq \varphi_{\omega}(\gamma_j(s)) - \varphi_{\omega}(\gamma_j(s)) - b_j \leq c_{\infty} - 2\varepsilon \), since \( \bar{\varepsilon} < b_j/4 \). On the other hand, if \( \gamma_j(s) \in B_{\rho_0}(A_j(\omega)) \), by the property (iii) of \( \eta_{j_\omega} \) and by Lemma 2.8 (ii), we have \( \varphi_{\omega}(\gamma_j(s)) \leq \varphi_{\omega}(\gamma_j(s)) - \bar{\rho}_0 \leq c_{\infty} - 2\varepsilon \), since \( \varepsilon \leq \bar{\rho}_0/9 \). Therefore, by Lemma 2.9, for all \( s \in [0, 1] \) we conclude that \( \varphi_{\omega}(\gamma_j(s)) \leq c_{\infty} - 2\varepsilon \leq c_j(\omega) - \varepsilon \). \( \square \)

We remark that by the arbitrariness of \( \pi > 0 \) and \( a \in L^\infty(\mathbb{R}^N) \) with ess inf \( a > 0 \), the above result shows that the problem \( (P_a) \) admits infinitely many solutions whenever \( a \) belongs to a dense subset of \( \{ a \in L^\infty(\mathbb{R}^N) : a \geq 0 \} \).

Then Theorem 1.1 follows by the next final Lemma.

**Lemma 2.11.** If \( \omega \in (0, \hat{\omega}) \), there exists \( \beta_0 > 0 \) such that if \( \| \beta \|_{L^\infty(\mathbb{R}^N)} \leq \beta_0 \) then the problem \( (P_{a + \alpha_{\omega} + \beta}) \) admits infinitely many solutions.

**Proof.** Given \( \beta \in L^\infty(\mathbb{R}^N) \) we denote \( \varphi_{\omega\beta}(u) = \varphi_{\omega}(u) - \int_{\mathbb{R}^N} \beta(x) F(u) \, dx \) and \( \mathcal{K}_{\omega\beta} = \{ u \in X \setminus \{ 0 \} : \varphi_{\omega\beta}'(u) = 0 \} \). We note that \( a + \alpha_{\omega} + \beta \in \mathcal{F} \) whenever \( \| \beta \|_{L^\infty(\mathbb{R}^N)} \leq \hat{\alpha}_0 \).

Letting \( M \) be the constant fixed before the definition of \( \Gamma_j(\omega) \), there exists \( C = C(M) > 0 \) such that

\[
\begin{align*}
(2.2) & \quad \sup_{\| u \| \leq M} \| \varphi_{\omega\beta}(u) - \varphi_{\omega}(u) \| \leq C \| \beta \|_{L^\infty(\mathbb{R}^N)}, \\
(2.3) & \quad \sup_{\| u \| \leq M} \| \varphi_{\omega\beta}'(u) - \varphi_{\omega}'(u) \| \leq C \| \beta \|_{L^\infty(\mathbb{R}^N)}.
\end{align*}
\]

We claim that if \( \omega \in (0, \hat{\omega}) \) and \( j \geq j(\omega) \) then \( \mathcal{K}_{\omega\beta} \cap A_j(\omega) \neq \emptyset \) whenever \( \| \beta \|_{L^\infty} \leq \beta_0 \), being \( \beta_0 = (1/2) \min \{ \alpha_{\omega}, \hat{\varepsilon}/C \} \) with \( \hat{\varepsilon} > 0 \) fixed above.

Indeed, arguing by contradiction, assume that \( \mathcal{K}_{\omega\beta} \cap A_j(\omega) = \emptyset \) for some \( \omega \in (0, \hat{\omega}) \) and \( j \geq j(\omega) \). Then, using (2.2) and (2.3), one can see that

1. There exists \( \nu_j > 0 \) such that \( \| \varphi_{\omega\beta}'(u) \| \geq \nu_j \) for all \( u \in A_j(\omega) \cap \{ \varphi_{\omega} \leq c_{\infty} + 2\bar{\beta}/3 \} \).
2. \( \| \varphi_{\omega\beta}'(u) \| \geq \bar{\nu}/2 \) for all \( u \in (B_{4\bar{\rho}_0}(A_j(\omega)) \setminus A_j(\omega)) \cap \{ \varphi_{\omega} \leq c_{\infty} + \bar{\beta}/2 \} \).

