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On the existence of infinitely many solutions for a class of semilinear elliptic equations in $\mathbb{R}^N$

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Analisi matematica. — *On the existence of infinitely many solutions for a class of semilinear elliptic equations in* $\mathbb{R}^N$. Nota di Francesca Alessio, Paolo Caldiroli e Piero Montecchiari, presentata (*) dal Corrisp. A. Ambrosetti.

Abstract. — We show, by variational methods, that there exists a set $\mathcal{A}$ open and dense in $\{a \in L^\infty(\mathbb{R}^N) : a \geq 0\}$ such that if $a \in \mathcal{A}$ then the problem $-\Delta u + u = a(x)|u|^{p-1}u$, $u \in H^1(\mathbb{R}^N)$, with p subcritical (or more general nonlinearities), admits infinitely many solutions.

Key words: Semilinear elliptic equations; Locally compact case; Minimax arguments; Multiplicity of solutions; Genericity.

Riassunto. — *Sull'esistenza di infinite soluzioni per una classe di equazioni ellittiche semilineari su* $\mathbb{R}^N$. Usando metodi variazionali, si dimostra che esiste un insieme $\mathcal{A}$ aperto e denso in $\{a \in L^\infty(\mathbb{R}^N) : a \geq 0\}$ tale che per ogni $a \in \mathcal{A}$ il problema $-\Delta u + u = a(x)|u|^{p-1}u$, $u \in H^1(\mathbb{R}^N)$, con p sottocritico (o con nonlinearità più generali), ammette infinite soluzioni.

1. Statement of the result

In this *Note* we state a result concerning the existence of infinitely many solutions for a class of semilinear elliptic problems of the form

$$(P_a) \qquad -\Delta u + u = a(x)f(u)\ , \quad u \in H^1(\mathbb{R}^N)$$

where $a \in L^\infty(\mathbb{R}^N)$, with ess inf $a > 0$, and $f : \mathbb{R} \to \mathbb{R}$ satisfies:

$(f1)$ $f \in C^1(\mathbb{R})$,

$(f2)$ there exists $C > 0$ such that $|f(t)| \leq C(1 + |t|^p)$ for any $t \in \mathbb{R}$, where $p \in (1\,,(N+2)/(N-2))$ if $N \geq 3$ and $p > 1$ if $N = 1\,,2$,

$(f3)$ there exists $\theta > 2$ such that $0 < \theta F(t) \leq f(t)t$ for any $t \neq 0$, where $F(t) = \int_0^t f(s)\,ds$,

$(f4)$ $f(t)/t < f'(t)$ for any $t \neq 0$.

Note that $f(t) = |t|^{p-1}t$ verifies $(f1)$–$(f4)$ whenever $p \in (1\,,(N+2)/(N-2))$ if $N \geq 3$ or $p > 1$ if $N = 1\,,2$.

Such kind of problem has been widely studied with variational methods and its main feature is given by a lack of global compactness due to the unboundedness of the domain. Indeed the imbedding of $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ is not compact and the Palais Smale condition fails.

The existence of nontrivial solutions of (P_a) strongly depends on the behaviour of a. We refer to [6-9, 15, 18, 27, 28] for existence results in the case in which a is a positive constant or $a(x) \to a_\infty > 0$ as $|x| \to \infty$.

(*) Nella seduta del 13 marzo 1998.

When a is periodic, the invariance under translations permits to prove existence, [24], and also multiplicity results, as in [1, 5, 13, 22], where, applying a technique developed in [26], infinitely many solutions (distinct up to translations) are found.

Multiplicity results have been obtained also without periodicity or asymptotic assumptions on a, in some «perturbative» settings, where concentration phenomena occur and a localization procedure can be used to get some compactness in the problem. We mention for instance [3, 4, 10-12, 14, 17, 19, 20, 23, 25].

