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3-dimensional physically consistent diffusion in anisotropic media with memory

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Fisica matematica. — *3-dimensional physically consistent diffusion in anisotropic media with memory.* Nota (*) del Socio MICHELE CAPUTO.

ABSTRACT. — Some data on the flow of fluids exhibit properties which may not be interpreted with the classic theory of propagation of pressure and of fluids [21] based on the classic D'Arcy's law which states that the flux is proportional to the pressure gradient. In order to obtain a better representation of the flow and of the pressure of fluids the law of D'Arcy is here modified introducing a memory formalisms operating on the flow as well as on the pressure gradient which implies a filtering of the pressure gradient without singularities; the properties of the filtering are also described. We shall also modify the second constitutive equation of diffusion, which relates the density variations of the fluid to its pressure variations, by introducing the rheology of the fluid also represented by derivatives of fractional order operating on the pressure as well as on the density. Moreover the medium will be considered anisotropic. We shall obtain the diffusion equation with these conditions in an anisotropic medium and find the Green function for a point source.

KEY WORDS: Diffusion; Filtering; Anisotropy; Memory; D'Arcy.

RIASSUNTO. — *Diffusione di fluidi in mezzi anisotropi, con meccanismi di memoria fisicamente accettabili.* Alcuni dati sperimentali sul flusso di fluidi in mezzi porosi mostrano proprietà che non possono essere spiegate in base alla legge di D'Arcy che stabilisce proporzionalità fra flusso e gradiente di pressione. Per spiegare questa fenomenologia in questa *Nota* si modifica la legge di D'Arcy introducendo un formalismo di memoria, che opera sia sul flusso che sul gradiente di pressione, che genera un filtraggio senza singolarità fisicamente inaccettabili. Nella *Nota* si modifica anche l'equazione che lega la variazione di pressione con quella del fluido mediante l'introduzione di un secondo formalismo di memoria. Entrambi i formalismi di memoria sono rappresentati da derivate di ordine frazionario. Il mezzo è assunto anisotropico. Si trovano la trasformata di Laplace della funzione di Green in generale e la sua antitrasformata in casi particolari.

GLOSSARY

$1/A$ $(m^2 s^{-2})$ see formula (2),
$p(x, y, z, t)$ $(kg\, s^{-2} m^{-1})$ pressure of the fluid,
$\bar{q}$ (x, y, z, t) $(kg\, s^{-1} m^{-2})$ fluid mass flow rate in the medium per unit area in the x, y, z direction,
$p(x, y, z, 0) = p(0)$ initial pressure in the medium,
$t(s)$ time,
x, y, z (m) , Cartesian coordinates,
n (dimensionless) fractional order of differentiation $(0 < n < 1)$,
h_{ij} dimensionless diffusivity tensor,
h_i normalized components of the diffusivity tensor,
κ $(kg^{-1} m^3 s^1)$ ratio of the permeability of the medium to the viscosity of the fluid (see formula (1)),
$\eta(kg^{-1} m^2 s^{1+n})$ ratio of the pseudopermeability of the medium (see formula (1′)) to the viscosity of the fluid,
$\rho(x, y, z, t)$ $(kg\, m^{-3})$ variation of the density of the fluid from its initial value,
ρ_0 density of the fluid in the initial condition.

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Introduction

The basic equations needed to study the flow of fluids in various media have been set through this century perhaps beginning with Terzaghi [27, 28]; since then a great progress has been made through the work of Biot [4-7], Biot and Willis [8], Boley and Tolins [9], Nowacki [23], McNamee and Gibson [22], Booker [10], Rice and Cleary [24], Bell and Nur [2] and Roeloffs [25] who contributed in various form to set the equations rigorously representing the interaction between the medium and the flow of the fluid through it and to obtain solutions of these equations in many interesting cases.

Most authors who studied diffusion problems used the classic empirical law of D'Arcy stating proportionality between the fluid mass flow rate and the gradient of the pressure in the same direction.

Concerning the observations of the diffusion however many problems are still unresolved [25]. To overcome these difficulties more general models of diffusion have been suggested. A set of models is based on memory as in the one dimensional work of Wyss [29], Mainardi [21] and Caputo [12, 15]. Another set of models is the classical convection-dispersion model (CDM), where the transport coefficients are time dependent, which are also based on the D'Arcy's law; they are well reviewed in [1].

The diffusion of fluids is based on two constitutive equations: the first is the law of D'Arcy relating the flux to the gradient of the pressure, the second is the relation between the density variations of the fluid to those of its pressure.