By (1) and (2), since \( a + \alpha_{\omega} + \beta \in \mathcal{F} \), it is possible to show the existence of a pseudogravitational field \( \tilde{V}_j : X \to X \) satisfying:

1. \( \| \tilde{V}_j(u) \| \leq 1 \), \( \varphi_{\omega\beta}'(u) \tilde{V}_j(u) \geq 0 \) for all \( u \in X \) and \( \tilde{V}_j(u) = 0 \) for all \( u \in X \setminus B_{4\bar{\rho}_0}(A_j(\omega)) \),
2. \( \varphi_{\omega\beta}'(u) \tilde{V}_j(u) \geq \nu_j > 0 \) if \( u \in B_{\rho_0}(A_j(\omega)) \cap \{ \varphi_{\omega} \leq c_{\infty} + \bar{\beta}/2 \} \),
3. \( \varphi_{\omega\beta}'(u) \tilde{V}_j(u) \geq \bar{\nu}/2 \) if \( u \in (B_{2\bar{\rho}_0}(A_j(\omega)) \setminus B_{\rho_0}(A_j(\omega))) \cap \{ \varphi_{\omega} \leq c_{\infty} + \bar{\beta}/2 \} \),
4. \( \langle u, \tilde{V}_j(u) \rangle_{\mathbb{R}^N \setminus \overline{B}_{\bar{\rho}_0}(A_j(\omega))} \geq 0 \) if \( \| u \|_{\mathbb{R}^N \setminus \overline{B}_{\bar{\rho}_0}(A_j(\omega))} \geq \hat{\varepsilon} \).

Considering the flow associated to the field \( \tilde{V}_j \), we obtain the existence of a continuous function \( \eta_j : X \to X \) which verifies:

1. \( \eta_j(u) = u \) for all \( u \in X \setminus B_{\rho_0}(A_j(\omega)) \),
2. \( \varphi_{\omega\beta}(\eta_j(u)) \leq \varphi_{\omega\beta}(u) \) for all \( u \in X \).
\[ \varphi_{\omega^j}(\eta_j(u)) \leq \varphi_{\omega^j}(u) - \bar{m}_\rho_0/2 \text{ if } u \in B_{\rho_0}(A_j(\omega)) \cap \{ \varphi_{\omega} \leq c_\infty + \tilde{h}/2 \}, \]

\[ \|\eta_j(u)\|_{N} \leq \varepsilon \text{ if } \|u\|_{N \backslash B_{\rho_0}(x_j)} \leq \varepsilon. \]

Then, considering the path \( \tilde{\gamma}_j(s) = \eta_j(\gamma_\omega(s)(-x_j)) \), \( s \in [0, 1] \), by (i)' and (iv)', \( \tilde{\gamma}_j \in \Gamma_j(\omega) \). Then, by (2.2), (iii)' and (iii)'', since \( \hat{\varepsilon} < (1/8) \min \{ h_{\rho_0}, \mu_{\rho_0} \} \), using Lemma 2.9, we get

\[ \max_{s \in [0, 1]} \varphi_{\omega}(\tilde{\gamma}_j(s)) \leq \max_{s \in [0, 1]} \varphi_{\omega}(\tilde{\gamma}_j(s)) + \hat{\varepsilon}/2 \leq \max \{ c_\infty - h_{\rho_0}/2 + \hat{\varepsilon}, c_\infty - \bar{m}_\rho_0/2 + \tilde{2}\hat{\varepsilon} \} < c_j^*(\omega), \]

a contradiction. \( \Box \)

References


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