Although some non existence examples are known (see [16]) we show that the existence of infinitely many solutions for the problem (P_a) is a generic property with respect to $a \in L^\infty(\mathbb{R}^N)$ with $a \geq 0$ a.e. in $\mathbb{R}^N$. Precisely we prove

THEOREM 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $(f1)$-$(f4)$. Then there exists a set $\mathcal{A}$ open and dense in $\{a \in L^\infty(\mathbb{R}^N) : a(x) \geq 0$ a.e. in $\mathbb{R}^N\}$ such that for every $a \in \mathcal{A}$ the problem (P_a) admits infinitely many solutions.*

In fact, given any $a \in L^\infty(\mathbb{R}^N)$, with $\operatorname{ess\,inf} a > 0$, for all $\overline{\alpha} > 0$ we are able to construct a family of functions $\{\alpha_\omega \in C(\mathbb{R}^N) : \omega \in (0, \hat{\omega})\}$ with $0 \leq \alpha_\omega(x) \leq \overline{\alpha}$ in $\mathbb{R}^N$ for which the problem $(P_{a+\alpha_\omega})$ admits infinitely many solutions. Then we show that this class of solutions is stable with respect to small L^∞-perturbations of the functions $a + \alpha_\omega$.

Let us note that the condition $\operatorname{ess\,inf} a \geq 0$ can be weakened by requiring just $\liminf_{|x|\to\infty} a(x) \geq 0$. We refer to [2] for the complete proof of the result.

2. OUTLINE OF THE PROOF OF THEOREM 1.1

Let us fix $\overline{\alpha} > 0$ and $a \in L^\infty(\mathbb{R}^N)$ with $\operatorname{ess\,inf} a > 0$ and let us denote $\mathcal{F} = \{b \in$ $\in L^\infty(\mathbb{R}^N) : a_0 \leq b(x) \leq a_1$ *a.e. in* $\mathbb{R}^N\}$ where $a_0 = \frac{1}{2}\operatorname{ess\,inf} a$ and $a_1 = 2(\|a\|_{L^\infty} + \overline{\alpha})$.

Let $X = H^1(\mathbb{R}^N)$ be endowed with its standard norm $\|u\| = (\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)dx)^{1/2}$ and, for every $b \in \mathcal{F}$ let us introduce the functional

$$\varphi_b(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} b(x)F(u(x))\,dx.$$

By $(f2)$ and $(f3)$, $\varphi_b \in C^1(X,\mathbb{R})$ for all $b \in \mathcal{F}$ and $\varphi_b'(u)v = \langle u, v\rangle - \int_{\mathbb{R}^N} b(x)f(u(x))v(x)\,dx$ where $\langle u, v\rangle = \int_{\mathbb{R}^N}(\nabla u \cdot \nabla v + uv)\,dx$. The critical points of φ_b are solutions of the problem (P_b) and we set $\mathcal{K}_b = \{u \in X : \varphi_b'(u) = 0, \ u \neq 0\}$.

Moreover let us denote $\langle u, v\rangle_\Omega = \int_\Omega(\nabla u \cdot \nabla v + uv)\,dx$ and $\|u\|_\Omega = \langle u, u\rangle_\Omega^{1/2}$ for all $u, v \in X$ and Ω measurable subset of $\mathbb{R}^N$.

We start by describing the behavior of any functional φ_b near the origin.

LEMMA 2.1. *$\varphi_b(u) = \|u\|^2/2 + o(\|u\|^2)$ and $\varphi_b'(u) = \langle u, \cdot\rangle + o(\|u\|)$ as $u \to 0$, uniformly with respect to $b \in \mathcal{F}$.*

Moreover there exists $\overline{\rho} \in (0, 1)$ such that if Ω is a regular open subset of $\mathbb{R}^N$ satisfying the uniform cone property with respect to the cone $\{x = (x_1, \ldots, x_N) \in B_1(0) : x_1 > |x|/2\}$ and *if* $\sup_{y\in\Omega} \|u\|_{B_1(y)} \leq 2\overline{\rho}$ *then*

$$\int_\Omega b(x)F(u)\,dx \leq \tfrac{1}{4}\|u\|_\Omega^2 \quad and \quad |\int_\Omega b(x)f(u)v\,dx| \leq \tfrac{1}{2}\|u\|_\Omega\|v\|_\Omega$$

for every $b \in \mathcal{F}$ *and for every* $u, v \in X$.

According to Lemma 2.1, 0 is a strict local minimum for φ_b. Moreover, by $(f3)$, for any $u \in X \setminus \{0\}$ there exists $s(u) > 0$ such that $\varphi_b(s(u)u) < 0$ for every $b \in \mathcal{F}$. Hence, any functional φ_b has the mountain pass geometry with mountain pass level

$$c(b) = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} \varphi_b(\gamma(s))$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \varphi_b(\gamma(1)) < 0\ \ \forall\ b \in \mathcal{F}\}$.