Because of the inadequacy of current theories in taking into account memory some authors developed also non-local flow theories (*e.g.* [19]) using general principles of statistical physics under appropriate limiting conditions from which the classical Darcy's law is derived for saturated flow.

In order to allow a better interpretation of the observations, in this *Note*, instead of introducing in D'Arcy's law the fractional order derivative operating on the pressure gradient only, as done in 1-Dimensional previous work of Wyss [29], Mainardi [21] and Caputo [12, 15], we shall introduce in the 3-Dimensional D'Arcy's law a memory formalism represented by derivatives of fractional order operating on the pressure gradient as well as on the flux. This overcomes a difficulty in the frequency domain physical interpretation of the results of Wyss [29], Mainardi [21] and Caputo [12, 15], which we will describe and discuss.

One more aspect of the diffusion which we will consider is that the constitutive equation relating the variations of the density to those of the pressure, is actually based on two relations; in fact it results from Hooke's law between the pressure and the implied volume changes and from the geometric relation between volume and density. The law of Hooke is here subject to rheological constraints which we will imply by including a memory formalism in the law itself [20] operating on the pressure as well as on the density variations of the fluid. Moreover the medium will be considered anisotropic.

The model

Many models, including the CDM type, imply that the permeability of the medium is variable with time and this phenomenon is taken into account when writing the law of D'Arcy which states that the fluid mass flow rate $\bar{q}$ per unit area in the x, y, z direction is proportional to the gradient of the pore pressure p

$$\bar{q} = -\rho_0 \kappa\, grad p \tag{1}$$

where κ , with dimension $m^3 kg^{-1} s$, is the ratio of the permeability of the medium to the viscosity of the fluid and ρ_0 is the density of the fluid in its undisturbed condition.

One more constitutive equation, often taken without discussion, relates the pressure to the variation ρ of the density from its undisturbed condition

$$\rho(x, y, z, t) = Ap(x, y, z, t)\,. \tag{2}$$

The continuity relation between the time variation of the density and the divergence of the flux allows to eliminate flux and density and to reduce the problem to the classic diffusion equation.

One way to take into account the observed deviations of the flow from those implied by the classic diffusion equation is to introduce a memory formalism in D'Arcy's law and consider that the flow depends on the history of the pressure gradient. A simple way of doing it is to write equation (1) as follows

$$\bar{q}_{\,i} = -\rho_0 \eta (\partial^n / \partial t^n) grad p \tag{1$'$}$$

where η , with dimensions $kg^{-1} m^2 s^{1+n}$, is the ratio of the pseudopermeability of the medium to the viscosity of the fluid and the definition of derivative of fractional order n is

$$\partial^n p(x, t)/\partial t^n = (1/\Gamma(1-n)) \int_0^t (t-u)^{-n} (\partial p(x, u)/\partial u) du \tag{3}$$

with $0 \leq n < 1$. In the definition (3) there is convergence at $t = u$ for any value of t since it is assumed $0 \leq n < 1$. It is clear that the memory formalism introduced by (1$'$) to describe the flow of the fluid implies the use of more than one parameter, namely n and $\eta\rho_0$, instead of the only one parameter κ as in D'Arcy's law. The Green function of the diffusion resulting from (2) has been obtained in various forms by several authors (*e.g.* [29, 21, 15]).

The extensions (1$'$) of the diffusion equation in 1 dimension used by Wyss [29], Schneider and Wyss [26], Mainardi [21] and Caputo [15], although mathematically consistent, does not give satisfactory results, when $0 < n < 1$ since, as we shall see, it implies a filter acting on the pressure gradient whose response curve is nil at zero frequency and infinite at infinite frequency. The case $-1 < n < 0$ implies a filter whose response curve is infinite at zero frequency and nil at infinite frequency. This difficulty will be overcome with the introduction of the fractional order derivative acting on the pressure gradient as well as on the flux.

An aspect of the diffusion which we will consider in this *Note* is also that the relation (2), between the density and the pressure, is actually based on two relations; in fact it results from Hooke's law between the pressure and the implied volume changes and the geometric relation between volume and density. The law of Hooke is here subject to rheological constraints which we will imply by including derivatives of fractional order in the law itself [20] writing equation (2) as follows

$$\alpha\rho + \beta\partial^{m_1}\rho/\partial t^{m_1} = A(ap + b\partial^{m_2}p/\partial t^{m_2}) \tag{4}$$

where the parameters α and a are positive and dimensionless, β and b are also positive but have dimension s^{m_1} and s^{m_2} respectively and $0 < n_1 < 1$, $0 < n_2 < 1$. The Laplace Transform (LT) of (4) to be used later is [11]

$$\alpha R + \beta s^{m_1} R = A\big(aP + bs^{m_2}P - s^{m_2-1}p(0)\big) \tag{4$'$}$$

where R and P are the LT of ρ and of p respectively, $p(0)$ is its initial pressure assumed independent of the coordinates and s is the LT variable.

