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Non-solvability of the tangential $\bar{\partial}_M$-systems

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Geometria. — *Non-solvability of the tangential $\bar{\partial}_M$-systems.* Nota di Giuseppe Zampieri, presentata (*) dal Socio E. Vesentini.

Abstract. — We prove that for a real analytic generic submanifold M of $\mathbb{C}^n$ whose Levi-form has constant rank, the tangential $\bar{\partial}_M$-system is non-solvable in degrees equal to the numbers of positive and negative Levi-eigenvalues. This was already proved in [1] in case the Levi-form is non-degenerate (with M non-necessarily real analytic). We refer to our forthcoming paper [7] for more extensive proofs.

Key words: CR manifolds; Tangential Cauchy-Riemann Complexes; Real/Complex symplectic structures.

Riassunto. — *Non risolubilità del sistema $\bar{\partial}_M$ tangenziale.* Si prova che per una sottovarietà analitica reale *generica* M di $\mathbb{C}^n$ la cui forma di Levi ha rango costante, il complesso $\bar{\partial}_M$ tangenziale è non risolubile nei gradi corrispondenti ai numeri di autovalori positivi e negativi. Per forme non-degeneri il risultato era già stato stabilito in [1] (senza l'ipotesi che M sia analitica reale).

1. Notations and basic language on derived categories [3, 5]

Let X be a complex analytic manifold of dimension n, $M \subset X$ a real submanifold of codimension l, $\pi : T^*X \to X$ and $\pi : T^*_MX \to M$ the cotangent bundle to X and the conormal bundle to M respectively. By $\dot{T}^*X$ we shall denote the cotangent bundle with the 0-section removed. Let $\mathrm{D}^b(X)$ denote the derived category of the category of complexes of sheaves with bounded cohomology, and $D^b(X;p)$ (p a point of T^*X) its localization at p in the sense of [3].

Let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on X, $\mathbb{Z}_M$ the constant sheaf along M, $\mu_M(\mathcal{O}_X) := \mu\mathrm{hom}(\mathbb{Z}_M, \mathcal{O}_X)$ (resp. $\mathbb{R}\Gamma_M(\mathcal{O}_X) := \mathbb{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}_X}(\mathbb{Z}_M, \mathcal{O}_X)$) the complexes of Sato's microfunctions and hyperfunctions along M respectively (up to a shift l). We recall that $\mathbb{R}\pi_*\mu_M(\mathcal{O}_X) = \mathbb{R}\Gamma_M(\mathcal{O}_X)$ (π_* being the direct image) and $\mathbb{R}\Gamma_{T^*_MM}\mu_M(\mathcal{O}_X)[l] = \mathcal{O}_X|_M$. This gives rise to the following (Sato's) triangle in $\mathrm{D}^b(X)$:

$$\mathcal{O}_X|_M \to \mathbb{R}\Gamma_M(\mathcal{O}_X)[l] \to \mathbb{R}\dot{\pi}_*\mu_M(\mathcal{O}_X)[l] \overset{+1}{\to} .$$

When M is real analytic, one can consider its complexification $M^{\mathbb{C}}$ (a $2n-l$-dimensional complex manifold) and define $\mathcal{B}_M := \mathbb{R}\Gamma_M(\mathcal{O}_{M^{\mathbb{C}}})[2n-l]$. If M is in addition *generic* (*i.e.* the embedding $M^{\mathbb{C}} \to X \times \bar{X}$ is non-characteristic for $\bar{\partial}_X$), then $\bar{\partial}_X$ induces a complex $\bar{\partial}_M$ on $M^{\mathbb{C}}$ and it turns out that the complex $\bar{\partial}_M$ over forms with coefficients in $\mathcal{B}_M$ is *quasi-isomorphic* (*i.e.* isomorphic in $\mathrm{D}^b(X)$) to the complex $\mathbb{R}\Gamma_M(\mathcal{O}_X)[l]$.