Note that $c(b_1) \geq c(b_2)$ if $b_1, b_2 \in \mathcal{F}$ with $b_1(x) \leq b_2(x)$ a.e. in $\mathbb{R}^N$. In particular $0 < c(a_1) \leq c(b) \leq c(a_0)$ for every $b \in \mathcal{F}$.

Remark 2.1. By $(f4)$ for every $u \in X \setminus \{0\}$ there exists a unique $s_u > 0$ such that $\frac{d}{ds}\varphi_b(su)|_{s=s_u} = 0$ and hence $c(b) = \inf_{\|u\|=1} \sup_{s\geq 0} \varphi_b(su)$ and $\inf_{\mathcal{K}_b} \varphi_b \geq c(b)$ for any $b \in \mathcal{F}$.

Now we state some properties of sequences $(u_n) \subset X$ such that $\varphi_{b_n}(u_n) \to l$ and $\varphi'_{b_n}(u_n) \to 0$ for some sequence $(b_n) \subset \mathcal{F}$ (generalized Palais Smale sequences for the class $\mathcal{F}$).

Remark 2.2. Letting $\overline{\lambda} = (1 - \frac{2}{\theta})\overline{\rho}^2$, by Lemma 2.1 if $(u_n) \subset X$ is a generalized Palais Smale sequence for the class $\mathcal{F}$, then

(i) (u_n) is bounded and $\lim \varphi_{b_n}(u_n) \geq 0$;

(ii) if $\lim \varphi_{b_n}(u_n) \in [0, \overline{\lambda})$ then $u_n \to 0$;

(iii) if $\lim \varphi_{b_n}(u_n) \geq \overline{\lambda}$ then there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that $\liminf \|u_n\|_{B_1(y_n)} \geq \overline{\rho}$.

Let us note that (i) follows by the fact that, thanks to $(f3)$, for every $b \in \mathcal{F}$

$$(\tfrac{1}{2} - \tfrac{1}{\theta})\|u\|^2 \leq \varphi_b(u) + \tfrac{1}{\theta}\|\varphi'_b(u)\|\,\|u\| \quad \forall\ u \in X. \tag{2.1}$$

Now, the following characterization holds for the generalized Palais Smale sequences for the class $\mathcal{F}$.

Lemma 2.2. *Let* $(b_n) \subset \mathcal{F}$, $(u_n) \subset X$ *and* $(y_n) \subset \mathbb{R}^N$ *be such that* $\varphi_{b_n}(u_n) \to l$, $\varphi'_{b_n}(u_n) \to 0$ *and* $\liminf \|u_n\|_{B_1(y_n)} \geq \overline{\rho}$. *Then there exists* $u \in X$ *with* $\|u\|_{B_1(0)} \geq \overline{\rho}$ *such that, up to a subsequence,*

(i) $u_n(\cdot + y_n) \to u$ *weakly in* X, $\varphi_b(u) \leq l$ *and* $\varphi'_b(u) = 0$, *where* $b = \lim b_n(\cdot + y_n)$ *in the* w^*-L^∞ *topology,*

(ii) $\varphi_{b_n}(u_n - u(\cdot - y_n)) \to l - \varphi_b(u)$ *and* $\varphi'_{b_n}(u_n - u(\cdot - y_n)) \to 0$.

According to the above result, it is convenient to introduce some definitions concerning the problems «at infinity» associated to any functional φ_b. Given $b \in \mathcal{F}$, let us denote

$$H_\infty(b) = \{h \in L^\infty(\mathbb{R}^N) : \exists\ (y_n) \subset \mathbb{R}^N \ s.t.\ |y_n| \to \infty\,,\ b(\cdot + y_n) \to h\ w^*\text{-}L^\infty\}$$

and $c_\infty(b) = \inf_{h\in H_\infty(b)} c(h)$.

Using the fact that $H_\infty(b)$ is sequentially closed with respect to the w^*-L^∞ topology, it is possible to prove that the value $c_\infty(b)$ is attained. In fact we have:

LEMMA 2.3. *For every $b \in \mathcal{F}$ there exist $b_\infty \in H_\infty(b)$ and $u_\infty \in X \setminus \{0\}$ such that $\varphi_{b_\infty}(u_\infty) = c(b_\infty) = c_\infty(b)$ and $\varphi'_{b_\infty}(u_\infty) = 0$.*

In particular we are interested in applying the above result with $b = a + \overline{\alpha}$ as follows.