One more aspect of the diffusion to consider is the anisotropy which occurs often in nature especially in fractured and porous media. In this *Note* we will assume that the medium is anisotropic and describe the anisotropy with a second order tensor. However we shall limit ourselves to the case when the anisotropy is independent of coordinates.

Although the extension (1′) of the diffusion equations may be useful in limited frequency ranges, as we shall see, a physically more satisfactory and general extension of the difffusion equation based on D'Arcy's law may be obtained by writing it in the following form

$$\gamma q_i + \varepsilon\partial^{n_1}q_i/\partial t^{n_1} = -h_{ij}(cp_j + d\partial^{n_2}p_j/\partial t^{n_2}) \tag{5}$$

where q_i are the components of $\bar{q}$, p_j are the components of *gradp*, h_{ij} is the symmetric dimensionless tensor, assumed independent of the coordinates, describing the anisotropy of the medium, $\bar{q}\ (x, y, z, t)$ is the flux, $p(x, y, z, t)$ is the fluid pressures in the medium and $0 < m_1 < 1$, $0 < m_2 < 1$. The parameter γ is positive and dimensionless while ε, c and d are also positive but have dimensions s^{n_1}, s and s^{1+n_2} respectively.

Since the anisotropy tensor is independent of the coordinates we may rotate the coordinates to the principal directions of the anisotropy and obtain the following simplified form of (5)

$$\gamma q_i + \varepsilon\partial^{n_1}q_i/\partial t^{n_1} = -h_i(cp_i + d\partial^{n_2}p_i/\partial t^{n_2}) \qquad (i = 1\,,2\,,3) \tag{5$'$}$$

where, in the right hand member, there is no summation relative to the index i, h_i are the principal values of h_{ij} and are assumed positive. The rotated coordinates will be again written as x, y, z.

We shall see in this *Note* that the introduction of the fractional order derivatives in both sides of (5) eliminates the singularities at zero and infinite frequencies.

The LT of (5′) is of interest for the discussion of the filtering acting on the gradient of the pressure and for obtaining the diffusion equation

$$(5'')\qquad Q_i = -(c + ds^{n_2})h_i P_i/(\gamma + \varepsilon s^{n_1}) + \\ + h_i\big(s^{n_1-1} bq_i(x, y, z, 0) + s^{n_2-1} dp_i(x, y, z, 0)\big)/(\gamma + \varepsilon s^{n_1})$$

where Q_i is the LT of q_i.

We shall discuss here the more general form (5′) of D'Arcy's law whose LT (5″) readily shows that, when the initial pressure gradient and flow are nil, that is when the initial pressure is independent of position, and b or a are not nil, then, assuming $s = iw$ and interpreting (5″) as a filter, the response of the filter at zero and infinite frequency are not nil or singular.

In order to find the pressure distribution in the medium let us associate to (4) and (5′) the continuity equation

$$(6)\qquad div\bar{q} + \partial\rho/\partial t = 0$$

whose LT form is

$$(6')\qquad div\overline{Q} + sR = 0$$

where it is assumed $\rho(x, y, z, 0) = 0$. Substituting (5″) in (6′) we find

$$(7)\qquad divQ = -sA(a + bs^{m_2})P/(\alpha + \beta s^{m_1}) + Abs^{m_2}p(0)/(\alpha + \beta s^{m_1}).$$

As usual in this type of problem we assume that the pressure in the medium is initially independent of x, y, z and therefore

$$(8)\qquad \begin{cases} p_i(x, y, z, 0) = 0\,, \quad \bar{q}\,(x, y, z, 0) = 0\,, \\ \partial p_1(x, y, z, 0)/\partial x = 0\,, \quad \partial p_2(x, y, z, 0)/\partial y = 0\,, \quad \partial p_3(x, y, z, 0)/\partial z = 0\,, \\ \partial q_1(x, y, z, 0)/\partial x = 0\,, \quad \partial q_2(x, y, z, 0)/\partial y = 0\,, \quad \partial q_3(x, y, z, 0)/\partial z = 0\,. \end{cases}$$

