Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

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A remark on a Theorem of J. G. Thompson

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **9** (1998), n.3, p. 145–148.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1998_9_9_3_145_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1998.

Rend. Mat. Acc. Lincei
s. 9, v. 9:145-148 (1998)

Teoria dei gruppi. — *A remark on a Theorem of J. G. Thompson.* Nota di Bertram Huppert, presentata (*) dal Socio G. Zappa.

Abstract. — An important theorem by J. G. Thompson says that a finite group G is p-nilpotent if the prime p divides all degrees (larger than 1) of irreducible characters of G. Unlike many other cases, this theorem does not allow a similar statement for conjugacy classes. For we construct solvable groups of arbitrary p-lenght, in which the lenght of any conjugacy class of non central elements is divisible by p.

Key words: Lenght of conjugacy classes; p-lenght.

Riassunto. — *Osservazione su un Teorema di J. G. Thompson.* Un importante teorema di J. G. Thompson afferma che un gruppo finito G è p-nilpotente se il primo p divide tutti i gradi (maggiori di 1) dei caratteri irriducibili di G. A differenza di vari altri casi, questo teorema non dà luogo ad una affermazione simile per le classi di coniugio. Infatti noi costruiamo un gruppo risolubile di p-lunghezza arbitraria in cui la lunghezza di una classe di coniugio di elementi non centrali è divisibile per p.

In 1970 J. G. Thompson [1] proved the following theorem:

Theorem. *Let G be a finite group and p a prime. Suppose that for every complex irreducible character χ of G of degree larger than* 1, $\chi(1)$ *is divisible by p. Then G is p-nilpotent, which means that G has a normal p-complement.*

In recent years several similarities between theorems about character degrees and lengths of conjugacy classes have been observed. Hence it seems natural to consider groups with the following property:

(p) Let p be a prime. Suppose that for every noncentral element g of G the length

$$|g^G| = |G : C_G(g)|$$

of the conjugacy class g^G of g is divisible by p.

In particular, it seems natural to consider the p-length of groups with property (p). The main result is negative: There are solvable groups with property (p) of arbitrary large p-length.

In Example 1 we shall construct groups with property (p) of p-length 2. By a wreath product construction we shall then obtain examples of arbitrary p-length.

Lemma. *Suppose that N is a normal p-subgroup of G such that*

$$C_G(N) = Z(N) = Z(G)\,.$$

Then G has property (p).

(*) Nella seduta del 24 aprile 1998.

Proof. Suppose $g \in G$ and $p \nmid |g^G|$. Then $C_G(g)$ contains some Sylow-p-subgroup of G, so $N \leq C_G(g)$. This proves

$$g \in C_G(N) = Z(G) ,$$

hence $|g^G| = 1$.

Example 1.

a) The group $GL(2, 3)$ has 2-length 2. It has a normal subgroup N of order 8 such that

$$C_G(N) = Z(N) = Z(G) .$$

Hence $GL(2, 3)$ has property (2) by the Lemma.

b) For $p > 2$ we construct similar groups in the following way: Let $K = GF(p)$ and $L = GF(p^{2p})$. We take $a \in GF(p^2)^\times$ such that $a^p = -a$, hence $a^{p^2} = a$ and $a^{p^p} = -a$. The set $P = L \times K$ becomes a p-group of order p^{2p+1} by

$$(x_1 , y_1)(x_2 , y_2) = \left(x_1 + x_2 , y_1 + y_2 + trax_1 x_2^{p^p}\right) ,$$

where tr is the trace of L over K. One easily checks that this is a group operation on P and

$$P' = Z(P) = \{(0 , y)| y \in K\}$$

has order p. So P is an extraspecial p-group.

Take $c \in L^\times$ such that $ord\ c = p^p + 1$. Then by

$$(x , y)^\alpha = (cx , y) \quad \text{and} \quad (x , y)^\beta = (x^{p^2} , y)$$

we define automorphisms α and β of P. (Observe that $a^{p^2} = a$). Then

$$\alpha^{p^p+1} = \beta^p = 1 , \qquad \beta^{-1}\alpha\beta = \alpha^{p^2} .$$

We form the semidirect product

$$H = P\langle\alpha , \beta\rangle$$

of order $p^{2p+2}(p^p + 1)$. Then H obviously has p-length 2. As

$$C_H(P) = Z(P) = Z(H) ,$$

so by the Lemma H has property (p).

