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On rates of propagation for Burgers' equation

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Analisi matematica. — *On rates of propagation for Burgers' equation.* Nota di William Alan Day e Giuseppe Saccomandi, presentata (*) dal Corrisp. C. Baiocchi.

Abstract. — We give asymptotic formulae for the propagation of an initial disturbance of the Burgers' equation.

Key words: Burgers' equation; Rate of propagation of a disturbance; Hopf-Cole transformation.

Riassunto. — *Proprietà di trasmissione per l'equazione di Burgers.* Si dimostrano alcune formule asintotiche che sono di interesse nello studio del problema di Cauchy per l'equazione di Burgers. Queste formule permettono di seguire l'evoluzione dei massimi di un dato iniziale con un supporto compatto.

1. Introduction

In an important paper [1] Fichera has defended Fourier's theory of heat propagation against the accusation that it produces the paradox according to which heat propagates with infinite speed. In the course of his argument Fichera drew attention to an observation of J. C. Maxwell's to the effect that, while an infinitesimally small amount of heat propagates at arbitrarily large speed, the bulk propagates in a characteristically diffusion-like way, *i.e.* in such a way that the time taken is proportional to the square of the distance traversed. This idea has been provided with a rigorous measure-theoretic foundation by Day and Saccomandi in two other contexts: for a hyperbolic equation with diffusive damping in [2], and for the parabolic Fokker-Planck equation with periodic coefficients in [3].

In [4, 5], Day has provided a somewhat different justification for Maxwell's observation by considering the initial-value problem for the linear heat equation, with an initial disturbance which is spatially localised, and studying the time taken for the maximum disturbance to reach a point lying at large distance from the support of the initial data. It turns out, once again, that the time taken is asymptotically proportional to the square of the distance from the point to the location of the initial disturbance.

Our present objective is to extend the considerations of [4, 5] to the initial-value problem for Burgers' equation. This equation serves as a prototype for other nonlinear convection-diffusion equations and, from our point of view, it has the great advantage that it can be reduced to the linear heat equation by means of the Hopf-Cole transformation (1)[6]. Although our argument is specific to Burgers' equation and relies heavily upon the Hopf-Cole transformation it yields insight into what may happen for more

(*) Nella seduta del 13 febbraio 1998.

(1) We point out that the Burgers'equation is not the only nonlinear equation linearizable via a suitable transformation of variables. The hyperbolic Thomas equation [6], which arises in the study of ion exchange processes, is also linearizable by a transformation of the Hopf-Cole type.

general nonlinear convection-diffusion equations, for which solutions to initial-value problems can be explicitly computed only in very special cases.

2. Preliminaries and main result

Let $f(x)$ be any function which is non-negative and continuous for $-\infty < x < \infty$ and which has compact support, and let

$$g(x) = \exp\left[\frac{1}{2D}\int_x^{\infty} f(y)\,dy\right], \qquad -\infty < x < \infty, \tag{1}$$

where D is a positive constant. Let

$$v(x,t) = \int_{-\infty}^{\infty} K(x-y,t)g(y)\,dy,$$

where

$$K(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right),$$

so that v is a solution of the initial value problem

$$v_t = Dv_{xx}, \qquad -\infty < x < \infty, \qquad t > 0$$

$$v(x,0) = g(x), \qquad -\infty < x < \infty,$$

for the heat equation. Furthermore, let u be obtained from v via the Hopf-Cole transformation

$$u = -2D\frac{v_x}{v}. \tag{2}$$

Then u is a solution of Burgers' equation

$$u_t + uu_x = Du_{xx}, \qquad -\infty < x < \infty, \qquad t > 0,$$

satisfying an initial condition

$$u(x,0) = f(x), \qquad -\infty < x < \infty,$$

which corresponds to an initial disturbance which is non-negative and is confined to a finite interval of the real line.

