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Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method

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Equazioni a derivate parziali. — *Monotonicity and symmetry of solutions of p-Laplace equations,* $1 < p < 2$*, via the moving plane method.* Nota di LUCIO DAMASCELLI e FILOMENA PACELLA, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — We present some monotonicity and symmetry results for positive solutions of the equation $-div\left(|Du|^{p-2}\, Du\right) = f(u)$ satisfying an homogeneous Dirichlet boundary condition in a bounded domain Ω. We assume $1 < p < 2$ and f locally Lipschitz continuous and we do not require any hypothesis on the critical set of the solution. In particular we get that if Ω is a ball then the solutions are radially symmetric and strictly radially decreasing.

KEY WORDS: *p*-Laplace equations; Monotonicity and symmetry of positive solutions; Moving plane method.

RIASSUNTO. — *Monotonia e simmetria di soluzioni di equazioni ellittiche quasilineari.* Dimostriamo alcuni risultati di monotonia e simmetria per soluzioni positive dell'equazione $-div\left(|Du|^{p-2}\, Du\right) = f(u)$ con condizioni di Dirichlet omogenee sul bordo in un dominio limitato Ω. Supponiamo che $1 < p < 2$ e che f sia localmente Lipschitziana e non facciamo alcuna ipotesi sui punti critici della soluzione. In particolare otteniamo che se Ω è una palla le soluzioni sono radiali e radialmente strettamente decrescenti.

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Δ_p denotes the *p*-laplacian operator $\Delta_p u = div(|Du|^{p-2})\, Du$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and f is a locally Lipschitz continuous function.

The aim of this *Note* is to present some monotonicity and symmetry results contained in [4] for $1 < p < 2$.

In the case $p = 2$ several results have been obtained starting with the famous paper [6] by Gidas, Ni and Nirenberg where, using the so called «moving plane methods» it is proved, among other things, that if Ω is a ball and $p = 2$, solutions of (1.1) are radially symmetric and strictly radially decreasing.

Very little is known about the monotonicity and symmetry of solutions of (1.1) when $p \neq 2$. In this case the solutions can only be considered in a weak sense since, generally, they belong to the space $C^{1,\alpha}(\Omega)$ (see [5, 9]).

The main difficulty with problem (1.1), for $p \neq 2$, is that the *p*-laplacian operator is

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degenerate in the critical points of the solutions, so that comparison principles (which could substitute the maximum principles in order to use the moving plane method when the operator is not linear) are not available in the same form as for $p = 2$. Actually counterexamples both to the validity of comparison principles and to the symmetry results are available (see [7, 2]) for any p with different degrees of regularity of f.

Before stating our main theorems let us recall that some partial results about (1.1) were previously obtained in [1, 3, 7, 8]. While we were completing this paper F. Brock told us that in [2] he gets the symmetry result in the ball in the case $1 < p < 2$ or $p > 2$ but f monotone. For other symmetric domains he shows that solutions are «locally symmetric» in a suitable sense defined in [2]. His method does not use comparison principles but the so called «continuous Steiner symmetrization».

A first step towards extending the moving plane method to solutions of problems involving the p-laplacian operator has been done in [3]. In this paper the author mainly proves some weak and strong comparison principles for solutions of differential inequalities involving the p-laplacian. Using these principles he adapts the moving plane method to solutions of (1.1) getting some monotonicity and symmetry results in the case $1 < p < 2$. Although the comparison principles of [3] are quite powerful for $1 < p < 2$, the symmetry result is not complete and relies on the assumption that the set of the critical points of u does not disconnect the caps which are constructed by the moving plane method.

In [4] we use the results of [3] to get monotonicity and symmetry for solutions u of (1.1) in smooth domains in the case $1 < p < 2$ without extra-assumptions on u.

To state our results we need some notations.

Let ν be a direction in $\mathbb{R}^N$, *i.e.* $\nu \in \mathbb{R}^N$ and $|\nu| = 1$. For a real number λ we define

$$T^\nu_\lambda = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\} \tag{1.2}$$

$$\Omega^\nu_\lambda = \{x \in \Omega : x \cdot \nu < \lambda\} \tag{1.3}$$

$$x^\nu_\lambda = R^\nu_\lambda(x) = x + 2\,(\lambda - x \cdot \nu)\,\nu\ , \qquad x \in \mathbb{R}^N \tag{1.4}$$

(*i.e.* R^ν_λ is the reflection through the hyperplane T^ν_λ)

