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Exact null controllability of structurally damped and thermo-elastic parabolic models

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Teoria dei controlli. — *Exact null controllability of structurally damped and thermo-elastic parabolic models.* Nota (*) di Irena Lasiecka e Roberto Triggiani, presentata dal Corrisp. G. Da Prato.

Abstract. — We show exact null-controllability for two models of non-classical, parabolic partial differential equations with distributed control: (*i*) second-order structurally damped equations, except for a limit case, where exact null controllability fails; and (*ii*) thermo-elastic equations with hinged boundary conditions. In both cases, the problem is solved by duality.

Key words: Exact null-controllability; Structurally damped parabolic equations; Thermo-elastic parabolic equations.

Riassunto. — *Controllabilità esatta all'origine di equazioni fortemente smorzate e di equazioni termo-elastiche.* In questa *Nota* dimostriamo la proprietà di controllabilità esatta all'origine per due modelli di equazioni alle derivate parziali, di tipo parabolico, non-classiche, con controllo distribuito: (*i*) equazioni del secondo ordine fortemente smorzate, eccetto che per un caso limite dove tale proprietà di controllabilità è falsa; (*ii*) equazioni termo-elastiche, con condizioni al contorno incernierate. In entrambi i casi, il problema è risolto per dualità.

1. Two classes of parabolic-like P.D.E.'s. Statement of main results

In this article we present a preliminary study exhibiting the property of exact null-controllability for two classes of non-classical, parabolic-like P.D.E.'s, with distributed $L_2(0, T; \cdot)$-control, along with an important limit case where this property fails. As a motivation for our study, we recall that the notion of exact null-controllability for deterministic parabolic-like P.D.E.'s plays an essential, critical role in connection with corresponding stochastic parabolic differential equations. In this context, it is known, in fact, that the notion of exact null-controllability is *equivalent* to the strong Feller property of the semigroup of transition of the corresponding stochastic differential equation, which is obtained from the deterministic one by simply replacing the deterministic control with stochastic noise (G. Da Prato, private communication, July 1997). The two non-classical, parabolic-like classes of P.D.E.'s studied in this article are: (*i*) structurally damped second-order abstract equations [2, 4, 5] (subsection 1.1), and (*ii*) some thermo-elastic abstract equations (subsection 1.2). Under the action of distributed control, it is proved below that each of these two classes is *exactly null-controllable at any finite time $T > 0$, by means of $L_2(0, T; \cdot)$-controls* except for the limit case $\alpha = 1$ in (1.1.1) below. This means that arbitrary initial data in the state (energy) space (or in compatible, rougher spaces, see Remark 2.1 below) can be steered to rest over the time interval $[0, T]$, $0 < T < \infty$, by means of $L_2(0, T; \cdot)$-distributed controls. The proof is, for each class, by duality. It exploits special structural/spectral properties exhibited

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by these two parabolic-like classes.

1.1. *First class: Structurally damped second-order abstract equations.*

Let X be a Hilbert space and let $S : X \supset \mathcal{D}(S) \to X$ be a strictly positive, self-adjoint unbounded operator, with compact resolvent. In this subsection, we consider the following second-order, structurally damped, abstract equation

$$(1.1.1) \qquad w_{tt} + Sw + \rho S^{\alpha} w_t = u\,, \;\; w(0) = w_0; \;\; w_t(0) = w_1; \;\; \rho > 0; \;\; 1/2 \leq \alpha \leq 1\,.$$

Here and throughout, $u \in L_2(0\,, T; X)$ is the control function, α is a real constant in the range $1/2 \leq \alpha \leq 1$, whereby the homogeneous problem corresponding to eq. (1.1.1) with $u \equiv 0$ is structurally damped [2, 4, 5]: this means, mathematically, that the operator

$$(1.1.2) \qquad A_{\rho\alpha} = \begin{bmatrix} 0 & I \\ -S & -\rho S^{\alpha} \end{bmatrix} : \; E \supset \mathcal{D}(A_{\rho\alpha}) \to E \; = \mathcal{D}(S^{1/2}) \times X;$$

$$(1.1.3) \qquad \mathcal{D}(A_{\rho\alpha}) = \big\{ [x_1\,, x_2] \in E : \; x_1 \in \mathcal{D}(S^{3/2-\alpha})\,, \;\; x_2 \in \mathcal{D}(S^{1/2})\,, S^{1-\alpha}x_1 + \\ + \rho x_2 \in \mathcal{D}(S^{\alpha}) \big\}$$

is the generator of a s.c. contraction semigroup $e^{A_{\rho\alpha} t} : \; \{w_0\,, w_1\} \to \{w(t)\,, w_t(t)\}$ with $u \equiv 0$ on the energy space E above, which, moreover, is *analytic* (holomorphic) here, $t > 0$ [4, 5]. Setting $z(t) \equiv [w(t)\,, w_t(t)]$, we may rewrite the second-order equation (1.1.1) as a first-order problem,

$$(1.1.4) \qquad z_t = A_{\rho\alpha} z + Bu\,, \quad z(0) = \{w_0\,, w_1\}\,, \;\; B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The extreme cases $\alpha = 1/2$ and $\alpha = 1$ are the two most important ones in applications. The main result of the present subsection is, perhaps, surprising in the split between a positive result for $1/2 \leq \alpha < 1$ on the one hand, and a negative result for $\alpha = 1$ on the other.

THEOREM 1.1.1. *With reference to problem* (1.1.1)*, let* $T > 0$*, and let* $\rho^2 \neq 4\mu_n^{1-2\alpha}$ *for all* $n = 1\,, 2\,, 3\,, \ldots$*, where the* μ_n *are the eigenvalues of the positive self-adjoint operator* S*; in particular, let* $\rho \neq 2$ *for* $\alpha = 1/2$.

(*i*) *Let* $1/2 \leq \alpha < 1$*. Then: Given arbitrary initial conditions* $\{w_0\,, w_1\} \in E$*, there exists a control function* $u \in L_2(0\,, T; X)$*, such that the corresponding solution* $\{w(t)\,, w_t(t)\} \in \in L_2(0\,, T; \mathcal{D}(A_{\rho\alpha})) \cap C([0\,, T]; \mathcal{D}(-A_{\rho\alpha})^{1/2})$*,* $\mathcal{D}((-A_{\rho\alpha})^{1/2}) = \mathcal{D}(S^{1-\alpha/2}) \times \mathcal{D}(S^{\alpha/2})$ [8, 6] *of* (1.1.1) *satisfies the terminal rest condition:* $w(T) = 0$*;* $w_t(T) = 0$*. In short: system* (1.1.1) *is exactly null controllable on the energy state space* E*, over any* $[0\,, T]$*, by means of* $L_2(0\,, T; X)$*-controls.*

(*ii*) *Let* $\alpha = 1$*. Then, system* (1.1.1) *is not exactly null controllable on the energy state space* E*, within the class of* $L_2(0\,, T; X)$*-controls [but is exactly null controllable on* E *within the larger class of* $L_2(0\,, T; [\mathcal{D}(S^{1/2})]')$*-controls].*

REMARK 1.1.1. For $\rho^2 = 4\mu_{n^*}^{1-2\alpha}$ for some n^*, in particular for $\rho = 2$ and $\alpha = 1/2$, we refer to Remark 2.2 at the end of Section 2. □

REMARK 1.1.2. The above results complement very recent investigations in S. A. Avdonin and S. A. Ivonov's manuscript, 1997, entitled *Controllability of the wave equation with structural damping* on reachability/controllability properties of a one-dimensional system of type (1.1.1), where $S = -d^2/dx^2$, with right-hand side u replaced, however, by the term $b(\cdot)u(t)$ with separated space and time variables, $b \in X$ and $u(t) \in L_2(0\,,\,T)$ a scalar control. By using the classical moment problem (which requires one-dimensional scalar controls, and hence cannot apply to (1.1.1)), and sharp non-harmonic analysis properties of families of exponential functions [1], this manuscript shows that for $1/2 \leq \alpha \leq 1$ the structurally damped equation there considered is *not* exactly null controllable by scalar $L_2(0\,,\,T)$-controls $u(t)$. Our results with distributed controls $u \in L_2(0\,,\,T; X)$ are, by contrast, positive for $1/2 \leq \alpha < 1$; moreover, for $\alpha = 1$ they are still negative as in Avdonin-Ivanov's work, but with a much larger class $L_2(0\,,\,T; X)$ of controls, while they are positive with the even larger class $L_2(0\,,\,T; [\mathcal{D}(S^{1/2})]')$ of controls. It should be possible to re-obtain the aforementioned negative results of Avdonin-Ivanov by using the approach of this article, which is based on the characterization (2.13). □

1.2. *Second class: An abstract thermo-elastic equation (hinged mechanical B.C./ Dirichlet thermal B.C.).*

In this subsection we consider a special abstract thermo-elastic equation, which corresponds, in particular, to a «concrete» thermo-elastic plate equation with hinged mechanical B.C./Dirichlet thermal B.C. [3]. The distributed control may either be a «mechanical» control, or else a «thermal» control, see eq. (1.2.3) below. Let throughout X be a Hilbert space and $\mathcal{A}: X \supset \mathcal{D}(\mathcal{A}) \to X$ a (strictly) positive, self-adjoint operator with compact resolvent. The equation here considered is

$$y_t = Ay + Bu; \quad y = [\mathcal{A}w\,,\, w_t\,,\, \theta]\,; \tag{1.2.1}$$

$$A = \begin{bmatrix} 0 & \mathcal{A} & 0 \\ -\mathcal{A} & 0 & \mathcal{A} \\ 0 & -\mathcal{A} & -\mathcal{A} \end{bmatrix} : \; H \equiv X \times X \times X \supset \mathcal{D}(A) \equiv \tag{1.2.2}$$

$$\equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \to H\,,$$

and the operator B is

$$\text{either } B = B_m = [0\,,\, I,\, 0]\,, \text{ or else } B = B_\theta = [0\,,\, 0\,,\, I]\,, \tag{1.2.3}$$

corresponding to the mechanical control or the thermal control, respectively. Physically, with reference to the three components of the vector y in (1.2.1), we have that w is the mechanical displacement, w_t its corresponding velocity, and θ the temperature. The main result of the present subsection is

Theorem 1.2.1. *With reference to problem* (1.2.1), *let* $T > 0$. *Given arbitrary initial conditions* $y_0 \in H$, *there exists a control function* $u \in L_2(0, T; X)$, *such that the corresponding solution* $y(t) \in L_2(0, T; \mathcal{D}(A)) \cap C([0, T]; \mathcal{D}(A^{1/2}))$, $\mathcal{D}(A^{1/2}) = \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \times \mathcal{D}(\mathcal{A}^{1/2})$ [6] *of* (1.2.1) *with either* $B = B_m$, *or else* $B = B_\theta$, *as in* (1.2.3) *satisfies the terminal rest condition* $y(T) = 0$. *In short: problem* (1.2.1) *is exactly null controllable on the space* H *within the class of* $L_2(0, T; X)$*-controls.*

Remark 1.2.1. The above result complements the very recent paper [7] which shows exact null controllability in the case of *a one-dimensional* thermo-elastic equation, with hinged B.C., under some restrictions which exclude our model (1.2.2), by means of a scalar boundary control. Reference [7] uses a moment problem approach. □

2. Proof of Theorem 1.1.1.
The case $\alpha = 1/2$: Factorization and diagonalization

Even though it is possible to give a proof of Theorem 1.1.1 which is valid for all values of α, $1/2 \leq \alpha < 1$, see the approach of the subsequent Section 3, it is, however, enlightening to single out the (important) case $\alpha = 1/2$, as it exhibits the simplifying structural/spectral properties of factorization followed by diagonalization (see eq. (2.2) below), which are not shared by the other values of $1/2 < \alpha \leq 1$. A proof based – ultimately – on the diagonalization of the infinitesimal generator A in (2.2) (hence of the analytic semigroup e^{At}) is given here when $\alpha = 1/2$. In this special case, we rewrite more conveniently eq. (1.1.1) with $S^{1/2} = \mathcal{A}$: positive, self-adjoint, unbounded operator on X, with compact resolvent, as

$$w_{tt} + \mathcal{A}^2 w + \rho \mathcal{A} w_t = u; \;\; w(0) = w_0; \;\; w_t(0) = w_1; \;\; \rho > 0. \tag{2.1}$$

The specialization of Theorem 1.1.1 to the dynamics (2.1) is then

Theorem 2.1. *With reference to* (2.1), *let* $T > 0$, *and let* $\rho \neq 2$. *Given arbitrary initial conditions* $\{w_0, w_1\} \in E \equiv \mathcal{D}(\mathcal{A}) \times X$, *there exists a control function* $u \in L_2(0, T; X)$, *such that the corresponding solution* $\{w(t), w_t(t)\} \in C([0, T]; \mathcal{D}(\mathcal{A}^{3/2}) \times \mathcal{D}(\mathcal{A}^{1/2}))$ *of problem* (2.1) *satisfies the terminal rest condition* $w(T) = 0$, $w_t(T) = 0$.

Proof of Theorem 2.1.