Let $\chi : \dot{T}^*X \to \dot{T}^*X$ be a germ of a complex symplectic homogeneous transformation. According to [3], we may let χ *act on sheaves* through a quantization

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by a kernel Φ_K. In particular if in a neighborhood of a point $q \in \dot{T}^* X$ we have $\chi(T^*_M X) = T^*_{\widetilde{M}} X$ (for a new real manifold $\widetilde{M}$), then we get an isomorphism in $D^b(X; q)$: $\chi_* \mu_M(\mathcal{O}_X) = \mu_{\widetilde{M}}(\mathcal{O}_X)[\widetilde{l} - l + s^-_{\widetilde{M}} - s^-_M]$ (where s^-_M and $s^-_{\widetilde{M}}$ are the numbers of negative eigenvalues of the Levi form of M and $\widetilde{M}$ respectively.

2. Statement and proof

Let M be a real analytic generic submanifold of $X = \mathbb{C}^n$ of codimension l. Let $\mathcal{B}^j_M$ denote the forms on M of bidegree $(0, j)$ with coefficients hyperfunctions, consider the tangential $\bar{\partial}$ -complex:

$$0 \to \mathcal{B}^0_M \xrightarrow{\bar{\partial}_M} \mathcal{B}^1_M \xrightarrow{\bar{\partial}_M} \dots \xrightarrow{\bar{\partial}_M} \mathcal{B}^n_M \to 0\,, \tag{1}$$

and denote by $\mathrm{H}^j_{\bar{\partial}_M}$ its cohomology in degree j. As we have already pointed out in §1, the genericity of M implies that $\mathrm{H}^j_{\bar{\partial}_M} = \mathrm{H}^j \mathbb{R}\Gamma_M(\mathcal{O}_X)[l]$ where $\mathcal{O}_X$ are the holomorphic functions on X. For $p \in \dot{T}^*_M X$ (the conormal bundle to M in X), let $s^+_M(p)$ and $s^-_M(p)$ denote the numbers of positive and negative eigenvalues respectively of the «microlocal» Levi-form of M at p. Let $z = \pi(p)$.

Theorem. *In the above situation, assume* $s^{\pm}_M \equiv \mathrm{const}$ *in a neighborhood of* p. *Then*

$$(\mathrm{H}^j_{\bar{\partial}_M})_z \neq 0 \text{ for } j = s^-_M(p)\,, s^+_M(p)\,, 0. \tag{2}$$

Proof. We first collect some classical tools for our proof.

(*a*) (cf. [3]) We can find a complex symplectic homogeneous transformation χ from a neighborhood of p to a neighborhood of $q := \chi(p)$, which interchanges $T^*_M X$ with $T^*_{\widetilde{M}} X$ where $\widetilde{M}$ is a pseudoconvex hypersurface in the side $-q$ (*i.e.* the open half-space $\widetilde{M}^-$ with inward conormal $-q$ is pseudoconvex). By quantization (cf. §1), we get a correspondence:

$$\mu_M(\mathcal{O}_X)_p[l + s^-_M] \xrightarrow{\sim} \mathbb{R}\Gamma_{\widetilde{M}^+}(\mathcal{O}_X)_y[1]\,, \tag{3}$$

where ($y = \pi(q)$) and $\mu_M(\mathcal{O}_X)$ is the Sato's microlocalization of $\mathcal{O}_X$ along M (cf. §1). In particular $\mathcal{F} := \mu_M(\mathcal{O}_X)[l + s^-_M]$ is concentrated in degree 0 [3, Th. 11.3.1] and, since $\widetilde{M}$ is a hypersurface, $\mathrm{H}^0(\mathcal{F}) \xrightarrow{\sim} \varinjlim_B \frac{\mathcal{O}_X(\widetilde{M}^- \cap B)}{\mathcal{O}_X(B)}$ (where $\{B\}$ is a system of neighborhoods of y).

(For this statement only the constancy of s^-_M and not necessarily of s^+_M at p is required).