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu\,. \tag{1.5}$$

If $\lambda > a(\nu)$ then Ω^ν_λ is nonempty, thus we set

$$\left(\Omega^\nu_\lambda\right)' = R^\nu_\lambda(\Omega^\nu_\lambda)\,. \tag{1.6}$$

Following [6] we observe that if Ω is smooth and $\lambda > a(\nu)$, with $\lambda - a(\nu)$ small, then the reflected cap $\left(\Omega^\nu_\lambda\right)'$ is contained in Ω and will remain in it, at least until one of the following occurs:

(*i*) $\left(\Omega^\nu_\lambda\right)'$ becomes internally tangent to $\partial\Omega$ at some point not on T^ν_λ;
(*ii*) T^ν_λ is orthogonal to $\partial\Omega$ at some point.

Let $\Lambda_1(\nu)$ be the set of those $\lambda > a(\nu)$ such that for each $\mu \in (a(\nu)\,,\,\lambda)$ none of the conditions (*i*) and (*ii*) holds and define

$$\lambda_1(\nu) = \sup \Lambda_1(\nu)\,. \tag{1.7}$$

The main result of the paper is the following.

THEOREM 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and $u \in C^1(\overline{\Omega})$ a weak solution of* (1.1) *with* $1 < p < 2$. *For any direction ν and for λ in the interval $(a(\nu)\,,\,\lambda_1(\nu)]$ we have*

$$u(x) \leq u(x_\lambda^\nu) \qquad \forall\ x \in \Omega_\lambda^\nu\,. \tag{1.8}$$

Moreover

$$\frac{\partial u}{\partial \nu}(x) > 0 \qquad \forall\ x \in \Omega^\nu_{\lambda_1(\nu)} \setminus Z \tag{1.9}$$

where $Z = \{x \in \Omega : Du(x) = 0\}$.

Easy consequences of Theorem 1.1 are the following.

COROLLARY 1.1. *If, for a direction ν, the domain Ω is symmetric with respect to the hyperplane $T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$ and $\lambda_1(\nu) = \lambda_1(-\nu) = 0$, then u is symmetric,* i.e. $u(x) = u(x_0^\nu)$ *for any $x \in \Omega$, and decreasing in the ν direction in Ω_0^ν. Moreover $\partial u/\partial \nu > 0$ in $\Omega_0^\nu \setminus Z$.*

Note that the class of the domains to which Corollary 1.1 applies is that of all symmetric sets Ω with smooth boundary such that the hyperplanes T_λ^ν are never orthogonal to $\partial\Omega$ for $\lambda \neq 0$.

COROLLARY 1.2. *Suppose that Ω is the ball $B_R(0)$ in $\mathbb{R}^N$ with center at the origin and radius R. Then u is radially symmetric and $\partial u/\partial r < 0$ for $0 < r < R$.*

Note that the previous theorem implies also a regularity result since from $Du \neq 0$ in $B_R(0)\setminus\{0\}$, by standard regularity results, we deduce that u belongs to $C^2\left(B_R(0) \setminus \{0\}\right)$.

2. SKETCH OF THE PROOF OF THEOREM 1.1

From now on p will belong to the interval $(1\,,2)$ and u will be a $C^1(\overline{\Omega})$ solution of (1.1) with Ω having a smooth boundary. For any direction ν let $a(\nu)$, Ω_λ^ν, $\lambda_1(\nu)$ be as defined in Section 1. If $a(\nu) < \lambda \leq \lambda_1(\nu)$ and $x \in \Omega_\lambda^\nu$ we set

$$u_\lambda^\nu(x) = u(x_\lambda^\nu) \tag{2.1}$$

where x_λ^ν is as in (1.4),

$$Z_\lambda^\nu = Z_\lambda^\nu(u) = \{x \in \Omega_\lambda^\nu : Du(x) = Du_\lambda^\nu(x) = 0\} \tag{2.2}$$

$$Z = Z(u) = \{x \in \Omega : Du(x) = 0\}\,. \tag{2.3}$$

Finally we define

$$\Lambda_0(\nu) = \left\{\lambda \in (a(\nu)\,,\lambda_1(\nu)] : u \leq u_\mu^\nu \text{ in } \Omega_\mu^\nu \text{ for any } \mu \in (a(\nu)\,,\lambda]\right\}.$$

If $\Lambda_0(\nu) \neq \emptyset$ we set

$$\lambda_0(\nu) = \sup \Lambda_0(\nu)\,. \tag{2.4}$$

A preliminary result which is crucial to prove Theorem 1.1 is the following Proposition which gives a useful information on how the set Z of the critical points of a solution u of (1.1) can intersect the cap $\Omega^\nu_{\lambda_0(\nu)}$.