Step 1. Setting $y = [\mathcal{A}w, w_t] \in H = X \times X$, we rewrite eq. (2.1) as a first-order equation on H as:

$$\dot{y} = Ay + Bu \text{ on } H = X \times X, \quad A = \begin{bmatrix} 0 & \mathcal{A} \\ -\mathcal{A} & -\rho\mathcal{A} \end{bmatrix} = \mathcal{A}M; \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}; \tag{2.2}$$

$$M = \begin{bmatrix} 0 & 1 \\ -1 & -\rho \end{bmatrix}; \quad \det(zI - M) = z^2 + \rho z + 1 = (z + r_1)(z + r_2) = \tag{2.3}$$

$$= z^2 + (r_1 + r_2)z + r_1 r_2;$$

$$r_1 r_2 = 1; \quad r_1 + r_2 = \rho; \quad \sqrt{r_1 r_2}/(r_1 + r_2) = 1/\rho; \tag{2.4}$$

$$-r_{1,2} = \left[-\rho \pm \sqrt{\rho^2 - 4}\right]/2, \text{ real, if } \rho \geq 2;$$

$$-r_{1,2} = \left[-\rho \pm i\sqrt{4 - \rho^2}\right]/2, \text{ if } 0 < \rho < 2. \tag{2.5}$$

Henceforth, unless otherwise stated, we shall take $0 < \rho$, $\rho \neq 2$, so that the matrix M has two distinct eigenvalues $-r_1$ and $-r_2$ (real if $\rho > 2$; complex conjugate if $0 < \rho < 2$) with corresponding eigenvectors $[1, -r_1]$ and $[1, -r_2]$. Then M is diagonalizable, with diagonalizing matrix P,

$$P^{-1}MP = \begin{bmatrix} -r_1 & 0 \\ 0 & -r_2 \end{bmatrix}; \quad P = \begin{bmatrix} 1 & 1 \\ -r_1 & -r_2 \end{bmatrix}. \tag{2.6}$$

We define the operator $\Pi \equiv P(\text{Identity on } X) \in \mathcal{L}(H)$ and obtain by (2.2), (2.6),

$$A = \begin{bmatrix} 0 & \mathcal{A} \\ -\mathcal{A} & -\rho\mathcal{A} \end{bmatrix} = \Pi\widetilde{A}\Pi^{-1}, \quad \widetilde{A} = \begin{bmatrix} -\mathcal{A}r_1 & 0 \\ 0 & -\mathcal{A}r_2 \end{bmatrix}; \quad \Pi = \begin{bmatrix} I & I \\ -r_1 I & -r_2 I \end{bmatrix}; \tag{2.7}$$

$$e^{At} = \Pi e^{\widetilde{A}t}\Pi^{-1}; \quad e^{\widetilde{A}t} = \begin{bmatrix} e^{-\mathcal{A}r_1 t} & 0 \\ 0 & e^{-\mathcal{A}r_2 t} \end{bmatrix}; \quad \text{Re } r_i > 0. \tag{2.8}$$

To proceed with the proof of Theorem 2.1, it is convenient to distinguish between real distinct eigenvalues ($\rho > 2$) and complex conjugate distinct eigenvalues ($0 < \rho < 2$).

2.1. *Continuation of proof of Theorem 2.1 for $\rho > 2$: Distinct negative roots.*

Step 2. In this case the eigenvalues $-r_1$, $-r_2$ are real negative and distinct and we have

$$\widetilde{A} = \widetilde{A}^* = \begin{bmatrix} -\mathcal{A}r_1 & 0 \\ 0 & -\mathcal{A}r_2 \end{bmatrix}; \quad e^{\widetilde{A}t} = e^{\widetilde{A}^*t} = \begin{bmatrix} e^{-\mathcal{A}r_1 t} & 0 \\ 0 & e^{-\mathcal{A}r_2 t} \end{bmatrix}; \tag{2.9}$$

$$\sqrt{r_1 r_2}/(r_1 + r_2) = 1/\rho < 1/2, \tag{2.10}$$

recalling (2.7), (2.8), (2.4). Applying Π^{-1} on both sides of equation $\dot{y} = Ay + Bu$ in (2.2) yields

$$\widetilde{y}_t = \widetilde{A}\widetilde{y} + \widetilde{B}u; \quad \widetilde{y} = \Pi^{-1}y = \Pi^{-1}\begin{bmatrix} \mathcal{A}w \\ w_t \end{bmatrix}; \tag{2.11}$$

$$\widetilde{B} = \Pi^{-1}B = \Pi^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} = \frac{1}{r_2 - r_1}\begin{bmatrix} I \\ -I \end{bmatrix}; \quad \widetilde{B}^*\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{r_2 - r_1}(y_1 - y_2);$$

$$\widetilde{B} \in \mathcal{L}(X; H), \quad \widetilde{B}^* \in \mathcal{L}(H; X). \tag{2.12}$$

The $\widetilde{y}$-problem in (2.11) on H represents the *diagonalized version* of the original y-equation in (2.2), through the bounded, boundedly invertible operator Π in H.

Accordingly, the exact null-controllability on H, *over* $[0\,,T]$, *of the original dynamics* $y_t = Ay + Bu$ *in* (2.2) *is equivalent to the same property for the diagonalized dynamics* $\widetilde{y}_t = \widetilde{A}\widetilde{y} + \widetilde{B}u$ *in* (2.11). $\square$

Step 3. It is well known that, via functional analytic arguments, exact null-controllability in H for the $\widetilde{y}$-dynamics in (2.11) within the class of $L_2(0\,,T;X)$-controls, is *equivalent* to the following inequality: there exists a constant $C_T > 0$ such that

$$\int_0^T \left\| \widetilde{B}^* e^{\widetilde{A}^* t} x \right\|_U^2 dt \geq C_T \left\| e^{\widetilde{A}^* T} x \right\|_H^2, \quad \forall\, x \in H, \tag{2.13}$$

where, in the present distributed case, $U = H$. Thus, the crux of our proof consists in verifying the validity of inequality (2.13).

Step 4. Let, at first, the positive self-adjoint operator $\mathcal{A}$ have *simple* eigenvalues $\{\mu_n\}$ with corresponding eigenvectors $\{e_n\}$ forming an orthonormal basis in X,

$$\mathcal{A}e_n = \mu_n e_n, \quad 0 < \mu_n \to +\infty. \tag{2.14}$$

For the multiple eigenvalue case, see Step 9 below. Then, by (2.14), we obtain

$$e^{-\mathcal{A}r_1 t}x_1 = \sum_{n=1}^{\infty} e^{-\mu_n r_1 t}(x_1\,, e_n)_X e_n, \quad x_1 \in X; \tag{2.15}$$

$$e^{-\mathcal{A}r_2 t}x_2 = \sum_{n=1}^{\infty} e^{-\mu_n r_2 t}(x_2\,, e_n)_X e_n, \quad x_2 \in X; \tag{2.16}$$

$$e^{-\mathcal{A}r_1 t}x_1 - e^{-\mathcal{A}r_2 t}x_2 = \sum_{n=1}^{\infty} \left[e^{-\mu_n r_1 t}(x_1\,, e_n)_X - e^{-\mu_n r_2 t}(x_2\,, e_n)_X \right] e_n. \tag{2.17}$$

Thus, by recalling $\widetilde{B}^*$ from (2.12), $e^{\widetilde{A}^* t}$ from (2.9), as well as (2.15), (2.16), we obtain with $x = [x_1\,, x_2] \in H$:

$$\widetilde{B}^* e^{\widetilde{A}^* t} x = [r_2 - r_1]^{-1} \left[e^{-\mathcal{A}r_1 t}x_1 - e^{-\mathcal{A}r_2 t}x_2 \right] \tag{2.18}$$

$$\left\| \widetilde{B}^* e^{\widetilde{A}^* t} x \right\|_H^2 = |r_2 - r_1|^{-2} \left\| e^{-\mathcal{A}r_1 t}x_1 - e^{-\mathcal{A}r_2 t}x_2 \right\|_X^2 \tag{2.19}$$

$$(\text{by } (2.17)) = |r_2 - r_1|^{-2} \sum_{n=1}^{\infty} \left| e^{-\mu_n r_1 t}(x_1\,, e_n)_X - e^{-\mu_n r_2 t}(x_2\,, e_n)_X \right|^2. \tag{2.20}$$

Hence, setting for $x_i \in X$,

$$\alpha_n \equiv (x_1\,, e_n)_X \in \ell_2; \;\; \beta_n = (x_2\,, e_n)_X \in \ell_2, \tag{2.21}$$

we obtain from (2.20), (2.21) the expression for the left-hand side of (2.13):

$$\int_0^T \left\| \widetilde{B}^* e^{\widetilde{A}^* t} x \right\|_H^2 dt = |r_2 - r_1|^{-2} \sum_{n=1}^{\infty} \int_0^T \left| e^{-\mu_n r_1 t}\alpha_n - e^{-\mu_n r_2 t}\beta_n \right|^2 dt. \tag{2.22}$$

Similarly, from $e^{\widetilde{A}^* T}$ in (2.9) and (2.15), (2.16), we obtain

$$\left\| e^{\widetilde{A}^* T} x \right\|_H^2 = \left\| e^{-\mathcal{A} r_1 T} x_1 \right\|_X^2 + \left\| e^{-\mathcal{A} r_2 T} x_2 \right\|_X^2 \tag{2.23}$$

$$(\text{by (2.15), (2.16)}) = \sum_{n=1}^{\infty} \left[e^{-2\mu_n r_1 T} |(x_1, e_n)_X|^2 + e^{-2\mu_n r_2 T} |(x_2, e_n)_X|^2 \right] \tag{2.24}$$

$$(\text{by (2.21)}) = \sum_{n=1}^{\infty} \left[e^{-2\mu_n r_1 T} |\alpha_n|^2 + e^{-2\mu_n r_2 T} |\beta_n|^2 \right], \tag{2.25}$$

where in the last step we have recalled (2.21). Using (2.22) and (2.25) in inequality (2.13), we obtain

Lemma 2.2. *When $\mathcal{A}$ has simple eigenvalues, the characterization for exact null-controllability in H with $L_2(0, T; X)$-controls of the $\widetilde{y}$-dynamics in* (2.11), *equivalently of the original y-dynamics in* (2.2); i.e., *of* (2.1), *is via* (2.21) *rewritten as the following inequality:*

$$\sum_{n=1}^{\infty} \int_0^T \left| e^{-\mu_n r_1 t} \alpha_n - e^{-\mu_n r_2 t} \beta_n \right|^2 dt \geq C_T \sum_{n=1}^{\infty} \left[e^{-2\mu_n r_1 T} |\alpha_n|^2 + e^{-2\mu_n r_2 T} |\beta_n|^2 \right]. \tag{2.26}$$

Step 5. Thus, the crux of our proof consists in showing the validity of inequality (2.26). To this end, the following lemma is fundamental.

Lemma 2.3. *Let $a, b > 0$ be two positive constants, α and β two possibly complex constants. Assume that $k \equiv \sqrt{ab}/(a+b) < 1/2$. Then, the following inequalities hold true:*

(*i*)

$$\int_0^T \left| e^{-at}\alpha - e^{-bt}\beta \right|^2 dt \geq \frac{1}{2}(1-2k) \left[\left| \frac{\alpha}{\sqrt{a}} \right|^2 + \left| \frac{\beta}{\sqrt{b}} \right|^2 \right] + -\frac{1}{2}(1+2k) \left[e^{-2aT} \left| \frac{\alpha}{\sqrt{a}} \right|^2 + e^{-2bT} \left| \frac{\beta}{\sqrt{b}} \right|^2 \right]; \tag{2.27}$$

(*ii*)

$$\int_0^T \left| e^{-at}\alpha - e^{-bt}\beta \right|^2 dt \geq \frac{1}{2} \left\{ \frac{(1-2k)e^{2aT}}{a} - \frac{(1+2k)}{a} \right\} e^{-2aT} |\alpha|^2 + + \frac{1}{2} \left\{ \frac{(1-2k)e^{2bT}}{b} - \frac{(1+2k)}{b} \right\} e^{-2bT} |\beta|^2. \tag{2.28}$$

Proof. We have, using $|z|^2 = z \cdot \bar{z}$:

$$|e^{-at}\alpha - e^{-bt}\beta|^2 = e^{-2at}|\alpha|^2 + e^{-2bt}|\beta|^2 - e^{-(a+b)t} 2 \operatorname{Re}(\alpha\overline{\beta}). \tag{2.29}$$

Setting

$$\widetilde{\alpha} \equiv e^{-aT}\alpha; \ \widetilde{\beta} \equiv e^{-bT}\beta; \text{ hence } \alpha = e^{aT}\widetilde{\alpha}, \ \beta = e^{bT}\widetilde{\beta}, \tag{2.30}$$

and integrating (2.29) over $[0\,,\,T]$ yields

$$(2.31)\qquad \int_0^T \left| e^{-at}\alpha - e^{-bt}\beta \right|^2 dt = \frac{(1-e^{-2aT})}{2}\left|\frac{\alpha}{\sqrt{a}}\right|^2 + \frac{(1-e^{-2bT})}{2}\left|\frac{\beta}{\sqrt{b}}\right|^2 + - 2\left(1-e^{-(a+b)T}\right)\frac{\sqrt{ab}}{a+b}\,\mathrm{Re}\left[\left(\frac{\alpha}{\sqrt{a}}\right)\left(\frac{\overline{\beta}}{\sqrt{b}}\right)\right]$$