Our purpose is to point out certain features of the way in which the disturbance propagates. To this end we consider a point x lying outside the support of f. At such a point u is zero initially; thereafter, it increases to its maximum value and decays again to zero. We denote by $\tau(x)$ the time at which the maximum is attained and, to avoid possible ambiguity, we take this to be the first of such time; thus

$$u(x,\tau(x)) = \max_t u(x,t),$$

$$u(x,t) < u(x,\tau(x)), \qquad 0 \le t < \tau(x).$$

The exact dependence of $\tau(x)$ upon x must be expected to be complicated but none the less it is possible to determine the asymptotic behaviour as $x \to \pm\infty$. Before we state what this is we recall that, in the case of the heat equation,

$$\tau(x) \backsim \frac{x^2}{2D} \quad as \quad |x| \to \infty.$$

For the proof of this and related results see [4, 5]. Here there is no distinction between the asymptotic behaviours as $x \to -\infty$ and as $x \to +\infty$, and the asymptotic relation expresses the characteristic diffusion-like law that the time elapsed is proportional to the square of the distance traversed.

For Burgers' equation the situation is more complicated. The asymptotic behaviour is still diffusion-like in that $\tau(x)$ is ultimately proportional to x^2, but the coefficient of proportionality now depends upon the integral

$$\int_{-\infty}^{\infty} f dx \quad (= Q, say).$$

Moreover, the behaviour as $x \to -\infty$, is different from the behaviour as $x \to +\infty$.

In order to state our conclusions we introduce a function

$$\Theta(\sigma) = \frac{\sigma^2 - \frac{1}{2}}{\int_\sigma^\infty (\rho^2 - \sigma^2)\exp(-\rho^2)d\rho}\,, \qquad \sigma \geq 0. \tag{3}$$

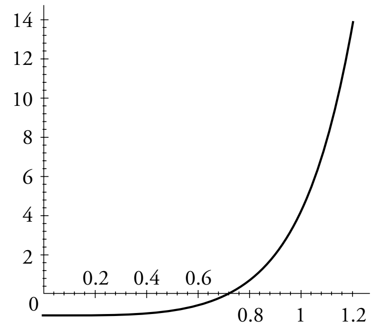

Fig. 1. – Plot of $\Theta(\sigma)$ versus σ.

This is strictly increasing and has the properties:

$$\Theta(0) = -\frac{2}{\sqrt{\pi}}\,, \quad \Theta\left(\frac{1}{\sqrt{2}}\right) = 0\,,$$
$$\Theta(\sigma) \to \infty \quad as \quad \sigma \to \infty.$$

We also define numbers $\sigma^+(Q, D)$ and $\sigma^-(Q, D)$ to be the unique roots of the equations

$$\Theta(\sigma) = \frac{\exp\left(\frac{Q}{2D}\right) - 1}{\sqrt{\pi}},$$

$$\Theta(\sigma) = -\frac{1 - \exp\left(-\frac{Q}{2D}\right)}{\sqrt{\pi}},$$

respectively and we define the number $\xi(\cong 0.59392)$ by requiring that

$$\Theta(\xi) = -\frac{1}{\sqrt{\pi}}.$$

Theorem 1. *Let*

$$T^{\pm}(Q, D) = \frac{1}{4D[\sigma^{\pm}(Q, D)]^2}.$$

Then

$$\tau(x) \sim T^{\pm}(Q, D)x^2 \quad \textit{as} \quad x \to \pm\infty.$$

Moreover

$$T^{+}(Q, D) < \frac{1}{2D} < T^{-}(Q, D). \tag{4}$$

As $Q \to 0$,

$$T^{\pm}(Q, D) \to \frac{1}{2D}, \tag{5}$$

and as $Q \to \infty$,

$$T^{+}(Q, D) \to 0, \tag{6}$$

$$T^{-}(Q, D) \to \frac{1}{4D\xi^2} \backsimeq \frac{0.70874}{D}. \tag{7}$$

According to the inequalities (4) propagation in the positive x-direction proceeds more rapidly than for the heat equation but propagation in the negative x-direction proceeds less rapidly. The relation (5) says that weak disturbances propagate as for the heat equation, as might be expected. The relations (6) and (7) tell us what happens for strong disturbances; in particular (6) says that for sufficiently strong disturbances the time taken to propagate to the place x, with $x > 0$, is arbitrarily small.

