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Non-solvability of the tangential $\bar{\partial}_M$-systems

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Abstract. — We prove that for a real analytic generic submanifold $M$ of $\mathbb{C}^n$ whose Levi-form has constant rank, the tangential $\partial_M$-system is non-solvable in degrees equal to the numbers of positive and negative Levi-eigenvalues. This was already proved in [1] in case the Levi-form is non-degenerate (with $M$ non-necessarily real analytic). We refer to our forthcoming paper [7] for more extensive proofs.

Key words: CR manifolds; Tangential Cauchy-Riemann Complexes; Real/Complex symplectic structures.

Riassunto. — Non risolubilità del sistema $\partial_M$ tangenziale. Si prova che per una sottovarietà analitica reale generica $M$ di $\mathbb{C}^n$ la cui forma di Levi ha rango costante, il complesso $\partial_M$ tangenziale è non risolubile nei gradi corrispondenti ai numeri di autovalori positivi e negativi. Per forme non-degeneri il risultato era già stato stabilito in [1] (senza l’ipotesi che $M$ sia analitica reale).

1. Notations and basic language on derived categories [3, 5]

Let $X$ be a complex analytic manifold of dimension $n$, $M \subset X$ a real submanifold of codimension $l$, $\pi : T^* X \to X$ and $\pi : T^*_M X \to M$ the cotangent bundle to $X$ and the conormal bundle to $M$ respectively. By $T^*_M X$ we shall denote the cotangent bundle with the 0-section removed. Let $D^b(X)$ denote the derived category of the category of complexes of sheaves with bounded cohomology, and $D^b(X; p)$ ($p$ a point of $T^* X$) its localization at $p$ in the sense of [3].

Let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on $X$, $\mathbb{Z}_M$ the constant sheaf along $M$, $\mu_M(\mathcal{O}_X) := \mu \text{hom}(\mathbb{Z}_M, \mathcal{O}_X)$ (resp. $\mathbb{R}\Gamma_M(\mathcal{O}_X) := \mathbb{R}\text{Hom}_{\mathcal{O}_X}(\mathbb{Z}_M, \mathcal{O}_X)$) the complexes of Sato’s microfunctions and hyperfunctions along $M$ respectively (up to a shift $l$). We recall that $\mathbb{R}\pi_*\mu_M(\mathcal{O}_X) = \mathbb{R}\Gamma_M(\mathcal{O}_X)$ ($\pi_*$ being the direct image) and $\mathbb{R}\Gamma_{T^*_M \mathcal{O}_X}(\mathcal{O}_X)[l] = \mathcal{O}_X|_M$. This gives rise to the following (Sato’s) triangle in $D^b(X)$:

$$
\mathcal{O}_X|_M \to \mathbb{R}\Gamma_M(\mathcal{O}_X)[l] \to \mathbb{R}\pi_*\mu_M(\mathcal{O}_X)[l] \overset{+l}{\cong}.
$$

When $M$ is real analytic, one can consider its complexification $M^\mathbb{C}$ (a $2n-l$-dimensional complex manifold) and define $B_M := \mathbb{R}\Gamma_M(\mathcal{O}_M)[2n-l]$. If $M$ is in addition generic (i.e. the embedding $M^\mathbb{C} \to X \times \hat{X}$ is non-characteristic for $\hat{\partial}_\chi$), then $\hat{\partial}_X$ induces a complex $\hat{\partial}_M$ on $M^\mathbb{C}$ and it turns out that the complex $\hat{\partial}_M$ over forms with coefficients in $B_M$ is quasi-isomorphic (i.e. isomorphic in $D^b(X)$) to the complex $\mathbb{R}\Gamma_M(\mathcal{O}_X)[l]$.

Let $\chi : \hat{T}^* X \to \hat{T}^* X$ be a germ of a complex symplectic homogeneous transformation. According to [3], we may let $\chi$ act on sheaves through a quantization

by a kernel $\Phi_K$. In particular if in a neighborhood of a point $q \in \tilde{T}^*_M X$ we have $\chi(T^*_M X) = T^*_M X$ (for a new real manifold $\tilde{M}$), then we get an isomorphism in $D^b(X; q)$: $\chi \mu_M(O_X) = \mu_{\tilde{M}}(O_{\tilde{X}})[\tilde{T} - 1 + \tilde{s}^- M - \tilde{s}^+ M]$ (where $\tilde{s}^- M$ and $\tilde{s}^+ M$ are the numbers of negative eigenvalues of the Levi form of $M$ and $\tilde{M}$ respectively).

2. Statement and proof

Let $M$ be a real analytic generic submanifold of $X = \mathbb{C}^n$ of codimension $l$. Let $B^j_M$ denote the forms on $M$ of bidegree $(0, j)$ with coefficients hyperfunctions, consider the tangential $\tilde{\partial}$-complex:

$$0 \rightarrow B^0_M \xrightarrow{\tilde{\partial} M} B^1_M \xrightarrow{\tilde{\partial} M} \ldots \xrightarrow{\tilde{\partial} M} B^n_M \rightarrow 0,$$

and denote by $H^j_{\tilde{\partial} M}$ its cohomology in degree $j$. As we have already pointed out in §1, the genericity of $M$ implies that $H^j_{\tilde{\partial} M} = H^j_M \Gamma_M(O_X)[l]$ where $O_X$ are the holomorphic functions on $X$. For $p \in \tilde{T}^*_M X$ (the conormal bundle to $M$ in $X$), let $s^+_M(p)$ and $s^-_M(p)$ denote the numbers of positive and negative eigenvalues respectively of the «microlocal» Levi-form of $M$ at $p$. Let $z = \pi(p)$.