(by (2.30))

$$(2.32)\qquad = \frac{1}{2}\left\{\left|\frac{\alpha}{\sqrt{a}}\right|^2 + \left|\frac{\beta}{\sqrt{b}}\right|^2 - \frac{4\sqrt{ab}}{a+b}\,\mathrm{Re}\left[\left(\frac{\alpha}{\sqrt{a}}\right)\left(\frac{\overline{\beta}}{\sqrt{b}}\right)\right]\right\} + -\frac{1}{2}\left\{\left|\frac{\widetilde{\alpha}}{\sqrt{a}}\right|^2 + \left|\frac{\widetilde{\beta}}{\sqrt{b}}\right|^2 - \frac{4\sqrt{ab}}{a+b}\,\mathrm{Re}\left[\left(\frac{\widetilde{\alpha}}{\sqrt{a}}\right)\left(\frac{\overline{\widetilde{\beta}}}{\sqrt{b}}\right)\right]\right\}$$

(by (2.34) below)

$$(2.33)\qquad \geq \frac{(1-2k)}{2}\left[\left|\frac{\alpha}{\sqrt{a}}\right|^2 + \left|\frac{\beta}{\sqrt{b}}\right|^2\right] - \frac{(1+2k)}{2}\left[\left|\frac{\widetilde{\alpha}}{\sqrt{a}}\right|^2 + \left|\frac{\widetilde{\beta}}{\sqrt{b}}\right|^2\right],$$

where, in the last step from (2.32) to (2.33), we have recalled $k \equiv [\sqrt{ab}/(a+b)] < 1/2$; and, moreover, we have used the inequalities

$$(2.34)\qquad |p|^2 + |q|^2 - 2k \cdot 2\,\mathrm{Re}(p\bar{q}\,) \begin{cases} \geq & (1-2k)[|p|^2 + |q|^2]; \\ \leq & (1+2k)[|p|^2 + |q|^2], \end{cases}$$

twice, once with $p = \alpha/\sqrt{a}$, $q = \beta/\sqrt{b}$; and once with $p = \widetilde{\alpha}/\sqrt{a}$ and $q = \widetilde{\beta}/\sqrt{b}$. Then, (2.33) shows (2.27), as desired, via $\widetilde{a}$ and $\widetilde{\beta}$ in (2.30). Then (2.27) readily yields (2.28). □

Step 6. We apply Lemma 2.3(*ii*), eq. (2.28) with

$$(2.35)\qquad \begin{cases} a = \mu_n r_1 > 0; \;\; b = \mu_n r_2 > 0; \\ k = \sqrt{ab}/(a+b) = \sqrt{r_1 r_2}/(r_1 + r_2) = 1/\rho < 1/2; \\ \alpha = (x_1\,,\,e_n)_X \equiv \alpha_n; \;\; \beta = (x_2\,,\,e_n)_X \equiv \beta_n\,, \end{cases}$$

where we have recalled (2.10), and obtain by (2.28), (2.35),

$$(2.36)\qquad \int_0^T \left| e^{-\mu_n r_1 t}\alpha_n - e^{-\mu_n r_2 t}\beta_n \right|^2 dt \geq$$

$$\geq \frac{1}{2}\left\{\left(1-\frac{2}{\rho}\right)\frac{e^{2\mu_n r_1 T}}{\mu_n r_1} - \left(1+\frac{2}{\rho}\right)\frac{1}{\mu_n r_1}\right\}\left[e^{-2\mu_n r_1 T}|\alpha_n|^2\right] + + \frac{1}{2}\left\{\left(1-\frac{2}{\rho}\right)\frac{e^{2\mu_n r_2 T}}{\mu_n r_2} - \left(1+\frac{2}{\rho}\right)\frac{1}{\mu_n r_2}\right\}\left[e^{-2\mu_n r_2 T}|\beta_n|^2\right].$$

Step 7. Given $T > 0$ fixed (but arbitrary), there exists a positive integer $N = N_T$, depending on T (as well as on ρ, which is fixed), such that

$$\frac{1}{2}\left\{\left(1-\frac{2}{\rho}\right)\frac{e^{2\mu_n r_i T}}{\mu_n r_i}-\left(1+\frac{2}{\rho}\right)\frac{1}{\mu_n r_i}\right\} \geq C_{T\rho N} > 0; \qquad \forall\ n \geq N_T;\ i = 1\,, 2\,, \tag{2.37}$$

where we recall that $2/\rho < 1$ by (2.35): this is possible, since $\mu_n r_i \to +\infty$ as $n \to +\infty$, $i = 1\,, 2$. Then, inequality (2.37) used in (2.36) yields (recalling α and β in (2.35)),

$$\int_0^T \left|e^{-\mu_n r_1 t}(x_1\,, e_n)_X - e^{-\mu_n r_2 t}(x_2\,, e_n)_X\right|^2 dt \geq \geq C_{T\rho N}\left[e^{-2\mu_n r_1 T}|(x_1\,, e_n)_X|^2 + e^{-2\mu_n r_2 T}|(x_2\,, e_n)_X|^2\right]\,, \quad \forall\ n \geq N_T. \tag{2.38}$$

A fortiori, (2.38) implies for $N = N_T$:

$$\sum_{n=N}^{\infty}\int_0^T \left|e^{-\mu_n r_1 t}(x_1\,, e_n)_X - e^{-\mu_n r_2 t}(x_2\,, e_n)_X\right|^2 dt \geq \geq C_{T\rho N}\sum_{n=N}^{\infty}\left[e^{-2\mu_n r_1 T}|(x_1\,, e_n)_X|^2 + e^{-2\mu_n r_2 T}|(x_2\,, e_n)_X|^2\right]. \tag{2.39}$$

Eq. (2.39) is the «right» sought-after inequality (2.26), only from $n = N_T$ on, however.

Step 8. We now analyze the same inequality (2.39) up to any positive integer N.

LEMMA 2.4. *Let $T > 0$ and let N be any positive integer. Then, there exists a positive constant $\epsilon_{TN} > 0$, such that:*

$$K_{TN} \equiv \sum_{n=1}^{N-1}\int_0^T \left|e^{-\mu_n r_1 t}(x_1\,, e_n)_X - e^{-\mu_n r_2 t}(x_2\,, e_n)_X\right|^2 dt \geq \geq \epsilon_{TN}\sum_{n=1}^{N-1}\left[e^{-2\mu_n r_1 T}|(x_1\,, e_n)_X|^2 + e^{-2\mu_n r_2 T}|(x_2\,, e_n)_X|^2\right]. \tag{2.40}$$

PROOF. First, for the left-hand side K_{TN} of inequality (2.40), the following alternative holds true: (*i*) either $K_{TN} = 0$, in which case – due to the linear independence of the exponentials $e^{-\mu_n r_1 t}$ and $e^{-\mu_n r_2 t}$ with $r_1 \neq r_2$ under present assumption ($\rho > 2$) – it would follow that $(x_1\,, e_n)_X = (x_2\,, e_n)_X \equiv 0\,,\ \forall\ n = 1\,, \dots\,, N-1$, and then the right-hand side of (2.40) would vanish as well; and (2.40) would hold true as an equality for all $\epsilon_{TN} > 0$; (*ii*) or else $K_{TN} > 0$, in which case there surely exists some constant $\epsilon_{TN} > 0$, such that (2.40) holds true. □

Step 9. We apply Lemma 2.4 for $N =$ the positive integer N_T provided by (2.39). Summing up (2.39) and (2.40) for such $N = N_T$, then yields inequality (2.26), as desired. Thus, *the exact null-controllability inequality* (2.13) *is proved*: problem (2.11) in $\widetilde{y}$, equivalently problem (2.2) in y, or (2.1) in w, is exactly null controllable in H, with $L_2(0, T; X)$-controls, at least in the present case $\rho > 2$, where $\mathcal{A}$ is assumed to have simple eigenvalues.

However, in general, if $\rho > 2$ and the eigenvalues $\mu_n > 0$ of the positive, self-adjoint operator $\mathcal{A}$ have eigenvectors e_{nk}, $n = 1, 2, \ldots$; $k = 1, \ldots, K_n$, forming an orthonormal basis on X, then expansions (2.15), (2.16), (2.18), (2.22), and (2.25) become

$$e^{-\mathcal{A}r_i t}x_i = \sum_{n=1}^{\infty}\sum_{k=1}^{K_n} e^{-\mu_n r_i t}(x_i, e_{nk})_X e_{nk}, \quad i = 1, 2; \tag{2.41}$$

$$\widetilde{B}^* e^{\widetilde{A}^* t}x = [r_2 - r_1]^{-1}\sum_{n=1}^{\infty}\sum_{k=1}^{K_n}\left[e^{-\mu_n r_1 t}(x_1, e_{nk})_X - e^{-\mu_n r_2 t}(x_2, e_{nk})_X\right] e_{nk}; \tag{2.42}$$

$$\int_0^T \left\|\widetilde{B}^* e^{\widetilde{A}^* t}x\right\|_X^2 dt = \tag{2.43}$$

$$= |r_2 - r_1|^{-2}\sum_{n=1}^{\infty}\sum_{k=1}^{K_n}\int_0^T \left|e^{-\mu_n r_1 t}(x_1, e_{nk})_X - e^{-\mu_n r_2 t}(x_2, e_{nk})_X\right|^2 dt;$$

$$\left\|e^{\widetilde{A}^* T}x\right\|_H^2 = \left\|e^{-\mathcal{A}r_1 T}x_1\right\|_X^2 + \left\|e^{-\mathcal{A}r_2 T}x_2\right\|_X^2 = \tag{2.44}$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{K_n}\left[e^{-\mu_n r_1 T}|(x_1, e_{nk})_X|^2 + e^{-2\mu_n r_2 T}|(x_2, e_{nk})_X|^2\right].$$

Then, the above argument for simple eigenvalues of $\mathcal{A}$, centered on Lemma 2.3 and Lemma 2.4, extends to the multiple eigenvalue case for $\mathcal{A}$. The proof for $\rho > 2$ is complete.

REMARK 2.1. Theorem 2.1 continues to hold true even if we take 'rough' (compatible) initial data; say,

$$y_0 \equiv \{w_0, w_1\} \in [\mathcal{D}(\mathcal{A}^{s-1})]' \times [\mathcal{D}(\mathcal{A}^s)]' \equiv E_s \text{ for any } s > 0, \tag{2.45}$$

rather than $y_0 \equiv \{w_0, w_1\} \in E \equiv \mathcal{D}(\mathcal{A}) \times X$. In (2.45), we have taken duality with respect to the pivot space X. This extension can be seen in two ways:

(1) in one approach, we first take control $u \equiv 0$ on $0 \le t \le \epsilon$, with $\epsilon > 0$ arbitrary, during which time the solution $y(t) = e^{At}y_0$ is regularized by the analytic semigroup (extended to E_s), so that $y(\epsilon) = e^{A\epsilon}y_0 \in E \equiv \mathcal{D}(\mathcal{A}) \times X$. Next, we use the above argument of exact null-controllability on the interval $[\epsilon, T + \epsilon]$.

(2) Alternatively, we can draw the same conclusion from the same estimates above: we return to (2.36), multiply and divide its right-hand side by μ_n^{2s}, $s > 0$ arbitrary but fixed, whereby then (2.37) is replaced by

$$\frac{1}{2}\left\{\left(1-\frac{2}{\rho}\right)\frac{e^{2\mu_n r_i T}}{\mu_n^{2s+1} r_i} - \left(1+\frac{2}{\rho}\right)\frac{1}{\mu_n^{2s+1} r_i}\right\} \geq C_{T\rho Ns} > 0\,, \qquad \forall\ n \geq N_T\,,\ \ i = 1\,,2\,, \tag{2.46}$$

and (2.38) then becomes

$$\int_0^T \left|e^{-\mu_n r_1 t}(x_1\,,e_n)_X - e^{-\mu_n r_2 t}(x_2\,,e_n)_X\right|^2 dt \geq \geq C_{T\rho Ns}\left[e^{-2\mu_n r_1 T}\mu_n^{2s}|(x_1\,,e_n)_X|^2 + e^{-2\mu_n r_2 T}\mu_n^{2s}|(x_2\,,e_n)_X|^2\right]\,, \qquad \forall\ n \geq N_T\,, \tag{2.47}$$

with, now, $\mu_n(x_i\,,e_n) \in \mathcal{D}(\mathcal{A}^s)$, as desired. □

2.2. *Continuation of proof of Theorem 2.1 for* $0 < \rho < 2$*: Distinct complex conjugate roots.*

Step 1. In this case, the eigenvalues $-r_1$, $-r_2$ in (2.5) are distinct, complex conjugate: $-r_{1,2} = [-\rho \pm i\sqrt{4-\rho^2}]/2$, $\bar{r}_2 = r_1$, in which case $\widetilde{A}^*$ and $e^{\widetilde{A}^* t}$ are now, recalling $\widetilde{A}$ and $e^{\widetilde{A}t}$ in (2.7), (2.8):

$$\widetilde{A}^* = \begin{bmatrix} -\mathcal{A}r_2 & 0 \\ 0 & -\mathcal{A}r_1 \end{bmatrix}; \quad e^{\widetilde{A}^* t} = \begin{bmatrix} e^{-\mathcal{A}r_2 t} & 0 \\ 0 & e^{-\mathcal{A}r_1 t} \end{bmatrix}; \tag{2.48}$$

i.e., r_1 and r_2 [or x_1 and x_2 in X] are interchanged with respect to (2.9). Via (2.15), (2.16), (2.12), we see that the characterization (2.13) for exact null controllability in H, with $L_2(0\,,T;X)$-controls, reads now

$$\int_0^T \left\|\widetilde{B}^* e^{\widetilde{A}^* t}x\right\|_{U=H}^2 dt = |r_2 - r_1|^{-2}\sum_{n=1}^{\infty}\int_0^T \left|e^{-\mu_n r_2 t}\alpha_n - e^{-\mu_n r_1 t}\beta_n\right|^2 dt \geq \geq C_T\left\|e^{\widetilde{A}^* T}x\right\|_H^2 = C_T\sum_{n=1}^{\infty}\left\{e^{-\mu_n\rho T}[|\alpha_n|^2 + |\beta_n|^2]\right\}; \tag{2.49}$$

$$\text{Re } r_1 = \text{Re } r_2 = \rho/2;\ \ \alpha_n = (x_1\,,e_n)_X\,,\ \ \beta_n = (x_2\,,e_n)_X\,, \tag{2.50}$$

which is the counterpart of eq. (2.26) in Section 2.1 ($\rho > 2$).