By Lemma 2.3, since $H_\infty(a + \overline{\alpha}) = H_\infty(a) + \overline{\alpha}$, there exist $a_\infty \in L^\infty(\mathbb{R}^N)$ and a sequence $(x_j) \subset \mathbb{R}^N$ such that $a(\cdot + x_j) \to a_\infty$ w^*-L^∞ $|x_{j+1}| - |x_j| \uparrow +\infty$ and $c_\infty(a + \overline{\alpha}) = c(a_\infty + \overline{\alpha})$. Then, for $\omega \in (0\,,1)$ we define $j(\omega) = \inf\{j \in \mathbb{N}\ :\ |x_j| - |x_{j-1}| \geq 4/\omega\}$ and

$$\alpha_\omega(x) = \begin{cases} \overline{\alpha}(1 - \omega^2|x - x_j|^2/4) & \text{for } |x - x_j| \leq 2/\omega\ ,\ \ j \geq j(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\max_{x\in\mathbb{R}^N} \alpha_\omega(x) = \overline{\alpha} = \alpha(x_j)$ for all $j \geq j(\omega)$ and $\alpha_\omega(x) \leq \frac{15}{16}\overline{\alpha}$ for every $x \in \mathbb{R}^N \setminus \bigcup_{j\in\mathbb{N}} B_{\frac{1}{2\omega}}(x_j)$.

To simplify the notation, for $\omega \in (0\,,1)$ we set $\varphi_\omega = \varphi_{a+\alpha_\omega}$, $\mathcal{K}_\omega = \mathcal{K}_{a+\alpha_\omega}$. In addition we denote $\varphi_\infty = \varphi_{a_\infty+\overline{\alpha}}$ and $c_\infty = c_\infty(a + \overline{\alpha})$.

REMARK 2.3. By definition of c_∞, if $b \in H_\infty(a)$ and $\beta \in L^\infty(\mathbb{R}^N)$ with $0 \leq \beta \leq \overline{\alpha}$ a.e. in $\mathbb{R}^N$, then $c(b + \beta) \geq c(b + \overline{\alpha}) \geq c_\infty$. Moreover, if $\beta \in (0\,,\overline{\alpha})$, then $c_\infty(a + \beta) > c_\infty(a + \overline{\alpha}) = c_\infty$. This is proved using suitable estimates on the critical points of the functionals φ_{b_∞}, being $b \in \mathcal{F}$.

In the following lemmas we state some properties concerning the sequences $(u_n) \subset X$ such that $\varphi'_{\omega_n}(u_n) \to 0$ and that «carry mass» at infinity, *i.e.*, for which $\|u_n\|_{B_1(y_n)} \geq \overline{\rho}$ for some sequence $|y_n| \to \infty$.

First, we give an estimate from below of the level of such sequences:

LEMMA 2.4. *Let $(\omega_n) \subset (0\,,1)$, $(u_n) \subset X$ and $(y_n) \subset \mathbb{R}^N$ be such that $\varphi'_{\omega_n}(u_n) \to 0$, $|y_n| \to \infty$ and $\|u_n\|_{B_1(y_n)} \geq \overline{\rho}$ for every $n \in \mathbb{N}$. Then $c_\infty \leq \liminf \varphi_{\omega_n}(u_n)$.*

Secondly, a compactness result holds for those sequences $(u_n) \subset X$ at a level close to c_∞ and such that $\varphi'_{\omega_n}(u_n) \to 0$ and every u_n has a «mass» located in $\overline{B}_{\frac{1}{\omega_n}}(x_{j_n})$.

LEMMA 2.5. *There exist $h_0 > 0$ and $\omega_0 \in (0\,,1)$ such that if $(\omega_n) \subset (0\,,\omega_0)$, $(u_n) \subset X$ and $(y_n) \subset \mathbb{R}^N$ satisfy $\varphi'_{\omega_n}(u_n) \to 0$, $\|u_n\|_{B_1(y_n)} \geq \overline{\rho}$, $y_n \in \overline{B}_{\frac{1}{\omega_n}}(x_{j_n})$ with $j_n \geq j(\omega_n)$, and $\limsup \varphi_{\omega_n}(u_n) \leq c_\infty + h_0$, then $(u_n(\cdot + y_n))$ is precompact in X.*