Eliminating Q in (5″) and (7), with the conditions (8), we find the following equation governing the diffusion of the pressure in the anisotropic medium with constitutive equations (4) and (5′)

$$(9)\qquad \big[\partial^2 h_1 P/\partial x^2 + \partial^2 h_2 P/\partial y^2 + \partial^2 h_3 P/\partial z^2\big] = \\ = sA(\gamma + \varepsilon s^{n_1})(a + bs^{m_2})P/(\alpha + \beta s^{m_1})(c + ds^{n_2}) - \\ -A(\gamma + \varepsilon s^{n_1})bs^{m_2}p(0)/(\alpha + \beta s^{m_1})(c + ds^{n_2})\,.$$

The solution of the diffusion equation

It is verified that a particular solutions of (9) is

$$(9')\qquad p(0)bs^{m_2}/s(a + bs^{m_2})\,.$$

The inverse LT of (9′) is the Mittag-Leffler function here presented in a different form [13]

$$(9'')\qquad \chi(t) = p(0)(\sin\pi m_2/\pi m_2)\int_0^\infty \exp\big(-(au/b)^{1/m_2} t\big)\,du/(u^2 + 2u\cos\pi m_2 + 1)\,.$$

The homogeneous version of equation (9) is

$$(10)\quad \begin{cases} h_1\partial^2 P(x,y,z,s)/\partial x^2 + h_2\partial^2 P(x,y,z,s)/\partial y^2 + h_3\partial^2 P(x,y,z,s)/\partial z^2 = H(s)\ , \\ H(s) = A\big[s(a+bs^{m_2})(\gamma+\varepsilon s^{n_1})/(\alpha+\beta s^{m_1})(c+ds^{n_2})\big]P(x,y,z,s)\,. \end{cases}$$

To solve equation (10) we shall proceed by separation of variables. Let us then set

$$(11)\qquad P(x,y,z,s) = U(x,s)V(y,s)W(z,s)$$

and assume

$$(11')\qquad \begin{aligned} h_1\partial^2 U/\partial x^2 &= r(s)U\ , \\ h_2\partial^2 V/\partial y^2 &= g(s)V\ , \\ h_3\partial^2 W/\partial z^2 &= l(s)W\ , \end{aligned}$$

where, at the moment, $r(s)$, $g(s)$ and $l(s)$ are positive functions of s but otherwise arbitrary, which yield the solution converging at infinite distance

$$(11'')\qquad \begin{aligned} U(x,s) &= B(s)\exp\{-(r(s)/h_1)^{1/2}x\}\ , \\ V(y,s) &= D(s)\exp\{-(g(s)/h_2)^{1/2}y\}\ , \\ W(z,s) &= E(s)\exp\{-(l(s)/h_3)^{1/2}z\}\ , \end{aligned}$$

which yield the solution of (10) if

$$(12)\qquad r(s)+g(s)+l(s) = H(s)\,.$$

The solution (11″) with the condition (12) allows to solve a variety of initial value problems for a point source. We note that combining the solution (11′) with the solution diverging at infinity one may solve a variety of initial value problems with a source more general than a point.

Since we are studying the point source it is no limitation to our problem to assume

$$r(s) = g(s) = l(s) = H(s)/3\,.$$

The solutions (11′) are then

$$(11''')\qquad \begin{aligned} U(x,s) &= B(s)\exp\{-\big[As(a+bs^{m_2})(\gamma+\varepsilon s^{n_1})/3h_1(\alpha+\beta s^{m_1})(c+ds^{n_2})\big]^{1/2}x\}\ , \\ V(y,s) &= D(s)\exp\{-\big[As(a+bs^{m_2})(\gamma+\varepsilon s^{n_1})/3h_2(\alpha+\beta s^{m_1})(c+ds^{n_2})\big]^{1/2}y\}\ , \\ W(z,s) &= E(s)\exp\{-\big[As(a+bs^{m_2})(\gamma+\varepsilon s^{n_1})/3h_3(\alpha+\beta s^{m_1})(c+ds^{n_2})\big]^{1/2}z\}\,. \end{aligned}$$

Where $B(s)$, $D(s)$ and $E(s)$ are functions admitting inverse LT but otherwise arbitrary. Assuming $B(s) = D(s) = E(s)$, which is no limitation since we consider a point source in the origin of the coordinates, we find

$$(13)\qquad \begin{aligned} P(x,y,z,s) = B(s)\exp\{-\big[As(a+bs^{m_2})(\gamma+\varepsilon s^{n_1})/3(c+ds^{n_2})(\alpha+\beta s^{m_1})\big]^{1/2}\cdot \\ \cdot(x/h_1^{1/2}+y/h_2^{1/2}+z/h_3^{1/2})\} \end{aligned}$$

where $B(s)$ is the LT of a function $f(t)$ which will be defined with the conditions at the boundary.