Theorem. *There do exist solvable groups with property* (p) *of arbitrary large p-length.*

Proof. Let already a solvable group G_k be constructed such that G_k has property (p) and $\ell_p(G_k) \geq k$. Take the group H as in Example 1 such that H has property (p) and $\ell_p(H) = 2$. Let Q be a p-complement of H and represent H faithfully as a transitive permutation group on the $|H : Q| = p^{2p+2} \doteqdot m$ cosets of Q in H. Any Sylow-p-subgroup of H is then regularly represented. Form the wreath product

$$G_{k+1} = G_k \wr H = BH ,$$

where

$$B = T_1 \times \ldots \times T_m$$

is the basic subgroup, $T_i \simeq G_k$ and H permutes the T_i.

(1) G_{k+1} has property (p):

Suppose at first that $g \notin B$. As $G_{k+1}/B \simeq H$ has property (p), so *either* p divides

$$|(gB)^{G_{k+1}/B}| ,$$

which divides $|g^{G_{k+1}}|$ *or*

$$gB \in Z(G_{k+1}/B) .$$

In the second case $g = g_0 h$, where $g_0 \in B$, $h \in Z(H)$. Suppose $p \nmid |g^{G_{k+1}}|$. Then some Sylow-p-subgroup $P = P_1 \times \ldots \times P_m$ of B is in $C_{G_{k+1}}(g)$, where $E \neq P_i \in Syl_p T_i$. Therefore

$$P_i^{h^{-1}} = P_i^{g_0} \leq T_i .$$

Hence the permutation induced by h on the T_i is trivial. But if $1 \neq h \in Z(H)$ then h has no fixed point on $\{1, \ldots, m\}$. Hence $h = 1$ and $g = g_0 \in B$, a contradiction.

Now suppose $g \in B$. If $g \notin Z(B)$, then as G_k and B have property (p), we obtain $p|\ |g^B|$. But $B \lhd G_{k+1}$, so $|g^B|$ divides $|g^{G_{k+1}}|$.

There remains the case that

$$g = g_1 \ldots g_m \in Z(B) ,$$

where $g_i \in Z(T_i)$. Then $C_G(g)$ permutes only those T_i with equal factors g_i. If not all the g_i are equal, none of the transitive Sylow-p-subgroup of H lies in $C_H(g)$, so p divides $|g^{G_{k+1}}|$. If $g = g_1 \ldots g_m$ and $g_1 = \ldots = g_m$, then $g \in Z(G_{k+1})$.

(2) We claim that $\ell_p(G_{k+1}) \geq k + 1$. We have

$$\ell_p(B) = \ell_p(G_k) \geq k .$$

Let $P_{k-1}(B)$ be the uniquely determined maximal normal subgroup of B of p-length $k-1$. As $\ell_p(B) \geq k$, so

$$P_{k-1}(B) = P_{k-1}(T_1) \times \ldots \times P_{k-1}(T_m) < B .$$

Now

$$\Big[P_{k-1}(G_{k+1}) , B\Big] \leq P_{k-1}(G_{k+1}) \cap B = P_{k-1}(B) .$$

Hence $P_{k-1}(G_{k+1})$ centralizes $B/P_{k-1}(B)$, hence does not permute the T_i. So

$$P_{k-1}(G_{k+1}) = P_{k-1}(B) \leq B .$$

But

$$\ell_p\Big(G_{k+1}/P_{k-1}(G_{k+1})\Big) \geq \ell_p(G_{k+1}/B) = 2$$

and therefore

$$\ell_p(G_{k+1}) \geq k + 1\,.$$

(If we start the construction with G_2 the group of Example 1, then $\ell_p(G_k) = k$).

REMARK. There are also insolvable groups with property (p).

a) The simplest case we obtain by extending a nonabelian group P of exponent p and order p^3 by $SL(2\,,p)$, so that $P' = Z(G)$. If $p > 3$, then G is not solvable.

b) For $p \geq 3$ similar examples can be obtained by extending an extraspecial group P of exponent p and order $p^{2m+1} (m > 1)$ by the symplectic group $Sp(2m\,,p)$.

c) For $p = 2$ we extend each of the extraspecial 2-groups P of order 2^{2m+1} $(m > 1)$ by the corresponding orthogonal group $O(2m\,,2)$. (Here both of the orthogonal groups have to be used, according to the choice of P).

This paper is dedicated to Karl Heinrich Hofmann on his 65 Birthday.

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Pervenuta il 24 aprile 1998.

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