PROPOSITION 2.1. *Suppose that $u \in C^1(\overline{\Omega})$ is a weak solution of (1.1), with $1 < p < 2$. For any direction ν the cap $\Omega^\nu_{\lambda_0(\nu)}$ does not contain any subset Γ of Z on which u is constant and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of $T^\nu_{\lambda_0(\nu)}$ (relatively to the induced topology).*

The proof of the previous proposition relies on a careful use of the Hopf's lemma [4, Proposition 3.1].

To prove Theorem 1.1 we show that $\Lambda_0(\nu) \neq \emptyset$ and $\lambda_0(\nu) = \lambda_1(\nu)$. The last thing will be proved by showing that if $\lambda_0(\nu) < \lambda_1(\nu)$ then there exists a «small» set Γ of critical points of u in the cap $\Omega^\nu_{\lambda_0(\nu)}$ on which u is constant and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of $T^\nu_{\lambda_0(\nu)}$. Of course this would be in contradiction with the statement of Proposition 2.1.

Let us remark that the existence of the set Γ (assuming $\lambda_0(\nu)$ smaller than $\lambda_1(\nu)$) will be deduced by applying the moving plane method not only with a fixed direction, but with all the directions in a neighborhood of a direction ν_0. This procedure of moving simultaneously hyperplanes orthogonal to different directions seems to be new and could probably be used also in other problems.

We start stating a result which is a different formulation and an extension of Theorem 1.5 in [3]. It essentially asserts that, once we start the moving plane procedure we must necessarily reach the position $T^\nu_{\lambda_1(\nu)}$ unless the set Z of the critical points of u creates a connected component C of the set where $Du \neq 0$ which is symmetric with respect to the hyperplane $T^\nu_{\lambda_0(\nu)}$ and where u coincides with the symmetric function $u^\nu_{\lambda_0(\nu)}$.

THEOREM 2.1. *For any direction ν we have that $\Lambda_0(\nu) \neq \emptyset$ and, if $\lambda_0(\nu) < \lambda_1(\nu)$ then there exists at least one connected component C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ such that $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν.*

For any such component C^ν we also get

$$Du(x) \neq 0 \qquad \forall\ x \in C^\nu\ , \tag{3.6}$$

$$Du(x) = 0 \qquad \forall\ x \in \partial C^\nu \setminus \left(T^\nu_{\lambda_0(\nu)} \cup \partial\Omega\right)\,. \tag{3.7}$$

Moreover for any λ with $a(\nu) < \lambda < \lambda_0(\nu)$ we have

$$u < u^\nu_\lambda \quad \text{in } \Omega^\nu_\lambda \setminus Z^\nu_\lambda \tag{3.8}$$

and finally

$$\frac{\partial u}{\partial \nu}(x) > 0 \qquad \forall\ x \in \Omega^\nu_{\lambda_0(\nu)} \setminus Z\,. \tag{3.9}$$

Now, for any direction ν let $\mathcal{F}_\nu$ be the collection of the connected components C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ such that $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν, $Du \neq 0$ in C^ν, $Du = 0$ on $\partial C^\nu \setminus \left(T^\nu_{\lambda_0(\nu)} \cup \partial\Omega\right)$.

If $\lambda_0(\nu) < \lambda_1(\nu)$ we deduce from Theorem 2.1 that $\mathcal{F}_\nu \neq \emptyset$. If this is the case and $C^\nu \in \mathcal{F}_\nu$ we also have $u \equiv u^\nu_{\lambda_0(\nu)}$ in $\overline{C}^\nu$, so that $\left(\overline{C}^\nu \cap \partial\Omega\right) \setminus T^\nu_{\lambda_0(\nu)} = \emptyset$ since $u = 0$ on $\partial\Omega$, while $u^\nu_{\lambda_0(\nu)} > 0$ in $\overline{C}^\nu \setminus T^\nu_{\lambda_0(\nu)}$, because by the definition of $\lambda_1(\nu)$ (see Section 1) we have that $\left(\overline{\Omega^\nu_{\lambda_0(\nu)}} \setminus T^\nu_{\lambda_0(\nu)}\right)' \subset \Omega$.