3. Proof of the asymptotic relations

The key to the proof is to derive appropriate estimates on v and v_x and, hence, on u, which show that, when x is large and positive, $|x|u/2D$ is approximately equal to

$$F(\sigma) = \frac{\sigma \exp(-\sigma^2)}{\frac{\sqrt{\pi}}{\exp(Q/2D)-1} + \int_\sigma^\infty \exp(-\rho^2)\,d\rho},$$

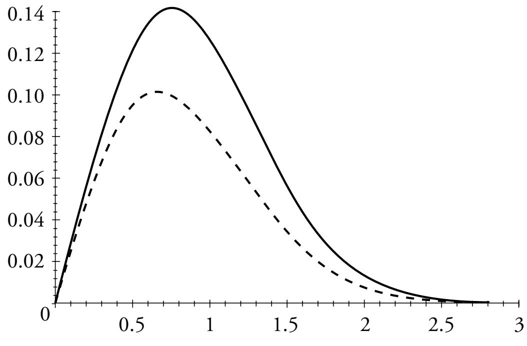

Fig. 2. – Plot of $F(\sigma)$ (line) and $G(\sigma)$ (dash) versus σ for $Q/D = 1$.

and, when x is large and negative, $|x|u/2D$ is approximately equal to

$$G(\sigma) = \frac{\sigma \exp(-\sigma^2)}{\frac{\sqrt{\pi}}{1-\exp(-Q/2D)} - \int_\sigma^\infty \exp(-\rho^2)\,d\rho}\,,$$

where σ (≥ 0) is the similarity variable $|x|/\sqrt{4Dt}$.

LEMMA 2. *Let $a > 0$ be such that the support of f is contained within the interval $[-a\,,\,a]$. Then for $x > a$*

$$F_1(\sigma\,,\,x) \leq \frac{1}{2D}|x|u(x\,,\,t) \leq F_2(\sigma\,,\,x)\,, \tag{8}$$

and for $x < -a$,

$$G_1(\sigma\,,\,x) \leq \frac{1}{2D}|x|u(x\,,\,t) \leq G_2(\sigma\,,\,x)\,, \tag{9}$$

where

$$F_1(\sigma\,,\,x) = \frac{\sigma \exp\left[-(1+a/x)^2\sigma^2\right]}{\frac{\sqrt{\pi}}{\exp(Q/2D)-1} + \int_{(1-a/x)\sigma}^\infty \exp(-\rho^2)\,d\rho}\,,$$

$$F_2(\sigma\,,\,x) = \frac{\sigma \exp\left[-(1-a/x)^2\sigma^2\right]}{\frac{\sqrt{\pi}}{\exp(Q/2D)-1} + \int_{(1+a/x)\sigma}^\infty \exp(-\rho^2)\,d\rho}\,,$$

$$G_1(\sigma\,,\,x) = \frac{\sigma \exp\left[-(1-a/x)^2\sigma^2\right]}{\frac{\sqrt{\pi}}{1-\exp(-Q/2D)} - \int_{(1-a/x)\sigma}^\infty \exp(-\rho^2)\,d\rho}\,,$$

$$G_2(\sigma\,,\,x) = \frac{\sigma \exp\left[-(1+a/x)^2\sigma^2\right]}{\frac{\sqrt{\pi}}{1-\exp(-Q/2D)} - \int_{(1+a/x)\sigma}^\infty \exp(-\rho^2)\,d\rho}.$$

PROOF. We prove only the inequality $F_1 \leq |x|u/2D$; the three remaining inequalities can be established by making appropriate minor modifications to the argument. It follows from equation (1) that

$$g(x) = \exp(Q/2D)\,, \quad x < -a\,,$$

$$1 \le g(x) \le \exp(Q/2D)\,, \quad -a \le x \le a\,,$$

$$g(x) = 1\,, \quad x > a.$$

Hence

$$v(x\,,\,t) \le \exp(Q/2D) \int_{-\infty}^{a} K(x-y\,,\,t)\,dy + \int_{a}^{\infty} K(x-y\,,\,t)\,dy\,,$$

and in view of the fact that

$$\int_{-\infty}^{\infty} K(x-y\,,\,t)\,dy = 1\,,$$

we can rewrite this as

$$v(x\,,\,t) \le 1 + \big(\exp(Q/2D) - 1\big) \int_{-\infty}^{a} K(x-y\,,\,t)\,dy.$$