Step 2. The crux of the proof now is the following Lemma 2.5, which is the counterpart of Lemma 2.3.

LEMMA 2.5. *Let* $a = a_1 + ia_2$ *and* $b = a_1 - ia_2 = \bar{a}$, $a_i \in \mathbf{R}$ *be two complex conjugate numbers,* $a_1 > 0$. *Let* α , β *be two possibly complex constants. Assume that* $2h \equiv |a_1/a| < 1$. *Then, the following inequalities hold true:*

(*i*)

$$(2.51)\qquad \int_0^T \left|e^{-at}\alpha - e^{-bt}\beta\right|^2 dt \ge (1-2h)\left[\left|\alpha/\sqrt{a_1}\right|^2 + \left|\beta/\sqrt{a_1}\right|^2\right]/2 + \\ -(1+2h)e^{-2a_1 T}\left[\left|\alpha/\sqrt{a_1}\right|^2 + \left|\beta/\sqrt{a_1}\right|^2\right]/2;$$

(*ii*)

$$(2.52)\qquad \int_0^T \left|e^{-at}\alpha - e^{-bt}\beta\right|^2 dt \ge \frac{1}{2}\left\{\frac{(1-2h)}{a_1}e^{2a_1 T} - \frac{(1+2h)}{a_1}\right\}e^{-2a_1 T}\left[|\alpha|^2 + |\beta|^2\right].$$

PROOF. We follow the proof of Lemma 2.3, as appropriately modified under present assumptions. The counterpart of eq. (2.29), now with $b = \bar{a}$ is

$$(2.53)\qquad \left|e^{-at}\alpha - e^{-bt}\beta\right|^2 = e^{-2a_1 t}[|\alpha|^2 + |\beta|^2] - 2\ \mathrm{Re}\left(e^{-2at}\alpha\overline{\beta}\right),$$

so that, setting now (compare with (2.30))

$$(2.54)\qquad \widetilde{\alpha} \equiv e^{-a_1 T}\alpha;\ \widetilde{\beta} \equiv e^{-a_1 T}\beta;\ \text{hence } \alpha = \widetilde{\alpha}e^{a_1 T};\ \beta = \widetilde{\beta}e^{a_1 T},$$

we have, by (2.53), that the counterpart of (2.31)-(2.33) is:

$$(2.55)\qquad \int_0^T \left|e^{-at}\alpha - e^{-bt}\beta\right|^2 dt = \\ = \frac{(1-e^{-2a_1 T})}{2}\left[\left|\frac{\alpha}{\sqrt{a_1}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2\right] - 2\ \mathrm{Re}\left[\frac{1-e^{-2aT}}{2\alpha}\alpha\overline{\beta}\right] =$$

$$(2.56)\qquad = \frac{1}{2}\left\{\left[\left|\frac{\alpha}{\sqrt{a_1}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 - 4\ \mathrm{Re}\left(\frac{\alpha}{\sqrt{a_1}}\frac{\overline{\beta}}{\sqrt{a_1}}\frac{a_1}{2a}\right)\right] + \\ -\left[\left|e^{-a_1 T}\frac{\alpha}{\sqrt{a_1}}\right|^2 + \left|e^{-a_1 T}\frac{\beta}{\sqrt{a_1}}\right|^2 - 4\ \mathrm{Re}\left(\frac{e^{-a_1 T}\alpha}{\sqrt{a_1}}\frac{e^{-a_1 T}\overline{\beta}}{\sqrt{a_1}}e^{-2ia_2 T}\frac{a_1}{2a}\right)\right]\right\}$$

$$(2.57)\qquad (\text{by } (2.54)) = \frac{1}{2}\left\{\left[\left|\frac{\alpha}{\sqrt{a_1}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 - 4\ \mathrm{Re}\left(\frac{\alpha}{\sqrt{a_1}}\frac{\overline{\beta}}{\sqrt{a_1}}\frac{a_1}{2a}\right)\right] + \\ -\left[\left|\frac{\widetilde{\alpha}}{\sqrt{a_1}}\right|^2 + \left|\frac{\widetilde{\beta}}{\sqrt{a_1}}\right|^2 - 4\ \mathrm{Re}\left(\frac{\widetilde{\alpha}}{\sqrt{a_1}}\frac{\overline{\widetilde{\beta}}}{\sqrt{a_1}}e^{-2ia_2 T}\frac{a_1}{2a}\right)\right]\right\}$$

$$\text{(2.58)} \qquad \geq \frac{1}{2}\left\{\left[\left|\frac{\alpha}{\sqrt{a_1}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 - 2\left|\frac{a_1}{2a}\right| 2 \left|\frac{\alpha}{\sqrt{a_1}}\right| \left|\frac{\beta}{\sqrt{a_1}}\right|\right] + \right.$$

$$\left. - \left[\left|\frac{\tilde{\alpha}}{\sqrt{a_1}}\right|^2 + \left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right|^2 + 2 \cdot 1 \cdot \left|\frac{a_1}{2a}\right| 2 \left|\frac{\tilde{\alpha}}{\sqrt{a_1}}\right| \left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right|\right]\right\}.$$

Recalling that $2h \equiv |a_1/a| < 1$ by assumption, we obtain from (2.58),

$$\text{(2.59)} \qquad \int_0^T \left|e^{-at}\alpha - e^{-bt}\beta\right|^2 dt \geq \frac{1}{2}(1-2h)\left[\left|\frac{\alpha}{\sqrt{a_1}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2\right] +$$

$$-\frac{1}{2}(1+2h)\left[\left|\frac{\tilde{\alpha}}{\sqrt{a_1}}\right|^2 + \left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right|^2\right].$$

Finally, (2.59) yields (2.51) as desired, by use of (2.54). Then (2.52) follows at once from (2.51). □

Step 3. We now verify that, in the present complex conjugate case where $0 < \rho < 2$, then $h \equiv |a_1/(2a)| < 1/2$, as required by Lemma 2.5, if we take (recall (2.5)):

$$\text{(2.60)} \qquad \begin{cases} a = \mu_n r_2 = \mu_n\left(\rho + i\sqrt{4-\rho^2}\right)/2\,, \ \text{hence } a_1 = \mu_n \rho/2\,, \\[2ex] 4|a|^2 = \mu_n^2[\rho^2 + 4 - \rho^2] = 4\mu_n^2; \ h^2 = \dfrac{a_1^2}{4|a|^2} = \dfrac{\mu_n^2\rho^2/4}{4\mu_n^2} < \dfrac{1}{4}\,, \end{cases}$$

i.e., $h < 1/2$, as desired, since $0 < \rho < 2$. We then apply Lemma 2.5 with $a = \mu_n r_2 = \bar{b}$, $b = \mu_n r_1$ as in (2.60), and obtain by (2.52),

$$\text{(2.61)} \qquad \int_0^T \left|e^{-\mu_n r_2 t}\alpha_n - e^{-\mu_n r_1 t}\beta_n\right|^2 dt \geq$$

$$\geq \left\{(1-2h)(\mu_n\rho)^{-1} e^{\mu_n\rho T} - (1+2h)(\mu_n\rho)^{-1}\right\} e^{-\mu_n\rho T}[|\alpha_n|^2 + |\beta_n|^2]\,, \qquad n = 1, 2, \dots\,,$$

with $\alpha_n = (x_1, e_n)_X$ and $\beta_n = (x_2, e_n)_X$ as in (2.50).

Step 4. As in Step 7 of Section 2.1, given $T > 0$, there exists a positive integer $N = N_T$, depending on T (as well as on ρ, which is fixed), such that

$$\text{(2.62)} \qquad \left\{(1-2h)(\mu_n\rho)^{-1} e^{\mu_n\rho T} - (1+2h)(\mu_n\rho)^{-1}\right\} \geq C_{T\rho N} > 0\,, \quad \forall\, n \geq N_T;$$

$$\text{(2.63)} \qquad \int_0^T \left|e^{-\mu_n r_2 t}\alpha_n - e^{-\mu_n r_1 t}\beta_n\right|^2 dt \geq C_{T\rho N} e^{-\mu_n\rho T}[|\alpha_n|^2 + |\beta_n|^2]\,, \ \forall\, n \geq N_T\,,$$

and hence

$$\sum_{n=N}^{\infty}\int_0^T \left|e^{-\mu_n r_2 t}\alpha_n - e^{-\mu_n r_1 t}\beta_n\right|^2 dt \geq C_{T\rho N}\sum_{n=N}^{\infty} e^{-\mu_n \rho T}[|\alpha_n|^2 + |\beta_n|^2]\,, \tag{2.64}$$

which is the counterpart of (2.39).

Step 5. The analysis of the same inequality (2.64), this time up to N, is the same as in Step 8 (Lemma 2.4) of Section 2.1; since the proof is based on the property $r_1 \neq r_2$, which still holds true.

LEMMA 2.6. *Let $T > 0$ and let N be any positive integer. Then there exists a positive constant $\epsilon_{TN} > 0$ such that*

$$\sum_{n=1}^{N-1}\int_0^T \left|e^{-\mu_n r_2 t}\alpha_n - e^{-\mu_n r_1 t}\beta_n\right|^2 dt \geq \epsilon_{TN}\sum_{n=1}^{N-1} e^{-\mu_n \rho T}[|\alpha_n|^2 + |\beta_n|^2]. \qquad \square \tag{2.65}$$

To obtain the sought-after estimate (2.49), and thus complete the proof of Theorem 2.1 in the present case $0 < \rho < 2$ with simple eigenvalues μ_n of $\mathcal{A}$, we sum up inequalities (2.64) and (2.65).

The multiple eigenvalue case can be dealt with as at the end of Section 2.1.

REMARK 2.2. In the case $\rho = 2$ so far excluded, we have $r_1 = r_2$ by (2.5) and so the matrix $P^{-1}MP$ in (2.6) becomes a Jordan cell. The operator $\widetilde{A}$ and its s.c. analytic semigroup $e^{\widetilde{A}t}$ have a Jordan structure. To analyze this case, we recall [5, Lemma A.1(*iv*)(*b*), p. 47] that then $\psi_n^- = [0\,,\, e_n]$ is a generalized eigenvector of $\mathcal{A}$; and that, moreover, $\overline{\text{span}}\{\Phi_n^{+,-}\,,\,\psi_n^-\} = E$, see [5, eq. (A.17)]. (In [5], the present ρ is replaced by 2ρ). This special case has not been studied in detail yet. $\square$

3. PROOF OF THEOREM 1.1.1: THE CASE $1/2 \leq \alpha \leq 1$

ORIENTATION. When $1/2 < \alpha \leq 1$, the factorization property followed by diagonalization, displayed by the operator A in (2.2) for $\alpha = 1/2$ is no longer true. However, for the entire range $1/2 \leq \alpha \leq 1$ of analyticity of the underlying semigroup $e^{A_{\rho\alpha}}$ [5] corresponding to the generator $A_{\rho\alpha}$ in (1.1.2), a different set of spectral properties of $A_{\rho\alpha}$ hold true, as pointed out in [5, Appendix A]. Accordingly, in the present proof we shall exploit a subset of the aforementioned spectral theory for $\rho^2 \neq 4\mu_n^{1-2\alpha}$, $\forall\, n$, where μ_n are the eigenvalues of the operator S, see (3.4). It turns out that there is a basic difference between the set of cases $1/2 \leq \alpha \leq 1$ on the one hand and the isolated case $\alpha = 1$ on the other hand. Indeed, in the former situation $1/2 \leq \alpha < 1$, the eigenvalues of $A_{\rho\alpha}$ consist of two branches, λ_n^- and λ_n^+ having the following properties: both of them are real negative for $\rho^2 > 4\mu_n^{1-2\alpha}$, where $\mu_n^{1-2\alpha} \searrow 0$ as $n \to \infty$ for $1/2 < \alpha \leq 1$; and, moreover, both of them tend to $-\infty$: λ_n^-, $\lambda_n^+ \to -\infty$, as $n \to +\infty$, for $1/2 < \alpha < 1$; and for $\alpha = 1/2$, $\rho > 2$. By contrast, in the isolated case $\alpha = 1$, we still have $\lambda_n^- \to -\infty$; however, now, $\lambda_n^+ \nearrow$ finite limit $-1/\rho < 0$ as

$n \to \infty$. (Thus, $A_{\rho\alpha}$ has compact resolvent for $\alpha < 1$, but not for $\alpha = 1$ on the energy space $E = \mathcal{D}(S^{1/2}) \times X$ [5, Appendix A]). The fact that the branch $\lambda_n^+ \to$ a *finite* limit $-\ell \neq 0$ for $\alpha = 1$ will be responsible for the failure of Theorem 1.1.1 in this case, see the counterexample below, eqs. (3.3.1) and (3.3.4).