Hence there are two alternatives: either $Du(x) = 0$ for all $x \in \partial C^\nu$, in which case we define $\widetilde{C}^\nu = C^\nu$, or there are points $x \in \partial C^\nu \cap T^\nu_{\lambda_0(\nu)}$ such that $Du(x) \neq 0$. In this latter case we define $\widetilde{C}^\nu = C^\nu \cup C^\nu_1 \cup C^\nu_2$ where $C^\nu_1 = R^\nu_{\lambda_0(\nu)}(C^\nu)$ and $C^\nu_2 = \left\{x \in \partial C^\nu \cap T^\nu_{\lambda_0(\nu)} : Du(x) \neq 0\right\}$. It is easy to check that $\widetilde{C}^\nu$ is open and connected, with $Du \neq 0$ in $\widetilde{C}^\nu$, $Du = 0$ on $\partial\widetilde{C}^\nu$.

Let us finally denote by $\widetilde{\mathcal{F}}_\nu$ the collection $\left\{\widetilde{C}^\nu : C^\nu \in \mathcal{F}_\nu\right\}$ and by $I_\delta(\nu)$ the set

$$I_\delta(\nu) = \left\{\mu \in \mathbb{R}^N : \ |\mu| = 1 \, , |\mu - \nu| < \delta\right\} .$$

As already observed Theorem 1.1 will be proved if we show that $\lambda_0(\nu) = \lambda_1(\nu)$ for any direction ν. Therefore suppose that ν_0 is a direction such that $\lambda_0(\nu) < \lambda_1(\nu)$. Then from Theorem 2.1 follows that $\mathcal{F}_{\nu_0} \neq \emptyset$ and, since $\mathbb{R}^N$ is a separable metric space and every component is open, $\mathcal{F}_{\nu_0}$ contains at most countably many components of $\Omega^{\nu_0}_{\lambda_0(\nu_0)} \setminus Z^{\nu_0}_{\lambda_0(\nu_0)}$, so $\mathcal{F}_{\nu_0} = \{C^{\nu_0}_i \ , \ i \in I \subseteq \mathbb{N}\}$.

The remaining part of the proof can be summarized in the following three steps whose proofs are omitted (see [4]).

STEP 1. *The function $\lambda_0(\nu)$ is continuous,* i.e. *for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\nu \in I_\delta(\nu_0)$ then*

$$\lambda_0(\nu_0) - \epsilon < \lambda_0(\nu) < \lambda_0(\nu_0) + \epsilon \, .$$

Moreover there exists $\delta_0 > 0$ such that for any $\nu \in I_{\delta_0}(\nu_0)$ there exists $i \in I$ with $\widetilde{C}^{\nu_0}_i \in \widetilde{\mathcal{F}}_\nu$.

The second part of the previous statement asserts that for any direction ν in a suitable neighborhood $I_{\delta_0}(\nu_0)$ there exists a set $\widetilde{C}^{\nu_0}_i$ in the collection $\widetilde{\mathcal{F}}_{\nu_0}$ which also belongs to $\widetilde{\mathcal{F}}_\nu$.

STEP 2. *There exist a direction $\nu_1 \in I_{\delta(\epsilon_0)}(\nu_0)$, a neighborhood $I_{\delta_1}(\nu_1)$ and an index $i_1 \in \left\{1 \, , \dots , \, n_0\right\}$ such that for any $\nu \in I_{\delta_1}(\nu_1)$ the set $\widetilde{C}^{\nu_0}_{i_1}$ belongs to the collection $\widetilde{\mathcal{F}}_\nu$.*

From this we deduce that in $\widetilde{C}^{\nu_0}_{i_1}$ the function u is symmetric with respect to all hyperplanes $T^\nu_{\lambda_0(\nu)}$ with $\nu \in I_{\delta_1}(\nu_1)$. It is this symmetry property which is exploited in the next step to conclude the proof of Theorem 1.1.

STEP 3 *Let ν_1, i_1, δ_1 be as in Step 2 and set $C = C_{i_1}^{\nu_0}$. Then $\partial C \cap \Omega_{\lambda_0(\nu_1)}^{\nu_1}$ contains a subset Γ on which u is constant and whose projection on the hyperplane $T_{\lambda_0(\nu_1)}^{\nu_1}$ contains an open subset of the hyperplane.*

Since $Du = 0$ on $\partial C \cap \Omega_{\lambda_0(\nu_1)}^{\nu_1}$ Step 3 gives a contradiction with Proposition 2.1 and ends the proof of Theorem 1.1.

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