On making the change of variable $y = x - \sqrt{4Dt}\rho$ in the integral and supposing that $x > a$ we arrive the inequality

$$v(x\,,\,t) \le 1 + \frac{\exp(Q/2D) - 1}{\sqrt{\pi}} \int_{(1-a/x)\sigma}^{\infty} \exp(-\rho^2)\,d\rho\,, \qquad x > a. \tag{10}$$

In order to estimate v_x we argue that

$$v_x(x\,,\,t) = \exp(Q/2D) \int_{-\infty}^{-a} K_x(x-y\,,\,t)\,dy +$$

$$+ \int_{-a}^{a} K_x(x-y\,,\,t)\,g(y)\,dy + \int_{a}^{\infty} K_x(x-y\,,\,t)\,dy =$$

$$= -\exp(Q/2D)K(x+a\,,\,t) + \int_{-a}^{a} K_x(x-y\,,\,t)\,g(y)\,dy + K(x-a\,,\,t).$$

If $x > a$ and $-a \le y \le a$ then

$$K_x(x-y\,,\,t) = -\frac{(x-y)}{2Dt} K(x-y\,,\,t) \le 0\,,$$

and so

$$\int_{-a}^{a} K_x(x-y\,,\,t)\,g(y)\,dy \le \int_{-a}^{a} K_x(x-y\,,\,t)\,dy = K(x+a\,,\,t) - K(x-a\,,\,t)$$

and, therefore,

$$v_x(x\,,\,t) \le -\big(\exp(Q/2D) - 1\big)\,K(x+a\,,\,t)\,,$$

that is to say

$$-v_x(x\,,\,t) \ge \frac{\big(\exp(Q/2D) - 1\big)\,\sigma \exp\big(-(1+a/x)^2\sigma^2\big)}{\sqrt{\pi}|x|}\,, \quad x > a. \tag{11}$$

The desired inequality now follows from equation (2) and the estimates (10) and (11). $\square$

We are now in position to complete the proof of the asymptotic relation

$$\tau(x) \sim T^{+}(Q, D)x^{2} \quad as \quad x \to \infty.$$

For any fixed large positive x, the graphs of $F_1(\sigma, x)$, $F_2(\sigma, x)$, and $F(\sigma)$, versus σ take the value 0 when $\sigma = 0$. As σ increases, each increases monotonely to its maximum value and thereafter decreases monotonely to 0 as $\sigma \to \infty$. Moreover $F_{1,2}(\sigma, x) \to F(\sigma)$ when $x \to \infty$. On noting that the integral on the right-hand side of equation (3) is equal to

$$\left(\frac{1}{2} - \sigma^2\right) \int_{\sigma}^{\infty} \exp(-\rho^2)\, d\rho + \frac{1}{2}\sigma \exp(-\sigma^2),$$

we see that, in fact, $F(\sigma)$ attains its maximum at $\sigma = \sigma^{+}(Q, D)$. Moreover, there are unique numbers $\phi(x)$ and $\psi(x)$ such that $0 < \phi < \psi$ and

$$F_2(\phi, x) = F_2(\psi, x) = \max_{\sigma} F_1(\sigma, x).$$

These numbers have the property that $\phi(x) \to \sigma^{+}$ and $\psi(x) \to \sigma^{+}$ as $x \to \infty$ and, in view of the inequalities (8), it must be that

$$\phi(x) \le \frac{|x|}{\sqrt{4D\tau(x)}} \le \psi(x).$$

Hence

$$\tau(x) \sim \frac{x^2}{4D[\sigma^{+}(Q, D)]^2} \quad as \quad x \to \infty$$

and this is the required relation.

The asymptotic relation

$$\tau(x) \sim T^{-}(Q, D)x^{2} \quad as \quad x \to -\infty$$

follows from the inequalities (9) by a similar argument on noting that $G(\sigma)$ attains its unique maximum at $\sigma^{-}(Q, D)$.

The remaining assertions of the theorem follows immediately from the definition of $\sigma^{\pm}(Q, D)$ and the properties of the function $\Theta(\sigma)$.

This paper is dedicated to the memory of Professor Gaetano Fichera.

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