From here, in order to simplify the computations we shall assume that $n_1 = n_2 = m_1 = m_2 = n$, which will save the non singular properties of the filtering introduced, and seek the LT^{-1} of (13) in some cases of particular interest and following the same procedure previously used [15].

A case of special interest is that when $\gamma = \alpha = 0$ which we will study in the following assembling the parameters A, ε and β assuming $\varepsilon = \beta = 1$. The equation (13) is then

$$P(x, y, z, s) = B(s) \exp\left\{ -\left[As(a + bs^n)/3(c + ds^n)\right]^{1/2} (x/h_1^{1/2} + y/h_2^{1/2} + z/h_3^{1/2})\right\}. \tag{14}$$

The solution of the homogeneous equation (10), that is the inverse LT of (14), is found in the appendix A. From the solution of the homogeneous equation (10) we obtain the solution of the non homogeneous equation (9)

$$\begin{aligned}
p(x, y, z, t) = {} & \chi(t) + \left(LT^{-1}(B(s))\right) * (1/\pi) \int_0^\infty \left\{ \exp\left[-rt- \right.\right. \\
& + r^{1/2}\left(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3}\right) K^{1/2} \cdot \\
& \cdot \left[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos n\pi)\right]^{1/4} \cdot \\
& \left. \cdot \cos\left(0.5\left(\tan^{-1}\left(r^n \sin n\pi/(L + r^n \cos n\pi)\right) - \tan^{-1}\left(r^n \sin n\pi/(M + r^n \cos n\pi)\right)\right)\right)\right] \cdot \\
& \cdot \sin\left[r^{1/2}\left(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3}\right) K^{1/2} \cdot \right. \\
& \cdot \left[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos n\pi)\right]^{1/4} \cdot \\
& \left.\left. \cdot \sin\left(0.5\left(\tan^{-1}\left(r^n \sin n\pi/(L + r^n \cos n\pi)\right) - \tan^{-1}\left(r^n \sin n\pi/(M + r^n \cos n\pi)\right)\right)\right)\right]\right\} dr
\end{aligned} \tag{15}$$

where in $\chi(t)$ it is assumed $n = m_2$.

Formula (15) may be numerically integrated for all values of x, y, z and t. It is to be noted that, since x, y, z and t are here assumed positive or nil, the integral in (15) is convergent.

When $\mathrm{LT}^{-1} B(s) = \delta(t)$ then formula (15) is the Green function of the problem and one may solve several problems of diffusion for a point source.

From (14), since $f(t) = \mathrm{LT}^{-1} B(s)$, with $f(t)$ limited, it is also easily seen with LT extreme values theorem that $p(x, y, z, 0) = 0\, f(0) + \chi(0) = p(0)$ for any x, y, z which verifies that the pressure, at the initial time, is the initial one; for $t = \infty$ it is $p(x, y, z, \infty) = f(\infty) + \chi(\infty) = f(\infty)$.

The inspection of formula (15) shows that for any value of t, however small, and of x, y, z however large, the value of the integral is always larger than zero which implies that the signal travels with infinite velocity.

Filtering properties

As we mentioned the memory formalism (5′) implies filtering of the pressure gradient with response $F(w)$ obtained substituting $s = iw$, with w frequency, in (5″) with the assigned initial conditions (8)

$$|F(w)| = |(c + ds^{n_1})/(\gamma + \varepsilon s^{n_2})| = \tag{16}$$
$$= \left|(c^2 + d^2 w^{2n_1} + 2cdw^{n_1} \cos \pi n_1)/(\gamma^2 + \varepsilon^2 w^{2n_2} + 2\gamma\varepsilon w^{n_2} \cos \pi n_2)\right|^{1/2}.$$

In the preceding work of Wyss [29], Mainardi [21] and Caputo [12, 15] it was assumed that the fluid is perfectly elastic, that (5) is reduced with $\varepsilon = c = 0$ and $n_1 = n_2$. This implies that the filter described by (16) is zero at zero frequency and infinite at infinite frequency which would be hardly acceptable from a physical point of view.