The above Lemma suggests to introduce the following sets

$$\mathcal{A}_j(\omega, h, \nu) = \{u \in X \ : \ \varphi_\omega(u) \le c_\infty + h, \ \|\varphi'_\omega(u)\| \le \nu \ \text{ and } \sup_{y \in \overline{B}_{\frac{1}{2\omega}}(x_j)} \|u\|_{B_1(y)} \ge \overline{\rho}\}$$

defined for every $\omega \in (0, 1)$, $h > 0$, $\nu > 0$ and $j \ge j(\omega)$. Let us note that, by Lemma 2.5, for $\omega \in (0, \omega_0)$ the functional φ_ω satisfies the Palais Smale condition in each set $\mathcal{A}_j(\omega, h, \nu)$ with $j \ge j(\omega)$ and $0 < h \le h_0$.

Hence, the next goal will be to construct a pseudogradient flow which leaves invariant suitable localized minimax classes, in order to get the existence of Palais Smale sequences for φ_ω in each set $\mathcal{A}_j(\omega, h, \nu)$.

To this extent, we need suitable estimates in neighborhoods of the sets $\mathcal{A}_j(\omega, h, \nu)$. In fact the following holds:

LEMMA 2.6. *There exist* $\overline{\omega} \in (0, \omega_0)$, $\overline{h} \in (0, h_0)$ *and* $\overline{\nu} > 0$ *such that*:

(i) *if* $u \in B_{4\rho_0}(\mathcal{A}_j(\omega))$ *for some* $\omega \in (0, \overline{\omega})$ *and* $j \ge j(\omega)$, *then* $\|u\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{2\omega}-1}(x_j)} \le 6\rho_0$;

(ii) *if* $u \in (B_{4\rho_0}(\mathcal{A}_j(\omega)) \setminus \mathcal{A}_j(\omega)) \cap \{\varphi_\omega \le c_\infty + \overline{h}\}$ *for some* $\omega \in (0, \overline{\omega})$ *and* $j \ge j(\omega)$, *then* $\|u\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{2\omega}-1}(x_j)} < \rho_0$ *and* $\|\varphi'_\omega(u)\| > \overline{\nu}$,

where $\mathcal{A}_j(\omega) = \mathcal{A}_j(\omega, \overline{h}, \overline{\nu})$ *and* $\rho_0 = \overline{\rho}/8$.

By the above listed properties of the sets $\mathcal{A}_j(\omega)$, we can state the existence of a pseudogradient vector field acting in $\mathcal{A}_j(\omega)$. Precisely:

LEMMA 2.7. *There exist* $\overline{\varepsilon} > 0$ *and* $\overline{\mu} > 0$ *such that for any* $\varepsilon \in (0, \overline{\varepsilon})$ *there is* $\omega_\varepsilon \in (0, \overline{\omega})$ *for which if* $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega = \emptyset$ *for some* $\omega \in (0, \omega_\varepsilon)$ *and* $j \ge j(\omega)$, *then there exist* $\mu_{j\omega} > 0$ *and a locally Lipschitz continuous function* $V_{j\omega}\colon X \to X$ *verifying*:

(i) $\|V_{j\omega}(u)\| \le 1$, $\varphi'_\omega(u) V_{j\omega}(u) \ge 0$ *for all* $u \in X$ *and* $V_{j\omega}(u) = 0$ *for all* $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,

(ii) $\varphi'_\omega(u) V_{j\omega}(u) \ge \mu_{j\omega}$ *if* $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \le c_\infty + \overline{h}/2\}$,

(iii) $\varphi'_\omega(u) V_{j\omega}(u) \ge \overline{\mu}$ *if* $u \in (B_{2\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \le c_\infty + \overline{h}/2\}$,

(iv) $\langle u, V_{j\omega}(u)\rangle_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \ge 0$ *if* $\|u\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \ge \varepsilon$.

Now we construct infinitely many minimax classes of mountain pass type for any functional φ_ω with $\omega > 0$ sufficiently small.