PROOF OF THEOREM 1.1.1(*i*) FOR $1/2 \le \alpha < 1$. With reference to eq. (1.1.1), we shall now set $y(t) = [S^{1/2} w(t), -w_t(t)]$ and obtain

$$y_t = \mathbf{A}^*_{\rho\alpha} y + Bu \text{ on } H \equiv X \times X; B = \begin{bmatrix} 0 \\ -I \end{bmatrix}; \; B^* \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -y_2; \tag{3.1}$$

$$\mathbf{A}^*_{\rho\alpha} = \begin{bmatrix} 0 & -S^{1/2} \\ S^{1/2} & -\rho S^{\alpha} \end{bmatrix}; \; \mathbf{A}_{\rho\alpha} = \begin{bmatrix} 0 & S^{1/2} \\ -S^{1/2} & -\rho S^{\alpha} \end{bmatrix}; \tag{3.2}$$

$$H \supset \mathcal{D}(\mathbf{A}^*_{\rho\alpha}) = \{[x_1, x_2] \in H: \; x_2 \in \mathcal{D}(S^{1/2}); [x_1 + \rho S^{\alpha - 1/2} x_2] \in \mathcal{D}(S^{1/2})\}. \tag{3.3}$$

As seen in Section 2, with no loss of generality we may let the positive self-adjoint (unbounded) operator S with compact resolvent have simple eigenvalues $\{\mu_n\}$, with *non-normalized* (in X) eigenvectors $\{e_n\}$ forming an orthogonal basis on X.

$$Se_n = \mu_n e_n, \quad 0 < \mu_n \nearrow +\infty. \tag{3.4}$$

Then, the operator $\mathbf{A}_{\rho\alpha}$ in (3.2) has eigenvalues $\{\lambda_n^{+,-}\}$ and *normalized* (in H) eigenvectors $\Phi_n^{+,-}$, as follows:

$$(\lambda_n^{+,-})^2 + \rho\mu_n^{\alpha}\lambda_n^{+,-} + \mu_n = 0, \; \lambda_n^+\lambda_n^- = \mu_n; \; -\lambda_n^+ - \lambda_n^- = \rho\mu_n^{\alpha}; \tag{3.5a}$$

$$\lambda_n^{+,-} = \left(-\rho\mu_n^{\alpha} \pm \sqrt{\rho^2\mu_n^{2\alpha} - 4\mu_n}\right)/2; \quad \lambda_n^+ = -2\mu_n/\left(\sqrt{\rho^2\mu_n^{2\alpha} - 4\mu_n} + \rho\mu_n^{\alpha}\right); \tag{3.5b}$$

$$\Phi_n^+ = \begin{bmatrix} \mu_n^{1/2} & e_n \\ \lambda_n^+ & e_n \end{bmatrix}; \; \Phi_n^- = \chi_n \begin{bmatrix} \mu_n^{1/2} & e_n \\ \lambda_n^- & e_n \end{bmatrix}; \tag{3.6}$$

$$\|\Phi_n^+\|_H^2 \equiv 1 \iff (\mu_n + |\lambda_n^+|^2)\|e_n\|_X^2 \equiv 1; \tag{3.7}$$

$$\|\Phi_n^-\|_H^2 \equiv 1 \iff \chi_n^2 = (\mu_n + |\lambda_n^+|^2)/(\mu_n + |\lambda_n^-|^2); \; \chi_n^2\|e_n\|_X^2 = 1/(\mu_n + |\lambda_n^-|^2); \tag{3.8}$$

$$\{\Phi_n^+\}_{n=1}^{\infty} \text{ and } \{\Phi_n^-\}_{n=1}^{\infty} \text{ each forms an orthonormal family on } H. \tag{3.9}$$

$$\{\Phi_n^{+,-}\} \text{ is a complete family on } H, \text{ under the assumption that } \rho^2 \neq 4\mu_n^{1-2\alpha}, \text{ so that } \lambda_n^+ \neq \lambda_n^-. \tag{3.10}$$

$$H = H^+ + H^- \text{(non-orthogonal, direct sum)}; \; x = x^+ + x^-, \; x^+ \in H^+, \; x^- \in H^-; \tag{3.11}$$

$$H^+ \equiv \overline{\text{span}}\{\Phi_n^+\}_{n=1}^{\infty}; \; H^- \equiv \overline{\text{span}}\{\Phi_n^-\}_{n=1}^{\infty}. \tag{3.12}$$

Accordingly, from the above properties (3.6)-(3.12), we obtain that $\mathbf{A}_{\rho\alpha}$ is the direct sum of two normal operators (on H^+ and on H^-) and

$$e^{\mathbf{A}_{\rho\alpha} t}x = \sum_{n=1}^{\infty} e^{\lambda_n^+ t}(x^+ , \Phi_n^+)_H \Phi_n^+ + \sum_{n=1}^{\infty} e^{\lambda_n^- t}(x^- , \Phi_n^-)_H \Phi_n^- ; \tag{3.13}$$

$$\left\| e^{\mathbf{A}_{\rho\alpha} t}x \right\|_H^2 \sim \sum_{n=1}^{\infty} e^{2(\mathrm{Re}\lambda_n^+)t} |(x^+ , \Phi_n^+)_H|^2 + \sum_{n=1}^{\infty} e^{2(\mathrm{Re}\lambda_n^-)t} |(x^- , \Phi_n^-)_H|^2 . \tag{3.14}$$

Recalling from (3.1) that $-B^*[y_1 , y_2] = y_2$, we obtain by (3.13),

$$-B^* e^{\mathbf{A}_{\rho\alpha} t}x = \left[e^{\mathbf{A}_{\rho\alpha} t}x \right]_{\text{second component}} = \tag{3.15}$$

$$\text{(by (3.13), (3.6))} = \sum_{n=1}^{\infty} \left[e^{\lambda_n^+ t}(x^+ , \Phi_n^+)_H \lambda_n^+ \|e_n\|_X \right] e_n/\|e_n\|_X + \tag{3.16}$$

$$+ \sum_{n=1}^{\infty} \left[e^{\lambda_n^+ t}(x^- , \Phi_n^-)_H \chi_n \lambda_n^- \|e_n\|_X \right] e_n/\|e_n\|_X$$

$$\|B^* e^{\mathbf{A}_{\rho\alpha} t}x\|_X^2 = \sum_{n=1}^{\infty} \Big| e^{\lambda_n^+ t}(x^+ , \Phi_n^+)_H \lambda_n^+ \|e_n\|_X + \tag{3.17}$$

$$+ \; e^{\lambda_n^- t}(x^- , \Phi_n^-)_H \chi_n \lambda_n^- \|e_n\|_X \Big|^2 .$$

Thus, the *characterization* for the property *of exact null controllability* of eq. (1.1.1) on $E \equiv \mathcal{D}(S^{1/2}) \times X$, with $L_2(0 , T; X)$-controls (equivalently, of eq. (3.1) on $H \equiv X \times X$),

$$\int_0^T \left\| B^* e^{\mathbf{A}_{\rho\alpha} t}x \right\|_X^2 dt \geq C_T \left\| e^{\mathbf{A}_{\rho\alpha} T}x \right\|_H^2 , \quad x \in H, \tag{3.18}$$

is equivalent, via (3.14), (3.17), to the inequality:

$$\sum_{n=1}^{\infty} \int_0^T \left| e^{\lambda_n^+ t}(x^+ , \Phi_n^+)_H \lambda_n^+ \|e_n\|_X + e^{\lambda_n^- t}(x^- , \Phi_n^-)_H \chi_n \lambda_n^- \|e_n\|_X \right|^2 dt \geq \tag{3.19}$$

$$\geq C_T \left\{ \sum_{n=1}^{\infty} e^{2(\mathrm{Re}\lambda_n^+)T} |(x^+ , \Phi_n^+)_H|^2 + \sum_{n=1}^{\infty} e^{2(\mathrm{Re}\lambda_n^-)T} |(x^- , \Phi_n^-)_H|^2 \right\} ,$$

for some constant $C_T > 0$. The crux of the proof now consists in showing the validity of inequality (3.19).

Step 2. We shall show (3.19) when $1/2 \leq \alpha < 1$ and $\rho^2 \neq 4\mu_n^{1-2\alpha}$, as assumed. We begin by working in the case when $\rho^2 > 4\mu_n^{1-2\alpha}$, $1/2 \leq \alpha < 1$, in which case $\lambda_n^{+,-}$ are *both real* (*negative*): this means for all $n \geq$ some $N_\alpha \geq 1$ when $1/2 < \alpha < 1$; and for $\rho > 2$ when $\alpha = 1/2$. We apply Lemma 2.3 with

$$\left\{ \begin{array}{l} a = -\lambda_n^+ > 0; \;\; b = -\lambda_n^- > 0; \\ \alpha_n = (x^+ , \Phi_n^+)_H \lambda_n^+ \|e_n\|_X; \;\; \beta_n = (x^- , \Phi_n^-)_H \chi_n \lambda_n^- \|e_n\|_X; \end{array} \right. \tag{3.20}$$

$$\begin{aligned} k_n \equiv \sqrt{ab}/(a+b) &= \sqrt{\lambda_n^+ \lambda_n^-}/(-\lambda_n^+ - \lambda_n^-) = \\ &= \sqrt{\mu_n}/(\rho\mu_n^\alpha) = \left\{ \begin{array}{ll} 1/\rho < 1/2\,, & \text{if } \alpha = 1/2\,,\ \rho > 2\,, \\ \searrow 0\,, & \text{if } 1/2 < \alpha < 1\,, \end{array} \right. \end{aligned} \tag{3.21}$$

recalling (3.5*a-b*), so that the required condition $\sqrt{ab}/(a+b) < 1/2$ of Lemma 2.3 is satisfied, at least for all n sufficiently large, say w.l.o.g. still $n \geq N_\alpha \geq 1$, for $1/2 < \alpha \leq 1$. Applying Lemma 2.3(*ii*), eq. (2.28), for $n \geq N_\alpha$, we obtain via (3.20), (3.21):

$$\begin{aligned} &\int_0^T \left| e^{\lambda_n^+ t}(x^+ , \Phi_n^+)_H \lambda_n^+ \|e_n\|_X + e^{\lambda_n^- t}(x^- , \Phi_n^-)_H \chi_n \lambda_n^- \|e_n\|_X \right|^2 dt \geq \\ &\geq \frac{1}{2}\left\{ \frac{(1-2k_n)e^{2|\lambda_n^+|T}}{|\lambda_n^+|} - \frac{(1+2k_n)}{|\lambda_n^+|} \right\} |\lambda_n^+|^2 \|e_n\|_X^2 e^{-2|\lambda_n^+|T} |(x^+ , \Phi_n^+)_H|^2 + \\ &+ \frac{1}{2}\left\{ \frac{(1-2k_n)e^{2|\lambda_n^-|T}}{|\lambda_n^-|} - \frac{(1+2k_n)}{|\lambda_n^-|} \right\} \chi_n^2 |\lambda_n^-|^2 \|e_n\|_X^2 e^{-2|\lambda_n^-|T} |(x^- , \Phi_n^-)_H|^2 \end{aligned} \tag{3.22}$$

(recalling the normalizations in (3.7) and (3.8))

$$\begin{aligned} &\geq \frac{1}{2}\left\{ \frac{(1-2k_n)|\lambda_n^+|e^{2|\lambda_n^+|T}}{\mu_n + |\lambda_n^+|^2} - \frac{(1+2k_n)|\lambda_n^+|}{\mu_n + |\lambda_n^+|^2} \right\} e^{-2|\lambda_n^+|T} |(x^+ , \Phi_n^+)_H|^2 + \\ &+ \frac{1}{2}\left\{ \frac{(1-2k_n)|\lambda_n^-|e^{2|\lambda_n^-|T}}{\mu_n + |\lambda_n^-|^2} - \frac{(1+2k_n)|\lambda_n^-|}{\mu_n + |\lambda_n^-|^2} \right\} e^{-2|\lambda_n^-|T} |(x^- , \Phi_n^-)_H|^2 , \\ &\hspace{20em} \forall\ n \geq N_\alpha \geq 1. \end{aligned} \tag{3.23}$$