**Theorem.** In the above situation, assume $s^\pm_M \equiv \text{const}$ in a neighborhood of $p$. Then

$$H^j_{\tilde{\partial} M}(z) \neq 0 \text{ for } j = s^-_M(p), s^+_M(p), 0.$$

**Proof.** We first collect some classical tools for our proof.

(a) (cf. [3]) We can find a complex symplectic homogeneous transformation $\chi$ from a neighborhood of $p$ to a neighborhood of $q := \chi(p)$, which interchanges $T^*_M X$ with $T^*_M X$ where $\tilde{M}$ is a pseudoconvex hypersurface in the side $-q$ (i.e. the open half-space $M^-$ with inward conormal $-q$ is pseudoconvex). By quantization (cf. §1), we get a correspondence:

$$\mu_M(O_X)_p[I + \tilde{s}_M] \xrightarrow{\tilde{\Gamma}_M} \Gamma_M^+(O_X)_p[1],$$

where $(y = \pi(q))$ and $\mu_M(O_X)$ is the Sato’s microlocalization of $O_X$ along $M$ (cf. §1). In particular $F := \mu_M(O_X)[I + \tilde{s}_M]$ is concentrated in degree $0$ [3, Th. 11.3.1] and, since $\tilde{M}$ is a hypersurface, $H^0(F) \xrightarrow{\sim} \lim_{B} \frac{O_X(\tilde{M} \cap B)}{O_X(\partial B)}$ (where $\{B\}$ is a system of neighborhoods of $y$).

(For this statement only the constancy of $\tilde{s}_M$ and not necessarily of $s^+_M$ at $p$ is required).

(b) (cf. [6]) We may assume that by the above transformation $T^*_X$ is transformed to $T^*_X \times T^* Y$ and $T^*_M X$ to $T^*_M X \times Y$. In other words the integral leaves of the Levi-kernel can be straightened in suitable complex symplectic coordinates of $T^*_X$ (not of $X$).

(c) (cf. [7]) Let $V = V' \times Y$ be an open neighborhood of $p$ s.t. (a) and (b) hold in $V_1 = V' \times Y_1$ for $Y_1 \supset\supset Y$ open, and take $Z = Z' \times Y$ with $Z'$ closed and $Z' \subset\subset V'$.
Let \( f \in \Gamma(V, \mathcal{H}^0(\mathcal{F})) \); then for any open neighborhood \( W = W' \times Y \) of \( p \) with \( W' \subset \subset \text{int} Z' \), there exists \( \tilde{f} \in \Gamma_Z(V, \mathcal{F}) \) such that \( \tilde{f} |_W = f |_W \).

\[(d)\] We are ready to conclude. We identify \( T^*_M X \) to \( M \times \mathbb{R}^l \) (by a choice of a system of \( l \) independent equations for \( M \)). We take \( f \in H^0(F) \) \( p; f \neq 0 \) by \((a)\), and modify to \( \tilde{f} \in \Gamma_Z(V; \mathcal{H}^0(F)) \) according to \((c)\). Since the complex leaves of the microlocal foliation of \( T^*_M X \) are transversal to the fibers of \( \pi \), then for suitable \( Z \) and for \( U_0 \subset \subset \text{int} Z \) open neighborhood of \( z = \pi(p) \) we have that \( Z \cap (U_0 \times \mathbb{R}^l) \) is closed in \( U_0 \times \mathbb{R}^l \). This enables us to identify \( \tilde{f} \) to a section of \( \Gamma(U_0 \times \mathbb{R}^l, H^0(F)) \simeq H^0R\Gamma(U_0 \times \mathbb{R}^l, \mathcal{F}) \). Let \( \{U_\nu\} \) (resp. \( \{W_\nu\} \)) be a system of neighborhoods of \( z \) (resp. \( p \)), with \( U_\nu \subset U_0 \) and \( W_\nu \subset (U_\nu \times \mathbb{R}^l) \cap \text{int} Z \). Note now that we have morphisms:

\[
\begin{align*}
H^0R\Gamma(U_\nu \times \mathbb{R}^l, \mathcal{F}) & \rightarrow H^0R\Gamma(U_\nu \times \mathbb{R}^l, \mathcal{F}) \rightarrow H^0R\Gamma(W_\nu, \mathcal{F}).
\end{align*}
\]

Since \( \tilde{f} \neq 0 \) in \( W_\nu \) (i.e. in the third term of \((4)\)), then \( \tilde{f} \neq 0 \) in:

\[
\lim_{\nu} H^0R\Gamma(U_\nu \times \mathbb{R}^l, \mathcal{F}) \simeq \lim_{\nu} H^0R\Gamma(U_\nu \times \mathbb{R}^l, \mu_M(\mathcal{O}_X))[l] \simeq \lim_{\nu} H^0R\Gamma(U_\nu, \mu_M(\mathcal{O}_X))[l] \simeq (\mathcal{H}^0_{\partial_M})_x,
\]

where the isomorphism between the two lines comes from \((H^j_{\partial_M})_x = 0 \forall j \geq 1\). Thus \((\mathcal{H}^j_{\partial_M})_x \neq 0\). (Similarly one proves that \((\mathcal{H}^j_{\partial_M})_x \neq 0\).)

\[\square\]

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References


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