When $n_1 = n_2 = n$ and depending on $c/\gamma > d/\varepsilon$ or $c/\gamma < d/\varepsilon$ we have a decreasing or an increasing response curve respectively as shown in fig. 1 where it is noted that unless $\gamma = 0$ or $\varepsilon = 0$ there are no singularities in the response curves.

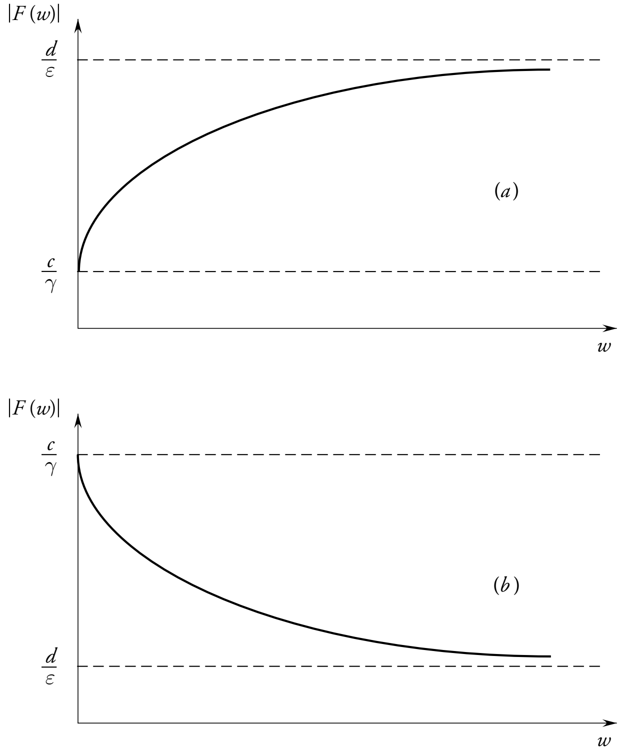

Fig. 1. – Response curves of the filter represented by the memory formalism introduced by the model defined in (5′). In (*a*) is the case when $d/\varepsilon > c/\gamma$, in (*b*) is the case when $d/\varepsilon < c/\gamma$.

A physically interesting case is when $n_1 = n_2 = n$ and $d = 0$ in which the filter response is finite at zero frequency and nil at infinite frequency which may be of use in some cases of geoelectric prospecting. The response to an infinite frequency input is d/ε, the relaxation time is $(\varepsilon/\gamma)^{1/n}$.

Another physically interesting case is when $n_1 = n_2 = n$ and $\gamma = 0$ [20], in which there is infinite, although integrable, response at zero frequency input and a finite response at infinite frequency input.

In the case when one of the parameters b or d is nil the procedure of appendix A is still valid and formulae (A7), (A8), (15) may be readily computed arriving to simpler results.

Conclusions

After the preceding discussion we may then extend also to the diffusion the formal mathematical analogy between anelastic and dielectric media assuming that the flux is the dual of the deformation (induction) and the pressure gradient is the dual of the stress (applied electric field) as shown by Caputo [17].

A particular case is when $n_1 = n_2 = n$, $\beta = b = 0$, $\alpha/a = 1$. In this case the rheology of the fluid is neglected and the solution is obtained from (15) where however $L = \gamma/\varepsilon$, $K = A\varepsilon/d$ and $\chi(t) = 0$.

The form of the Green function of the case when no singularities are present in the response curve of the filter implied by the memory inserted in D'Arcy's law is more complicated than that of the case when singularities are present [15]; this is the price for physical consistency. The same conclusion is valid also for the inclusion of the rheology in the constitutive equations of the fluid.

The physically consistent solution obtained in the present *Note* may contribute to give a satisfactory explanation to the variable velocity of migration of earthquake's foci and to the variable velocity of migration, in the crust of the Earth, of the precursory phenomena of strong earthquakes [13], which are possibly due to the diffusion of subterranean waters [3].