Step 3. We now complete the proof of Theorem 1.1.1 for $1/2 < \alpha < 1$, $\rho^2 \neq \neq 4\mu_n^{1-2\alpha}$; or $\alpha = 1/2$, $\rho > 2$. In this case, by recalling (3.5*b*), we obtain

$$\lambda_n^- \sim \mu_n^\alpha; \;\; \lambda_n^- \searrow -\infty; \;\; \lambda_n^+ \sim \mu_n^{1-\alpha}; \;\; \lambda_n^+ \searrow -\infty; \tag{3.24}$$

$$\frac{|\lambda_n^-|}{\mu_n + |\lambda_n^-|^2} \sim \frac{\mu_n^\alpha}{\mu_n + \mu_n^{2\alpha}} \sim \frac{1}{\mu_n^\alpha} \searrow 0; k_n \searrow 0 \text{ for } \frac{1}{2} < \alpha < 1; k_n \equiv \frac{1}{\rho}\,, \alpha = \frac{1}{2}; \tag{3.25}$$

$$\frac{|\lambda_n^+|}{\mu_n + |\lambda_n^+|^2} \sim \frac{\mu_n^{1-\alpha}}{\mu_n + \mu_n^{2-2\alpha}} \sim \frac{\mu_n^{1-\alpha}}{\mu_n} = \frac{1}{\mu_n^{\alpha}} \searrow 0. \tag{3.26}$$

Thus, by (3.24)- (3.26), given $T > 0$ fixed (but arbitrary), there exists a positive integer $N_{T\alpha}$ depending on T and α (as well as on ρ, which is fixed), such that

$$\frac{1}{2}\left\{\frac{(1-2k_n)|\lambda_n^{+,-}|e^{2|\lambda_n^{+,-}|T}}{\mu_n + |\lambda_n^{+,-}|^2} - \frac{(1+2k_n)|\lambda_n^{+,-}|}{\mu_n + |\lambda_n^{+,-}|^2}\right\} \geq C_{T\rho N_T} > 0\,, \tag{3.27}$$

$$\forall\ n \geq N_{T\alpha} \geq N_{\alpha} \geq 1.$$

Thus, *a fortiori*, inequality (3.23) implies by virtue of (3.27)

$$\int_0^T \left| e^{\lambda_n^+ t}(x^+\,, \Phi_n^+)_H \lambda_n^+ \|e_n\|_X + e^{\lambda_n^- t}(x^-\,, \Phi_n^-)_H \chi_n \lambda_n^- \|e_n\|_X \right|^2 dt \geq \tag{3.28}$$

$$\geq C_{T\rho N_{T\alpha}} \left\{ e^{-2|\lambda_n^+|T} |(x^+\,, \Phi_n^+)_H|^2 + e^{-2|\lambda_n^-|T} |(x^-\,, \Phi_n^-)_H|^2 \right\}\,, \quad \forall\ n \geq N_{T\alpha}\,,$$

and hence

$$\sum_{n=N_{T\alpha}}^{\infty} \int_0^T \left| e^{\lambda_n^+ t}(x^+\,, \Phi_n^+)_H \lambda_n^+ \|e_n\|_X + e^{\lambda_n^- t}(x^-\,, \Phi_n^-)_H \chi_n \lambda_n^- \|e_n\|_X \right|^2 dt \geq \tag{3.29}$$

$$\geq C_{T\rho N_T} \sum_{n=N_T}^{\infty} \left\{ e^{-2|\lambda_n^+|T} |(x^+\,, \Phi_n^+)_H|^2 + e^{-2|\lambda_n^-|T} |(x^-\,, \Phi_n^-)_H|^2 \right\}.$$

Thus, for $\alpha = 1/2$ and $\rho > 2$; or else for $1/2 < \alpha < 1$ where $\mu_n^{1-2\alpha} \searrow 0$, and so $\rho^2 > 4\mu_n^{1-2\alpha}$, in which cases $\lambda_n^{+,-}$ are real negative for n large enough (without loss of generality for all $n \geq N_{T\alpha}$), we see that eq. (3.29) proves the required inequality (3.19) with $\mathrm{Re}\,\lambda_n^{+,-} = -|\lambda_n^{+,-}|$, at least from $n = N_{T\alpha}$ on. Then, we can obtain the inequality corresponding to (3.29) with $\sum_{n=1}^{N-1}$ replacing $\sum_{n=N}^{\infty}$, $N = N_{T\alpha}$, when $\rho^2 \neq 4\mu_n^{1-2\alpha}$, in which case λ_n^+ and λ_n^- are distinct, and the argument of Step 8 in Section 2 (Lemma 2.4) continues to work. This way, inequality (3.19) is proved. □

It remains to handle the case $\alpha = 1/2$, $0 < \rho < 2$. Here we proceed as above by invoking however, Lemma 2.5(*ii*), eq. (2.5.2) (instead of Lemma 2.3, eq. (2.28)), as well as Lemma 2.6 (instead of Lemma 2.4).

The case $\alpha = 1$, $\rho^2 \neq 4\mu_n^{1-2\alpha} = 4\mu_n^{-1}$: Counterexample to inequality (3.18). The above argument, leading to the key inequality (3.27), was carried out for $1/2 \leq \alpha < 1$, $\rho^2 > 4\mu_n^{1-2\alpha}$, in which case then $\lambda_n^{+,-}$ (real negative) $\searrow -\infty$. In the case $\alpha = 1$, this argument breaks down, as we now have:

$$\lambda_n^- \sim \mu_n\,,\ \ \lambda_n^- \searrow -\infty\,,\ \text{ but } \lambda_n^+ \nearrow \text{ limit } -\ell = -1/\rho \neq 0. \tag{3.30}$$

Thus, the argument leading to inequality (3.27) continues to hold true for λ_n^-, when $\alpha = 1$. However, regarding λ_n^+ and $\alpha = 1$, we obtain for $T > 0$, via (3.30), and

$k_m \searrow 0$ by (3.21):

$$(3.31)\qquad \left[(1-2k_n)e^{2|\lambda_n^+|T}-(1+2k_n)\right]|\lambda_n^+|/(\mu_n+|\lambda_n^+|^2)\to[e^{2\ell T}-1]\cdot 0=0\,,$$

as $n\to+\infty$, which denies (3.27). Indeed, for $\alpha=1$, *the critical inequality* (3.18) (or (3.19)) *fails to hold true.* To show this, we take the sequence of elements $x_m\equiv\Phi_m^+\in H^+$, so that by (3.11), and (3.6), (3.7), (3.9) we have:

$$(3.32)\qquad x_m^+\equiv\Phi_m^+\,,\ \ x_m^-\equiv 0\,,\ \text{ and } (x_m^+\,,\Phi_n^+)_H=\begin{cases}0 & \text{if } m\neq n\,,\\ 1 & \text{if } m=n.\end{cases}$$

Then, via (3.32), and (3.17), we obtain

$$(3.33)\qquad \int_0^T\left\|B^*e^{\mathbf{A}_{\rho\alpha}t}x_m\right\|_X^2dt=\int_0^T\left|e^{\lambda_m^+t}\cdot 1\cdot\lambda_m^+\|e_m\|_X+0\right|^2dt=|\lambda_m^+|^2\|e_m\|_X^2\,\frac{1-e^{2\lambda_m^+T}}{2|\lambda_m^+|}$$

$$(3.34)\qquad (\text{by } (3.7))\quad =\frac{|\lambda_m^+|}{\mu_m+|\lambda_m^+|^2}\,\frac{(1-e^{2\lambda_m^+T})}{2}\to 0\cdot\frac{(1-e^{-2\ell T})}{2}=0\,,$$

as $\mu_m\nearrow\infty$, $\lambda_m^+\nearrow-\ell$, when $m\to\infty$. Moreover, by (3.32) and (3.14), we obtain

$$(3.35)\qquad \left\|e^{\mathbf{A}_{\rho\alpha}T}x_m\right\|_H^2=e^{-2|\lambda_m^+|T}1+0=e^{-2|\lambda_m^+|T}\to e^{-2\ell T}\,,$$

as $m\to\infty$. Thus, (3.34) and (3.35) combined show that the characterizing inequality (3.18), or (3.19), *is false* for $\alpha=1$ and $\rho^2\neq 4\mu_n^{1-2\alpha}$; and thus *exact null controllability of* eq. (1.1.1) on the energy space $E\equiv\mathcal{D}(S^{1/2})\times X$ [or, equivalently, of system (3.1) on $H\equiv X\times X$] over the time $[0\,,T]$, by means of $L_2(0\,,T;X)$-controls *is false* in this case.

Remark 3.1. By techniques similar to the ones employed above, one can prove that exact null controllability of eq. (1.1.1) with $\alpha=1$ on the energy space $E\equiv\mathcal{D}(S^{1/2})\times X$ [equivalently, of system (3.1) on the space $H\equiv X\times X$] over the time $[0\,,T]$ by means of the larger class of $L_2(0\,,T;[\mathcal{D}(S^{1/2})]')$-controls is true, when $\alpha=1$, $\rho^2\neq 4\mu_n^{1-2\alpha}$. Indeed, in this case, the corresponding exact null controllability characterization is

$$(3.36)\qquad \int_0^T\left\|S^{1/2}B^*e^{\mathbf{A}_{\rho\alpha}t}x\right\|_X^2dt\geq C_T\left\|e^{\mathbf{A}_{\rho\alpha}T}x\right\|_X^2\,,\ \ x\in X,$$

in lieu of (3.18). Then, the coordinate terms on the left-hand side of inequality (3.19) are now multiplied by μ_n. Thus, we now have that the key inequality (counterpart of (3.27) is)

$$(3.37)\qquad \frac{1}{2}\left\{\frac{(1-2k_n)\mu_n|\lambda_n^{+,-}|e^{2|\lambda_n^{+,-}|T}}{\mu_n+|\lambda_n^{+,-}|^2}-\frac{(1+2k_n)\mu_n|\lambda_n^{+,-}|}{\mu_n+|\lambda_n^{+,-}|^2}\right\}\geq C_{T\rho N_T}>0\,,$$

$$\forall\ n\geq N_{T\alpha}\,,$$

i.e., (3.27) with the left-hand side multiplied by μ_n. Thus, we now have $[\mu_n|\lambda_n^+|/(\mu_n + |\lambda_n^+|^2)] \nearrow \ell \neq 0$, and (3.36) hold true also for the λ_n^+-branch, as desired. This way, inequality (3.36) is proved as before. □

4. Thermo-elastic case: Proof of Theorem 1.2.1

Step 1. The proof of Theorem 1.2.1 in the thermo-elastic case (1.2.1)-(1.2.3) is conceptually similar to that of Theorem 1.1.1, case $\alpha = 1/2$, Section 2, in that both cases share the property that the basic operator A can be factored and is diagonalizable. Indeed, the operator A in (1.2.2) can be rewritten as [3]

$$A = \begin{bmatrix} 0 & \mathcal{A} & 0 \\ -\mathcal{A} & 0 & \mathcal{A} \\ 0 & -\mathcal{A} & -\mathcal{A} \end{bmatrix} = \mathcal{A}M; \tag{4.1}$$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}; \det(zI - M) = z^3 + z^2 + 2z + 1 = (z - z_1)(z - z_2)(z - z_3) = 0; \tag{4.2}$$

$$z_1 = -0.56984; \;\; z_2 = -0.21508 + i(1.30714); \;\; z_3 = \bar{z}_2, \tag{4.3}$$

with accuracy up to the fifth decimal point. Then the matrix M is diagonalizable, with diagonalizing matrix P:

$$P^{-1}MP = \begin{bmatrix} z_1 & & 0 \\ & z_2 & \\ 0 & & z_3 \end{bmatrix}; \qquad P = \begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \dfrac{-z_1}{1+z_1} & \dfrac{-z_2}{1+z_2} & \dfrac{-z_3}{1+z_3} \end{bmatrix}. \tag{4.4}$$

We define the operator $\Pi = P(I)$, Identity on X, and obtain

$$A = \begin{bmatrix} 0 & \mathcal{A} & 0 \\ -\mathcal{A} & 0 & \mathcal{A} \\ 0 & -\mathcal{A} & -\mathcal{A} \end{bmatrix} = \Pi \widetilde{A} \Pi^{-1}; \qquad \widetilde{A} = \begin{bmatrix} \mathcal{A}z_1 & & 0 \\ & \mathcal{A}z_2 & \\ 0 & & \mathcal{A}z_3 \end{bmatrix}; \tag{4.5}$$

$$e^{At} = \Pi e^{\widetilde{A}t} \Pi^{-1}; \;\; e^{\widetilde{A}t} = \begin{bmatrix} e^{\mathcal{A}z_1 t} & & 0 \\ & e^{\mathcal{A}z_2 t} & \\ 0 & & e^{\mathcal{A}z_3 t} \end{bmatrix}, \;\; \text{Re } z_i < 0. \tag{4.6}$$

Applying Π^{-1} on both sides of the equation $\dot{y} = Ay + Bu$ in (1.2.1) yields

$$\widetilde{y}_t = \widetilde{A}\widetilde{y} + \widetilde{B}u, \quad \widetilde{y} = \Pi^{-1}y, \quad \widetilde{B} = \Pi^{-1}B. \tag{4.7}$$