First, we point out that, by Lemma 2.3, there exists $u_\infty \in X$ such that $\varphi_\infty(u_\infty) = c_\infty$ and $\varphi'_\infty(u_\infty) = 0$. Moreover, by Remark 2.1, there exists $\gamma_\infty \in \Gamma$, with range $\gamma_\infty \subset$ $\subset \{su_\infty : s \ge 0\}$, satisfying:

(i) $\max_{s \in [0,1]} \varphi_\infty(\gamma_\infty(s)) = \varphi_\infty(u_\infty)$,

(ii) for every $r > 0$ there is $h_r > 0$ such that $\varphi_\infty(u) \le c_\infty - h_r$ for any $u \in$ range γ_∞ with $\|u - u_\infty\| \ge r$.

Let us fix $M > 0$ such that $\sup_{u \in B_{4\rho_0}(\mathcal{A}_j(\omega))} \|u\| \le M$ for all $\omega \in (0, \overline{\omega})$, $j \ge j(\omega)$ and $\max_{s \in [0,1]} \|\gamma_\infty(s)\| \le M$. This is possible because of (2.1).

Then, fixing $\hat{\varepsilon} > 0$ small enough (precisely $\hat{\varepsilon} < (1/8)\min\{\overline{\varepsilon}, h_{\rho_0}, \overline{\mu}\rho_0\}$ where h_{ρ_0} is defined in the above property (*ii*) and $\overline{\mu}$ and $\overline{\varepsilon}$ in Lemma 2.7), let us define

$$\Gamma_j(\omega) = \{\gamma \in \Gamma : \|\gamma(s)\| \leq M \text{ and } \|\gamma(s)\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \leq \hat{\varepsilon} \ \ \forall\ s \in [0\,,1]\}.$$

The classes of mountain pass paths $\Gamma_j(\omega)$ satisfy the following properties:

Lemma 2.8. *There exists $\hat{\omega} \in (0\,, \omega_{\hat{\varepsilon}})$ such that for all $\omega \in (0\,, \hat{\omega})$ and $j \geq j(\omega)$, setting $\gamma_j(s) = \gamma_\infty(s)(\cdot - x_j)$ for all $s \in [0\,, 1]$, there results:*

(*i*) $\gamma_j \in \Gamma_j(\omega)$,

(*ii*) $\max_{s\in[0\,,1]} \varphi_\omega(\gamma_j(s)) \leq c_\infty + \hat{\varepsilon}$,

(*iii*) *if* $\gamma_j(s) \notin B_{\rho_0}(\mathcal{A}_j(\omega))$ *then* $\varphi_\omega(\gamma_j(s)) \leq c_\infty - h_{\rho_0}/2$.

In particular $\Gamma_j(\omega) \neq \emptyset$ for all $\omega \in (0\,, \hat{\omega})$ and $j \geq j(\omega)$, and we can define the corresponding minimax values

$$c_j(\omega) = \inf_{\gamma\in\Gamma_j(\omega)} \max_{s\in[0\,,1]} \varphi_\omega(\gamma(s)).$$

These mountain pass levels are close to the mountain pass level c_∞ in the sense explained by the following Lemma.

Lemma 2.9. *For all $\omega \in (0\,, \hat{\omega})$ there exists $\hat{\jmath}(\omega) \geq j(\omega)$ such that $|c_j(\omega) - c_\infty| \leq \hat{\varepsilon}$ for all $j \geq \hat{\jmath}(\omega)$.*

Now we can prove that for $\omega > 0$ sufficiently small, the functional φ_ω admits infinitely many critical points. More precisely we show that:

Lemma 2.10. *If $\omega \in (0\,, \hat{\omega})$ then $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega \neq \emptyset$ for every $j \geq \hat{\jmath}(\omega)$.*

Proof. Arguing by contradiction, suppose that there exist $\omega \in (0\,, \hat{\omega})$ and $j \geq \hat{\jmath}(\omega)$ such that $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega = \emptyset$. Let $V_{j\omega}\colon X \to X$ be the pseudogradient vector field given by Lemma 2.7 and let $\eta \in C(\mathbb{R} \times X, X)$ be the associated flow, given by the solution of the Cauchy problem

$$\begin{cases} \dfrac{d\eta(t\,, u)}{dt} = -V_{j\omega}(\eta(t\,, u)) \\ \eta(0\,, u) = u\,. \end{cases}$$

Note that η is well defined and continuous in $\mathbb{R}\times X$ because the field $V_{j\omega}$ is a bounded, locally Lipschitz continuous function. Moreover, by the properties of $V_{j\omega}$ stated in Lemma 2.7, for a fixed $\tau > 0$ large enough, the function $\eta_{j\omega}(u) = \eta(\tau\,, u)$ satisfies:

(*i*) $\eta_{j\omega}(u) = u$ for all $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,

(*ii*) $\varphi_\omega(\eta_{j\omega}(u)) \leq \varphi_\omega(u)$ for all $u \in X$,

(*iii*) $\varphi_\omega(\eta_{j\omega}(u)) \leq \varphi_\omega(u) - \overline{\mu}\rho_0$ if $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \overline{h}/2\}$,

(*iv*) $\|\eta_{j\omega}(u)\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \leq \varepsilon$ if $\|u\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \leq \varepsilon$.