(*b*) (cf. [6]) We may assume that by the above transformation T^*X is transformed to $T^*X' \times T^*Y$ and $T^*_{\widetilde{M}} X$ to $T^*_{\widetilde{M}'} X' \times Y$. In other words the integral leaves of the Levi-kernel can be straightened in suitable complex symplectic coordinates of T^*X (not of X).

(*c*) (cf. [7]) Let $V = V' \times Y$ be an open neighborhood of p s.t. (*a*) and (*b*) hold in $V_1 = V' \times Y_1$ for $Y_1 \supset\supset Y$ open, and take $Z = Z' \times Y$ with Z' closed and $Z' \subset\subset V'$.

Let $f \in \Gamma(V_1, \mathrm{H}^0(\mathcal{F}))$; then for any open neighborhood $W = W' \times Y$ of p with $W' \subset\subset \mathrm{int}Z'$, there exists $\widetilde{f} \in \Gamma_Z(V, \mathcal{F})$ such that $\widetilde{f}|_W = f|_W$.

(*d*) We are ready to conclude. We identify $T^*_M X$ to $M \times \mathbb{R}^l$ (by a choice of a system of l independent equations for M). We take $f \in \mathrm{H}^0(\mathcal{F})_p$, $f \neq 0$ by (*a*), and modify to $\widetilde{f} \in \Gamma_Z(V, \mathrm{H}^0(\mathcal{F}))$ according to (*c*). Since the complex leaves of the microlocal foliation of $T^*_M X$ are transversal to the fibers of π, then for suitable Z and for $U_o \subset \mathrm{int}Z$ (U_o open neighborhood of $z = \pi(p)$) we have that $Z \cap (U_o \times \dot{\mathbb{R}}^l)$ is closed in $U_o \times \dot{\mathbb{R}}^l$. This enables us to identify $\widetilde{f}$ to a section of $\Gamma(U_o \times \dot{\mathbb{R}}^l, \mathrm{H}^0(\mathcal{F})) \simeq \mathrm{H}^0\mathbb{R}\Gamma(U_o \times \dot{\mathbb{R}}^l, \mathcal{F})$. Let $\{U_\nu\}$ (resp. $\{W_\nu\}$) be a system of neighborhoods of z (resp. p), with $U_\nu \subset U_o$ and $W_\nu \subset (U_\nu \times \dot{\mathbb{R}}^l) \cap \mathrm{int}Z$. Note now that we have morphisms:

$$\mathrm{H}^0\mathbb{R}\Gamma(U_o \times \dot{\mathbb{R}}^l, \mathcal{F}) \rightarrow \mathrm{H}^0\mathbb{R}\Gamma(U_\nu \times \dot{\mathbb{R}}^l, \mathcal{F}) \rightarrow \mathrm{H}^0\mathbb{R}\Gamma(W_\nu, \mathcal{F}). \tag{4}$$

Since $\widetilde{f} \neq 0$ in W_ν (*i.e.* in the third term of (4)), then $\widetilde{f} \neq 0$ in:

$$\begin{aligned}\varinjlim_\nu \mathrm{H}^0\mathbb{R}\Gamma(U_\nu \times \dot{\mathbb{R}}^l, \mathcal{F}) &\simeq \varinjlim_\nu \mathrm{H}^{s^-_M}\mathbb{R}\Gamma(U_\nu \times \dot{\mathbb{R}}^l, \mu_M(\mathcal{O}_X))[l] \\ &\simeq \varinjlim_\nu \mathrm{H}^{s^-_M}\mathbb{R}\Gamma(U_\nu, \mathbb{R}\Gamma_M(\mathcal{O}_X)[l]) \simeq (\mathrm{H}^{s^-_M}_{\bar{\partial}_M})_z,\end{aligned}$$

where the isomorphism between the two lines comes from $(\mathrm{H}^j_{\bar{\partial}_X})_z = 0 \ \forall j \geq 1$. Thus $(\mathrm{H}^{s^-_M}_{\bar{\partial}_M})_z \neq 0$. (Similarly one proves that $(\mathrm{H}^{s^+_M}_{\bar{\partial}_M})_z \neq 0$). □

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