Appendix A

Assuming

$$s = r\exp(i\theta) \tag{A1}$$

the exponential factor of $B(s)$ appearing in (14) may be written

$$\begin{aligned}\exp\Big\{-\big[As(a+bs^n)/(c+ds^n)\big]^{1/2}\big(x/\sqrt{3h_1}+y/\sqrt{3h_2}+z/\sqrt{3h_3}\big)\Big\} &= \\ = \exp\Big(-\big(x/\sqrt{3h_1}+y/\sqrt{3h_2}+z/\sqrt{3h_3}\big)\,T\exp(i\Theta)\Big)\end{aligned} \tag{A2}$$

where

$$\begin{aligned}&T(r,\theta) = (Kr)^{1/2}\Big[(L^2+r^{2n}+2Lr\cos n\theta)/(M^2+r^{2n}+2Mr\cos n\theta)\Big]^{1/4},\\ &\Theta(r,\theta) = 0.5\big(\tan^{-1}(r^n\sin n\theta/(L+r^n\cos n\theta)) - \tan^{-1}(r^n\sin n\theta/(M+r^n\cos n\theta))\big),\\ &K = Ab/d\,,\quad L = a/b\,,\quad M = c/d\,.\end{aligned} \tag{A3}$$

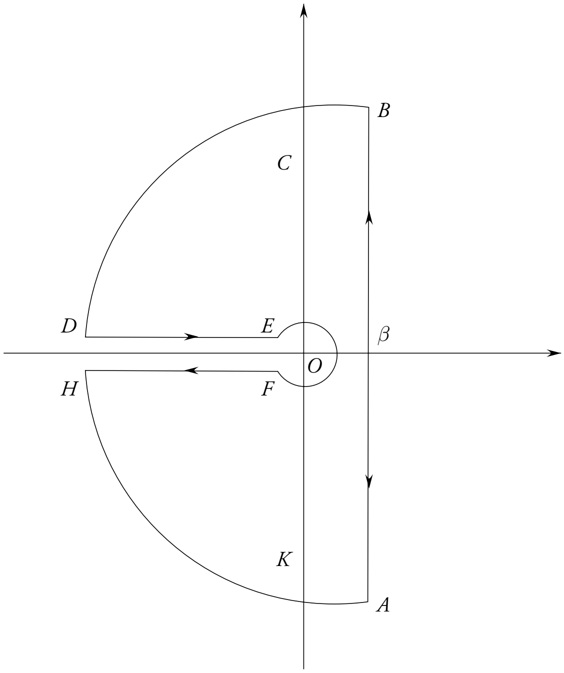

Fig. 2. – Path of integration of formula (14).

It is to be noted that

$$T(r,\pi) = T(r,-\pi)\ , \qquad \Theta(r,\pi) = -\Theta(r,-\pi)\,. \tag{A4}$$

The LT^{-1} of the exponential factor of $B(s)$, appearing in (8), is computed integrating along the closed path of fig. 2, inside which there are no poles of the exponential because this has no poles in the negative complex plane of s. The integral is therefore nil because the residuals are nil and we may write

$$\begin{aligned}(1/2ip)\lim_{u\to\infty}\Bigg[&\int_{b-iu}^{b+iu}\exp\left(st-\left(x/\sqrt{3h_1}+y/\sqrt{3h_2}+z/\sqrt{3h_3}\right)\cdot\right.\\ &\left.\cdot\left[As(a+bs^n)/(c+ds^n)\right]^{1/2}\right)ds+\\ &+\int_D^E\exp\left(st-\left(x/\sqrt{3h_1}+y/\sqrt{3h_2}+z/\sqrt{3h_3}\right)\left[As(a+bs^n)/(c+ds^n)\right]^{1/2}\right)ds+\\ &+\int_F^H\exp\left(st-\left(x/\sqrt{3h_1}+y/\sqrt{3h_2}+z/\sqrt{3h_3}\right)\left[As(a+bs^n)/(c+ds^n)\right]^{1/2}\right)ds\Bigg]=0.\end{aligned} \tag{A5}$$

Noting that in the integration on $DE:\theta=\pi$ and on $FH:\theta=-\pi$, then

$$\theta=\pm\pi\ ,\ s=-r\ ,\ ds=-dr\ ,\ s^{z/2}=r^{z/2}\left(\cos(\pi z/2)\pm i\sin(\pi z/2)\right) \tag{A6}$$

and finally

(A7)

$$p(x, y, z, t) = -(1/2i\pi) \int_0^{\infty} \Big\{ \exp \Big[- rt -$$

$$+ r^{1/2} \big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3} \big) K^{1/2} \cdot$$

$$\cdot \big[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos \pi n) \big]^{1/4} \cdot$$

$$\cdot \cos \big(0.5 \big(\tan^{-1}\big(-r^n \sin n\pi/(L + r^n \cos n\pi) \big) - \tan^{-1}\big(-r^n \sin n\pi/(M + r^n \cos n\pi) \big) - \pi \big) \big) \Big] \Big\}$$