The $\widetilde{y}$-problem in (4.7) represents the *diagonalized version* of the original y-equation in

(1.2.1). We have, by (4.3),

$$\widetilde{A}^* = \begin{bmatrix} \mathcal{A}z_1 & & 0 \\ & \mathcal{A}z_3 & \\ 0 & & \mathcal{A}z_2 \end{bmatrix}; \quad e^{\widetilde{A}^* t} = \begin{bmatrix} e^{\mathcal{A}z_1 t} & & 0 \\ & e^{\mathcal{A}z_3 t} & \\ 0 & & e^{\mathcal{A}z_2 t} \end{bmatrix}; \tag{4.8}$$

$$\widetilde{B} = [b_1\,,\,b_2\,,\,b_3]\,, \qquad \widetilde{B}^* y = b_1 y_1 + b_2 y_2 + b_3 y_3\,, \tag{4.9}$$

the exact expressions for $b_1\,,\,b_2\,,\,b_3$ in terms of $z_1\,,\,z_2\,,\,z_3$ in either case $B = B_m = [0\,,\,I,\,0]$, or else $B = B_\theta = [0\,,\,0\,,\,I]$, see (1.2.3), being non-critical in the argument below, except for being all non-zero. [*E.g.*, for $B = B_m = [0\,,\,I,\,0]$, we find

$$b_1 = (1+z_1)/\big((z_1 - z_2)(z_1 - z_3)\big);$$
$$b_2 = (1+z_2)/\big((z_2 - z_1)(z_2 - z_3)\big);$$
$$b_3 = (1+z_3)/\big((z_3 - z_1)(z_3 - z_2)\big).$$

Instead, for $B = B_\theta = [0\,,\,0\,,\,I]$, we find

$$b_1 = (1+z_1)(1+z_2)(1+z_3)/\big((z_1 - z_2)(z_1 - z_3)\big);$$
$$b_2 = (1+z_1)(1+z_2)(1+z_3)/\big((z_2 - z_3)(z_2 - z_1)\big);$$
$$b_3 = (1+z_1)(1+z_2)(1+z_3)/\big((z_3 - z_1)(z_3 - z_2)\big)].$$

Thus, if $x = [x_1\,,\,x_2\,,\,x_3] \in H$, we have by (4.8), (4.9):

$$\widetilde{B}^* e^{\widetilde{A}^* t} x = b_1 e^{\mathcal{A}z_1 t} x_1 + b_2 e^{\mathcal{A}z_3 t} x_2 + b_3 e^{\mathcal{A}z_2 t} x_3. \tag{4.10}$$

As seen in Section 2, without loss of generality, we may let the positive, self-adjoint operator $\mathcal{A}$ with compact resolvent have simple eigenvalues $\{\mu_n\}$ with corresponding eigenvectors $\{e_n\}$ forming an orthonormal basis in X,

$$\mathcal{A}e_n = \mu_n e_n\,, \quad 0 < \mu_n \to +\infty. \tag{4.11}$$

Then

$$e^{\mathcal{A}z_i t} x_i = \sum_{n=1}^{\infty} e^{\mu_n z_i t} (x_i\,,\,e_n)_X e_n\,, \;\; x_i \in X, \;\; i = 1\,,2\,,3. \tag{4.12}$$

Hence by (4.10)-(4.12), we obtain

$$\left\| \widetilde{B}^* e^{\widetilde{A}^* t} x \right\|_H^2 = \sum_{n=1}^{\infty} \left| e^{\mu_n z_1 t} b_1 (x_1\,,\,e_n)_X + e^{\mu_n z_3 t} b_2 (x_2\,,\,e_n)_X + e^{\mu_n z_2 t} b_3 (x_3\,,\,e_n)_X \right|^2, \tag{4.13}$$

while also by (4.8),

$$\left\| e^{\widetilde{A}^* T} x \right\|_H^2 = \sum_{n=1}^{\infty} \left\{ e^{2\mu_n z_1 T} |(x_1\,,\,e_n)_X|^2 + e^{2\mu_n (\mathrm{Re} z_2) T} [|(x_2\,,\,e_n)_X|^2 + |(x_3\,,\,e_n)_X|^2] \right\}. \tag{4.14}$$

Thus, the property of exact null-controllability on H, over $[0\,,\,T]$, with $L_2(0\,,\,T; X)$-controls of the original problem $\dot{y} = Ay + Bu$ in (1.2.1) is *equivalent* to the same property for the diagonalized problem (4.7); and this, in turn, is *equivalent* to the following inequality: there exists a constant $c_T > 0$ such that

$$\int_0^T \left\| \widetilde{B}^* e^{\widetilde{A}^* t} x \right\|_H^2 dt \geq c_T \left\| e^{A^* T} x \right\|^2 , \ \forall \ x \in H, \tag{4.15}$$

i.e., by (4.13) and (4.14),

$$\sum_{n=1}^{\infty} \int_0^T \left| e^{\mu_n z_1 t} b_1 (x_1 , e_n)_X + e^{\mu_n z_3 t} b_2 (x_2 , e_n)_X + e^{\mu_n z_2 t} b_3 (x_3 , e_n)_X \right|^2 dt \geq \tag{4.16}$$

$$\geq c_T \sum_{n=1}^{\infty} \left\{ e^{2\mu_n z_1 T} |(x_1 , e_n)_X|^2 + e^{2\mu_n \mathrm{Re}(z_2) T} [|(x_2 , e_n)_X|^2 + |(x_3 , e_n)_X|^2] \right\}.$$

Step 2. The crux of the proof consists in showing the validity of inequality (4.16). To this end, critical is the following lemma, the counterpart of Lemma 2.3.

Lemma 4.1. *Let* $r > 0$ *and* $a = a_1 + ia_2$ *and* $b = a_1 - ia_2$ *be two complex conjugate constants with* $a_1 > 0$. *Let* α , β , γ *be possibly complex constants. Assume that*

$$k_1 \equiv 2 \sqrt{ra_1}/|r + a| < 1/2; \qquad k_2 \equiv a_1/|a| < 1 - k_1 . \tag{4.17}$$

Then, the following inequalities hold true:

(*i*)

$$\int_0^T \left| e^{-rt}\alpha + e^{-at}\beta + e^{-bt}\gamma \right|^2 dt \geq \frac{1}{2} c_1 \left[\left| \frac{\alpha}{\sqrt{r}} \right|^2 + \left| \frac{\beta}{\sqrt{a_1}} \right|^2 + \left| \frac{\gamma}{\sqrt{a_1}} \right|^2 \right] \tag{4.18a}$$

$$- \frac{1}{2} c_2 \left\{ e^{-2rT} \left| \frac{\alpha}{\sqrt{r}} \right|^2 + e^{-2a_1 T} \left[\left| \frac{\beta}{\sqrt{a_1}} \right|^2 + \left| \frac{\gamma}{\sqrt{a_1}} \right|^2 \right] \right\};$$

$$c_1 \equiv \min\{[1 - 2k_1] , [1 - k_1 - k_2]\} > 0 , \ c_2 \equiv \max\{[1 + 2k_1] , [1 + k_1 + k_2]\} > 0; \tag{4.18b}$$

(*ii*)

$$\int_0^T \left| e^{-rt}\alpha + e^{-at}\beta + e^{-bt}\gamma \right|^2 dt \geq 1/2 \left[c_1 e^{2rT} - c_2 \right] e^{-2rT} \left| \frac{\alpha}{\sqrt{r}} \right|^2 + \tag{4.19}$$

$$+ \ 1/2 \left[c_1 e^{2a_1 T} - c_2 \right] e^{-2a_1 T} \left[\left| \frac{\beta}{\sqrt{a_1}} \right|^2 + \left| \frac{\gamma}{\sqrt{a_1}} \right|^2 \right].$$

Proof. (*i*) Writing $|z|^2 = z\bar{z}$ we obtain with $b = \bar{a}$,

$$\left| e^{-rt}\alpha + e^{-at}\beta + e^{-\bar{a}t}\gamma \right|^2 = e^{-2rt}|\alpha|^2 + e^{-2a_1 t}[|\beta|^2 + |\gamma|^2] + \tag{4.20}$$

$$+ 2 \ \mathrm{Re} \left[e^{-(r+a)t} \overline{\alpha}\beta \right] + 2 \ \mathrm{Re} \left[e^{-(r+\bar{a})t} \overline{\alpha}\gamma \right] + 2 \ \mathrm{Re} \left[e^{-2at} \beta\overline{\gamma} \right] .$$

Hence, from (4.20) we obtain

$$\int_0^T \left| e^{-rt}\alpha + e^{-at}\beta + e^{-\bar{a}t}\gamma \right|^2 dt = \tag{4.21}$$

$$= \frac{(1-e^{-2rT})}{2r}\,|\alpha|^2 + \frac{(1-e^{-2a_1T})}{2a_1}\,[|\beta|^2+|\gamma|^2] + 2\ \mathrm{Re}\left[\frac{1-e^{-(r+a)T}}{r+a}\,\overline{\alpha}\beta\right] +$$

$$+ 2\ \ \mathrm{Re}\left[\frac{1-e^{-2aT}}{2a}\,\beta\overline{\gamma}\right] + 2\ \mathrm{Re}\left[\frac{1-e^{-(r+\bar{a})T}}{r+\bar{a}}\,\overline{\alpha}\gamma\right] =$$

$$= \frac{1}{2}\left\{\left[\left|\frac{\alpha}{\sqrt{r}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2\right] + 4\ \mathrm{Re}\left[\frac{\overline{\alpha}}{\sqrt{r}}\ \frac{\beta}{\sqrt{a_1}}\ \frac{\sqrt{ra_1}}{r+a}\right] + \right. \tag{4.22}$$

$$\left. +\ \ 4\ \mathrm{Re}\left[\frac{\beta}{\sqrt{a_1}}\ \frac{\overline{\gamma}}{\sqrt{a_1}}\ \frac{a_1}{\sqrt{2a}}\right] + 4\ \mathrm{Re}\left[\frac{\overline{\alpha}}{\sqrt{r}}\ \frac{\gamma}{\sqrt{a_1}}\ \frac{\sqrt{ra_1}}{r+\bar{a}}\right]\right\} +$$

$$-\frac{1}{2}\left\{\left|\frac{e^{-rT}\alpha}{\sqrt{r}}\right|^2 + \left|\frac{e^{-a_1T}\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{e^{-a_1T}\gamma}{\sqrt{a_1}}\right|^2 + \right.$$

$$+4\ \mathrm{Re}\left[\frac{e^{-rT}\overline{\alpha}}{\sqrt{r}}\ \frac{e^{-a_1T}\beta}{\sqrt{a_1}}e^{-ia_2T}\frac{\sqrt{ra_1}}{r+a}\right] + 4\ \mathrm{Re}\left[\frac{e^{-a_1T}\beta}{\sqrt{a_1}}\ \frac{e^{-a_1T}\overline{\gamma}}{\sqrt{a_1}}e^{-2ia_2T}\frac{a_1}{2a}\right] +$$

$$\left. +\ \ 4\ \mathrm{Re}\left[\frac{e^{-rT}\overline{\alpha}}{\sqrt{r}}\ \frac{e^{-a_1T}\gamma}{\sqrt{a_1}}\ e^{ia_2T}\ \frac{\sqrt{ra_1}}{r+\bar{a}}\right]\right\}.$$