Let now $\hat{\gamma}_j(s) = \eta_{j\omega}(\gamma_j(s))$ for $s \in [0\,, 1]$, where $\gamma_j \in \Gamma_j(\omega)$ is defined as in Lemma 2.8. By the above listed properties (*i*) and (*iv*) of $\eta_{j\omega}$, the class $\Gamma_j(\omega)$ is invariant under the deformation $\eta_{j\omega}$ and then $\hat{\gamma}_j \in \Gamma_j(\omega)$. We claim that $\max_{s\in[0\,,1]} \varphi_\omega(\hat{\gamma}_j(s)) \leq c_j(\omega) - \hat{\varepsilon}$

and therefore we get a contradiction with the definition of $c_j(\omega)$. Indeed, if $\gamma_j(s) \notin$ $\notin B_{\rho_0}(\mathcal{A}_j(\omega))$, by the property (*ii*) of $\eta_{j\omega}$ and by Lemma 2.8 (*iii*), we have $\varphi_\omega(\hat{\gamma}_j(s)) \leq$ $\leq \varphi_\omega(\gamma_j(s)) \leq c_\infty - h_{\rho_0}/2 \leq c_\infty - 2\hat{\varepsilon}$, since $\hat{\varepsilon} < h_{\rho_0}/4$. On the other hand, if $\gamma_j(s) \in$ $\in B_{\rho_0}(\mathcal{A}_j(\omega))$, by the property (*iii*) of $\eta_{j\omega}$ and by Lemma 2.8 (*ii*), we have $\varphi_\omega(\hat{\gamma}_j(s)) \leq$ $\leq \varphi_\omega(\gamma_j(s)) - \overline{\mu}\rho_0 \leq c_\infty + \hat{\varepsilon} - \overline{\mu}\rho_0 \leq c_\infty - 2\hat{\varepsilon}$, since $\hat{\varepsilon} \leq \overline{\mu}\rho_0/9$. Therefore, by Lemma 2.9, for all $s \in [0\,,1]$ we conclude that $\varphi_\omega(\hat{\gamma}_j(s)) \leq c_\infty - 2\hat{\varepsilon} \leq c_j(\omega) - \hat{\varepsilon}$. □

We remark that by the arbitrariness of $\overline{\alpha} > 0$ and $a \in L^\infty(\mathbb{R}^N)$ with ess inf $a > 0$, the above result shows that the problem (P_a) admits infinitely many solutions whenever a belongs to a dense subset of $\{a \in L^\infty(\mathbb{R}^N) : a \geq 0\}$.

Then Theorem 1.1 follows by the next final Lemma.

Lemma 2.11. *If $\omega \in (0\,,\hat{\omega})$, there exists $\beta_0 > 0$ such that if $\|\beta\|_{L^\infty(\mathbb{R}^N)} \leq \beta_0$ then the problem $(P_{a+\alpha_\omega+\beta})$ admits infinitely many solutions.*

Proof. Given $\beta \in L^\infty(\mathbb{R}^N)$ we denote $\varphi_{\omega\beta}(u) = \varphi_\omega(u) - \int_{\mathbb{R}^N} \beta(x) F(u)\, dx$ and $\mathcal{K}_{\omega\beta} =$ $= \{u \in X \setminus \{0\} : \varphi'_{\omega\beta}(u) = 0\}$. We note that $a + \alpha_\omega + \beta \in \mathcal{F}$ whenever $\|\beta\|_{L^\infty(\mathbb{R}^N)} \leq$ $\leq a_0$.
Letting M be the constant fixed before the definition of $\Gamma_j(\omega)$, there exists $C = C(M) >$ > 0 such that

$$\sup_{\|u\| \leq M} |\varphi_{\omega\beta}(u) - \varphi_\omega(u)| \leq C\|\beta\|_{L^\infty(\mathbb{R}^N)}\,, \tag{2.2}$$

$$\sup_{\|u\| \leq M} \|\varphi'_{\omega\beta}(u) - \varphi'_\omega(u)\| \leq C\|\beta\|_{L^\infty(\mathbb{R}^N)}\,. \tag{2.3}$$