$$\cdot \Big\{ \cos \Big[r^{1/2} \big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3} \big) K^{1/2} \cdot$$

$$\cdot \Big[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos \pi n) \Big]^{1/4} \cdot$$

$$\cdot \sin \big(0.5 \big(\tan^{-1}\big(-r^n \sin n\pi/(L + r^n \cos n\pi) \big) - \tan^{-1}\big(-r^n \sin n\pi/(M + r^n \cos n\pi) \big) - \pi \big) \big) \Big] +$$

$$+ i \sin \Big[r^{1/2} \big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3} \big) K^{1/2} \cdot$$

$$\cdot \Big[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos \pi n) \Big]^{1/4} \cdot$$

$$\cdot \sin \big(0.5 \big(\tan^{-1}\big(-r^n \sin n\pi/(L + r^n \cos n\pi) \big) - \tan^{-1}\big(-r^n \sin n\pi/(M + r^n \cos n\pi) \big) - \pi \big) \big) \Big] \Big\} dr +$$

$$- (1/2i\pi) \int_{\infty}^{0} \Big\{ \exp \Big[- rt -$$

$$+ r^{1/2} \big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3} \big) K^{1/2} \cdot$$

$$\cdot \big[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos \pi n) \big]^{1/4} \cdot$$

$$\cdot \cos \big(0.5 \big(\tan^{-1}\big(r^n \sin n\pi/(L + r^n \cos n\pi) \big) - \tan^{-1}\big(r^n \sin n\pi/(M + r^n \cos n\pi) \big) + \pi \big) \big) \Big] \Big\} \cdot$$

$$\cdot \Big\{ \cos \Big[r^{1/2} \big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3} \big) K^{1/2} \cdot$$

$$\cdot \Big[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos \pi n) \Big]^{1/4} \cdot$$

$$\cdot \sin \big(0.5 \big(\tan^{-1} \big(r^n \sin n\pi/(L + r^n \cos n\pi) \big) - \tan^{-1} \big(r^n \sin n\pi/(M + r^n \cos n\pi) \big) + \pi \big) \big) \Big] +$$

$$+ i \sin \Big[r^{1/2} \big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3} \big) K^{1/2} \cdot$$

$$\cdot \Big[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos \pi n) \Big]^{1/4} \cdot$$

$$\cdot \sin \big(0.5 \big(\tan^{-1} \big(r^n \sin n\pi/(L + r^n \cos n\pi) \big) - \tan^{-1} \big(r^n \sin n\pi/(M + r^n \cos n\pi) \big) + \pi \big) \big) \Big] \Big\} dr.$$

Using (A4) we may then obtain from (A7) the general solution of the homogeneous equation (13) in the following simpler form

$$
\begin{aligned}
p(x, y, z, t) = & \left(LT^{-1}(B(s))\right) * (1/\pi) \int_0^\infty \Big\{ \exp \Big[-rt- \\
& + r^{1/2}\big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3}\big) K^{1/2} \cdot \\
& \cdot \left[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos n\pi)\right]^{1/4} \cdot \\
\text{(A8)} \quad & \cdot \cos\Big(0.5\Big(\tan^{-1}\big(r^n \sin n\pi/(L + r^n \cos n\pi)\big) - \tan^{-1}\big(r^n \sin n\pi/(M + r^n \cos n\pi)\big)\Big)\Big)\Big]\Big\} \cdot \\
& \cdot \sin\Big\{ r^{1/2}\big(x/\sqrt{3h_1} + y/\sqrt{3h_2} + z/\sqrt{3h_3}\big) K^{1/2} \cdot \\
& \cdot \Big[(L^2 + r^{2n} + 2Lr^n \cos n\pi)/(M^2 + r^{2n} + 2Mr^n \cos n\pi)\Big]^{1/4} \cdot \\
& \cdot \sin\Big(0.5\Big(\tan^{-1}\big(r^n \sin n\pi/(L + r^n \cos n\pi)\big) - \tan^{-1}\big(r^n \sin n\pi/(M + r^n \cos n\pi)\big)\Big)\Big)\Big\} \, dr.
\end{aligned}
$$

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Note added to proofs. – The relations (4) and (5) are a generalisation of the relations introduced in elasticity by M. Caputo and F. Mainardi (*A new dissipation model based on memory mechanism*. Pageoph., 91, 1971, 134-147) and have been used already, in the frequency domain, by R.L. Bagley and P.J. Torvik (*On the fractional calculus model of viscolelastic behaviour*. Journal of Rheology, 30, 1986, 133-155).