Setting

$$\begin{cases} \widetilde{\alpha} \equiv e^{-rT}\alpha;\ \widetilde{\beta} \equiv e^{-a_1T}\beta;\ \widetilde{\gamma} \equiv e^{-a_1T}\gamma, \text{ hence} \\ \alpha = e^{rT}\widetilde{\alpha};\ \beta = e^{a_1T}\widetilde{\beta};\ \gamma = e^{a_1T}\widetilde{\gamma}, \end{cases} \tag{4.23}$$

we rewrite eq. (4.22) more conveniently as

$$\int_0^T \left| e^{-rt}\alpha + e^{-at}\beta + e^{-\bar{a}t}\gamma \right|^2 dt = \frac{1}{2}\left\{\left|\frac{\alpha}{\sqrt{r}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2 + \right. \tag{4.24}$$

$$\left. +4\ \mathrm{Re}\left[\frac{\overline{\alpha}}{\sqrt{r}}\frac{\beta}{\sqrt{a_1}}\frac{\sqrt{ra_1}}{r+a}\right] + 4\ \mathrm{Re}\left[\frac{\beta}{\sqrt{a_1}}\frac{\overline{\gamma}}{\sqrt{a_1}}\frac{a_1}{2a}\right] + 4\ \mathrm{Re}\left[\frac{\overline{\alpha}}{\sqrt{r}}\frac{\gamma}{\sqrt{a_1}}\ \frac{\sqrt{ra_1}}{r+\bar{a}}\right]\right\} +$$

$$-\frac{1}{2}\left\{\left|\frac{\widetilde{\alpha}}{\sqrt{r}}\right|^2 + \left|\frac{\widetilde{\beta}}{\sqrt{a_1}}\right|^2 + \left|\frac{\widetilde{\gamma}}{\sqrt{a_1}}\right|^2 + 4\ \mathrm{Re}\left[\frac{\overline{\widetilde{\alpha}}}{\sqrt{r}}\ \frac{\widetilde{\beta}}{\sqrt{a_1}}\ e^{-ia_2T}\ \frac{\sqrt{ra_1}}{r+a}\right] + \right.$$

$$\left. + 4\ \mathrm{Re}\left[\frac{\widetilde{\beta}}{\sqrt{a_1}}\ \frac{\overline{\widetilde{\gamma}}}{\sqrt{a_1}}\ e^{-2ia_2T}\ \frac{a_1}{2a}\right] + 4\ \mathrm{Re}\left[\frac{\overline{\widetilde{\alpha}}}{\sqrt{r}}\ \frac{\widetilde{\gamma}}{\sqrt{a_1}}\ e^{ia_2T}\ \frac{\sqrt{ra_1}}{r+\bar{a}}\right]\right\} =$$

$$= \frac{1}{2}\,①-\frac{1}{2}\,②, \tag{4.25}$$

where

$$\text{①} \geq \left|\frac{\alpha}{\sqrt{r}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2 - 2\,\frac{\sqrt{ra_1}}{|r+a|}\,2\left|\frac{\alpha}{\sqrt{r}}\right|\,\left|\frac{\beta}{\sqrt{a_1}}\right| + \tag{4.26}$$

$$- \;2\left|\frac{a_1}{2a}\right|\cdot 2\left|\frac{\beta}{\sqrt{a_1}}\right|\,\left|\frac{\gamma}{\sqrt{a_1}}\right| - 2\,\frac{\sqrt{ra_1}}{|r+\bar{a}\,|}\,2\left|\frac{\alpha}{\sqrt{r}}\right|\,\left|\frac{\gamma}{\sqrt{a_1}}\right| \geq$$

$$\geq \left|\frac{\alpha}{\sqrt{r}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2 - 2\,\frac{\sqrt{ra_1}}{|r+a|}\left[\left|\frac{\alpha}{\sqrt{r}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2\right] +$$

$$- \left|\frac{a_1}{a}\right|\left[\left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2\right] - 2\,\frac{\sqrt{ra_1}}{|r+\bar{a}\,|}\left[\left|\frac{\alpha}{\sqrt{r}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2\right].$$

Hence since $|r+a| = |r+\bar{a}\,|$, we obtain from (4.26),

$$\text{①} \geq \left\{1 - 2\,\frac{\sqrt{ra_1}}{|r+a|} - 2\,\frac{\sqrt{ra_1}}{|r+\bar{a}\,|}\right\}\left|\frac{\alpha}{\sqrt{r}}\right|^2 + \tag{4.27}$$

$$+ \;\left\{1 - 2\,\frac{\sqrt{ra_1}}{|r+a|} - \left|\frac{a_1}{a}\right|\right\}\left[\left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2\right] =$$

$$= \{1 - 2k_1\}\left|\frac{\alpha}{\sqrt{r}}\right|^2 + \{1 - k_1 - k_2\}\left[\left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2\right].$$

Hence, recalling the constant $c_1 > 0$ defined in (4.18b), we obtain from (4.27),

$$\text{①} \geq c_1\left[\left|\frac{\alpha}{\sqrt{r}}\right|^2 + \left|\frac{\beta}{\sqrt{a_1}}\right|^2 + \left|\frac{\gamma}{\sqrt{a_1}}\right|^2\right]. \tag{4.28}$$

Similarly, from (4.24), (4.25), we estimate

$$\text{②} \leq \left|\frac{\tilde{\alpha}}{\sqrt{r}}\right|^2 + \left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right|^2 + \left|\frac{\tilde{\gamma}}{\sqrt{a_1}}\right|^2 + 2\,\frac{\sqrt{ra_1}}{|r+a|}\cdot 2\left|\frac{\tilde{\alpha}}{\sqrt{r}}\right|\,\left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right| + \tag{4.29}$$

$$+ \;2\left|\frac{a_1}{2a}\right|\;2\left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right|\,\left|\frac{\tilde{\gamma}}{\sqrt{a_1}}\right| + 2\,\frac{\sqrt{ra_1}}{|r+\bar{a}\,|}\cdot 2\left|\frac{\tilde{\alpha}}{\sqrt{r}}\right|\,\left|\frac{\tilde{\gamma}}{\sqrt{a_1}}\right| \leq$$

$$\leq \left\{1 + 2\,\frac{\sqrt{ra_1}}{|r+a|} + 2\left|\frac{\sqrt{ra_1}}{|r+\bar{a}\,|}\right|\right\}\left|\frac{\tilde{\alpha}}{\sqrt{r}}\right|^2 +$$

$$+ \;\left\{1 + 2\,\frac{\sqrt{ra_1}}{|r+a|} + \left|\frac{a_1}{a}\right|\right\}\left[\left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right|^2 + \left|\frac{\tilde{\gamma}}{\sqrt{a_1}}\right|^2\right] =$$

$$= \{1 + 2k_1\}\left|\frac{\tilde{\alpha}}{\sqrt{r}}\right|^2 + \{1 + k_1 + k_2\}\left[\left|\frac{\tilde{\beta}}{\sqrt{a_1}}\right|^2 + \left|\frac{\tilde{\gamma}}{\sqrt{a_1}}\right|^2\right].$$

Hence, recalling the constant $c_2 > 0$ defined in (4.18b), we obtain from (4.29),

$$\text{(4.30)} \qquad ② \le c_2 \left[\left| \frac{\widetilde{\alpha}}{\sqrt{r}} \right|^2 + \left| \frac{\widetilde{\beta}}{\sqrt{a_1}} \right|^2 + \left| \frac{\widetilde{\gamma}}{\sqrt{a_1}} \right|^2 \right].$$

Thus, using (4.28) and (4.30) in (4.25), we obtain

$$\text{(4.31)} \qquad \int_0^T \left| e^{-rt}\alpha + e^{-at}\beta + e^{-\bar{a}t}\gamma \right|^2 dt \ge \frac{1}{2}\, c_1 \left[\left| \frac{\alpha}{\sqrt{r}} \right|^2 + \left| \frac{\beta}{\sqrt{a_1}} \right|^2 + \left| \frac{\gamma}{\sqrt{a_1}} \right|^2 \right] + $$
$$ -\frac{1}{2}\; c_2 \left[\left| \frac{\widetilde{\alpha}}{\sqrt{r}} \right|^2 + \left| \frac{\widetilde{\beta}}{\sqrt{a_1}} \right|^2 + \left| \frac{\widetilde{\gamma}}{\sqrt{a_1}} \right|^2 \right].$$

Recalling (4.23), we see that (4.31) coincides with the desired estimate (4.18a). Then (4.18a) readily implies (4.19). Lemma 4.1 is proved. □

REMARK 4.1. In going from (4.26) to (4.27), if we use $2ab \le \epsilon a^2 + b^2/\epsilon$, we obtain a more general set of sufficient conditions for Lemma 4.1 which reduce to (4.17) for $\epsilon = 1$. However, (4.17) is sufficient for our purposes in Step 3 below. □

Step 3. We apply Lemma 4.1(ii), eq. (4.19) with

$$\text{(4.32)} \qquad \begin{cases} r = \mu_n|z_1| = \mu_n(-z_1) > 0; \;\; a = \mu_n(-z_2); \;\; a_1 = \mu_n \;\operatorname{Re}(-z_2) > 0; \\ |r + a| = \mu_n|z_1 + z_2|; \end{cases}$$

and recalling (4.3):

$$\text{(4.33)} \quad k_1 = \frac{2\sqrt{ra_1}}{|r+a|} = \frac{2\sqrt{(-z_1)\operatorname{Re}(-z_2)}}{|z_1 + z_2|} \sim \frac{2\sqrt{(0.56984)(0.21508)}}{|0.78492 + i(1.30714)|} \sim 0.45922 < \frac{1}{2};$$

$$\text{(4.34)} \quad k_2 = \frac{a_1}{|a|} = \left| \frac{\operatorname{Re}\, z_2}{z_2} \right| \sim \frac{0.21508}{\sqrt{(0.21508)^2 + (1.30714)^2}} = \frac{0.21508}{\sqrt{0.04625 + 1.70861}} =$$
$$ = \frac{0.21508}{\sqrt{1.75486}} = \frac{0.21508}{1.32471} \sim 0.16236 < 1 - k_1 \sim 0.54078.$$

Thus, the assumptions (4.17) of Lemma 4.1 are satisfied.

We obtain from (4.19), (4.32):

$$\text{(4.35)} \qquad \int_0^T \left| e^{\mu_n z_1 t}\alpha_n + e^{\mu_n z_2 t}\beta_n + e^{\mu_n z_3 t}\gamma_n \right|^2 dt \ge \frac{1}{2} \left[\frac{c_1 e^{2\mu_n |z_1| T} - c_2}{\mu_n |z_1|} \right] e^{2\mu_n z_1 T}|\alpha_n|^2 + $$
$$ + \frac{1}{2} \left[\frac{c_1 e^{2\mu_n \operatorname{Re}(-z_2) T} - c_2}{\mu_n \operatorname{Re}(-z_2)} \right] e^{2\mu_n (\operatorname{Re}\, z_2) T} [|\beta_n|^2 + |\gamma_n|^2],$$

with

$$\alpha_n = b_1(x_1\,, e_n)_X\,,\ \ \beta_n = b_3(x_3\,, e_n)_X\,,\ \ \gamma_n = b_2(x_2\,, e_n)_X. \tag{4.36}$$

Step 4. Starting from (4.35), we can complete the proof of the sought-after estimate (4.16), by proceeding as in Steps 7, 8, and 9 of Section 2.1. Given $T > 0$ fixed (but arbitrary), there exists a positive integer $N = N_T$ depending on T, such that

$$\frac{1}{2}\left[\frac{c_1 e^{2\mu_n|z_1|T} - c_2}{\mu_n|z_1|}\right]\,,\quad \frac{1}{2}\left[\frac{c_1 e^{2\mu_n \mathrm{Re}(-z_2)T} - c_2}{\mu_n \mathrm{Re}(-z_2)}\right] \geq C_{TN} > 0, \quad \forall\ n \geq N_T. \tag{4.37}$$

This is surely possible since $\mu_n \nearrow +\infty$, $|z_1| > 0\,,$ $\mathrm{Re}(-z_2) > 0$. Then (4.35) and (4.37) imply

$$\sum_{n=N}^{\infty}\int_0^T \left|e^{\mu_n z_1 t} b_1(x_1\,, e_n)_X + e^{\mu_n z_3 t} b_2(x_2\,, e_n)_X + e^{\mu_n z_2 t} b_3(x_3\,, e_n)_X\right|^2 dt \geq \tag{4.38}$$

$$\geq C'_{TN}\sum_{n=N}^{\infty} e^{2\mu_n z_1 T}|(x_1\,, e_n)_X|^2 + e^{2\mu_n(\mathrm{Re}\,z_2)T}[|(x_2\,, e_n)_X|^2 + |(x_3\,, e_n)_X|^2];$$

$$C'_{TN} = C_{TN}\min\{|b_1|^2\,, |b_2|^2\,, |b_3|^2\} > 0\,, \tag{4.39}$$

as all $b_i \neq 0$ (recall the statements below (4.9)).

To obtain the corresponding estimate of (4.38) for $\sum_{n=1}^{N=1}$, we use the same argument as in Lemma 2.4, which relies on the same three roots $z_1\,, z_2\,, z_3$ being distinct, hence on the corresponding exponentials $e^{\mu_n z_1 t}\,,$ $e^{\mu_n z_2 t}\,,$ $e^{\mu_n z_3 t}$ being linearly independent. We omit the details, and refer to Step 8 and Step 9 of Section 2.1, to complete the proof of Theorem 1.2.1, at least when the eigenvalues of $\mathcal{A}$ are simple. The case of multiple eigenvalues also works along similar lines, as pointed out at the end of Section 2.1, below Step 9.

5. Use of rank conditions

With reference to both problems, the structurally damped equation (1.1.1) and the thermo-elastic equation (1.2.1)-(1.2.3), rewritten as $\dot{y} = Ay + Bu$ on the space H (see (3.1), (3.2), in the first case), suppose that we attempt to use the rank condition $\mathrm{span}\{BU_1\,, ABU_2\} = H$, which is sufficient for steering any initial point in $\mathcal{D}(A)$ to the origin on an arbitrary time T, by means of an explicit smooth control, see [9, Theorem 3.1(*ii*), p. 362]. We then find that the above rank condition never holds true, *except in the case of the structurally damped equation* (1.1.1) *with* $\alpha = 1/2$. In this case, since e^{At} is analytic, we then let $u_1 \equiv 0$ on $0 \leq t \leq \epsilon$, so that $y(\epsilon) = e^{A\epsilon}y_0 \in \mathcal{D}(A)$. Then, the steering control u_2 in [9] steers $y(\epsilon)$ to the origin over an additional time interval T (arbitrarily small). Thus, u_1 followed by u_2 steers y_0 to the origin on an arbitrarily small time. This re-proves exact null-controllability for (1.1.1) with $\alpha = 1/2$. $\square$

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