We claim that if $\omega \in (0\,,\hat{\omega})$ and $j \geq \hat{\jmath}(\omega)$ then $\mathcal{K}_{\omega\beta} \cap \mathcal{A}_j(\omega) \neq \emptyset$ whenever $\|\beta\|_{L^\infty} \leq \beta_0$, being $\beta_0 = (1/2)\min\{a_0\,, \hat{\varepsilon}/C\}$ with $\hat{\varepsilon} > 0$ fixed above.
Indeed, arguing by contradiction, assume that $\mathcal{K}_{\omega\beta} \cap \mathcal{A}_j(\omega) = \emptyset$ for some $\omega \in (0\,,\hat{\omega})$ and $j \geq \hat{\jmath}(\omega)$. Then, using (2.2) and (2.3), one can see that

(1) there exists $\nu_j > 0$ such that $\|\varphi'_{\omega\beta}(u)\| \geq \nu_j$ for all $u \in \mathcal{A}_j(\omega) \cap \{\varphi_\omega \leq c_\infty + 2\overline{h}/3\}$.
(2) $\|\varphi'_{\omega\beta}(u)\| \geq \overline{\nu}/2$ for all $u \in (B_{4\rho_0}(\mathcal{A}_j(\omega)) \setminus \mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \overline{h}\}$.

By (1) and (2), since $a + \alpha_\omega + \beta \in \mathcal{F}$, it is possible to show the existence of a pseudogradient vector field $\widetilde{V}_j : X \to X$ satisfying:

(*i*) $\|\widetilde{V}_j(u)\| \leq 1$, $\varphi'_{\omega\beta}(u)\widetilde{V}_j(u) \geq 0$ for all $u \in X$ and $\widetilde{V}_j(u) = 0$ for all $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,
(*ii*) $\varphi'_{\omega\beta}(u)\widetilde{V}_j(u) \geq \mu_j > 0$ if $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \overline{h}/2\}$,
(*iii*) $\varphi'_{\omega\beta}(u)\widetilde{V}_j(u) \geq \overline{\mu}/2$ if $u \in (B_{2\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \overline{h}/2\}$,
(*iv*) $\langle u, \widetilde{V}_j(u)\rangle_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \geq 0$ if $\|u\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \geq \hat{\varepsilon}$.

Considering the flow associated to the field $\widetilde{V}_j$, we obtain the existence of a continuous function $\eta_j : X \to X$ which verifies:

(*i*)′ $\eta_j(u) = u$ for all $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,
(*ii*)′ $\varphi_{\omega\beta}(\eta_j(u)) \leq \varphi_{\omega\beta}(u)$ for all $u \in X$,

$(iii)'$ $\varphi_{\omega\beta}(\eta_j(u)) \le \varphi_{\omega\beta}(u) - \overline{\mu}\rho_0/2$ if $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \le c_\infty + \overline{h}/2\}$,

$(iv)'$ $\|\eta_j(u)\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \le \varepsilon$ if $\|u\|_{\mathbb{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \le \varepsilon$.

Then, considering the path $\widetilde{\gamma}_j(s) = \eta_j(\gamma_\infty(s)(\cdot - x_j))$, $s \in [0, 1]$, by $(i)'$ and $(iv)'$ $\widetilde{\gamma}_j \in \Gamma_j(\omega)$. Then, by (2.2), $(ii)'$ and $(iii)'$, since $\hat{\varepsilon} < (1/8)\min\{h_{\rho_0}, \overline{\mu}\rho_0\}$, using Lemma 2.9, we get $\max_{s\in[0,1]} \varphi_\omega(\widetilde{\gamma}_j(s)) \le \max_{s\in[0,1]} \varphi_{\omega\beta}(\widetilde{\gamma}_j(s)) + \hat{\varepsilon}/2 \le \max\{c_\infty - h_{\rho_0}/2 + \hat{\varepsilon}, c_\infty - -\overline{\mu}\rho_0/2 + 2\hat{\varepsilon}\} < c_j(\omega)$, a contradiction. □

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