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Maximal regularity for stochastic convolutions in L^p spaces

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Analisi matematica. — *Maximal regularity for stochastic convolutions in L^p spaces.* Nota (*) di Giuseppe Da Prato e Alessandra Lunardi, presentata dal Corrisp. G. Da Prato.

Abstract. — We prove an optimal L^p regularity result for stochastic convolutions in certain Banach spaces. It is stated in terms of real interpolation spaces.

Key words: Stochastic convolution; Analytic semigroups; Interpolation spaces.

Riassunto. — *Regolarità massimale per convoluzioni stocastiche negli spazi L^p.* Si dimostra un risultato di regolarità ottimale L^p per convoluzioni stocastiche in spazi di interpolazione fra opportuni spazi di Banach.

1. Introduction

This *Note* is concerned with optimal regularity of the stochastic convolution

$$(W_\beta(\varphi))(t) = \int_0^t e^{(t-s)A}\varphi(s)\,d\beta(s) \ ,$$

where A: $D(A) \subset X \mapsto X$ is the generator of an analytic semigroup e^{tA} in a Banach space X, β is a standard brownian motion in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and φ is a L^p integrable adapted stochastic process in $(0, T)$.

The smoothing effect of the stochastic convolution is known if X is a Hilbert space and $p = 2$ [1]: if $\varphi \in L^2_\beta(0, T; D_A(\theta, 2))$ with $0 < \theta < 1$ then $W_\beta(\varphi) \in L^2_\beta(0, T; D_A(\theta + 1/2, 2))$ if $\theta \neq 1/2$, $W_\beta(\varphi) \in L^2_\beta(0, T; D(A))$ if $\theta = 1/2$. Here $D_A(\alpha, 2)$ is the real interpolation space $(X, D(A))_{\alpha,2}$ if $0<\alpha<1$, or $(D(A), D(A^2))_{\alpha-1,2}$ if $1 < \alpha < 2$. See [8].

This result is useful in the study of partial differential stochastic equations such as for instance the Zakai equations arising in Filtering Theory [6, 2].

In this *Note* we generalize this result to a wide class of Banach spaces, $p \geq 1$, and $0 \leq \theta < 1$. Precisely, we show that if $\varphi \in L^p_\beta(0, T; D_A(\theta, p))$ then $W_\beta(\varphi) \in L^p_\beta(0, T; D_A(\theta + 1/2, p))$. Note however that in general $D_A(0, p)$ does not coincide with X, and $D_A(1, p)$ does not coincide with $D(A)$.

The class of Banach spaces that we consider are those spaces where the Burkholder inequality (see next section) holds. Such inequality is known to be true in the 2-uniformly smooth Banach spaces, and so in particular in the Lebesgue spaces $L^q(\mathbb{R}^d)$ and in the Sobolev spaces $W^{k,q}(\mathbb{R}^d)$, $q \geq 2$. This follows from [7, Proposition 2.4; 5, Lemma 1.1].

In the case where A is the realization of a second order elliptic operator with regular

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coefficients in $L^p(\mathbb{R}^d)$ with $p \geq 2$, Krylov [4] proved that if $\varphi \in L^p_\beta(0, T; W^{1,p}(\mathbb{R}^d))$ then $W_\beta(\varphi) \in L^p_\beta(0, T; W^{2,p}(\mathbb{R}^d)) = L^p_\beta(0, T; D(A))$. Our method does not allow us to prove such a result, since for $p \neq 2$ $W^{1,p}(\mathbb{R}^d)$ is not a real interpolation space between $L^p(\mathbb{R}^d)$ and $D(A) = W^{2,p}(\mathbb{R}^d)$. We have in fact $D_A(1/2, p) = B^1_{p,p}(\mathbb{R}^d)$, and $D_A(1, p) = B^2_{p,p}(\mathbb{R}^d)$, so that we get an optimal regularity result in Besov spaces rather than in Sobolev spaces.

2. Optimal regularity

We recall the definition and some properties of the interpolation spaces which will be used in the following, referring to [8] for an extensive treatment of interpolation theory.

Let X be a Banach space with norm $\|\cdot\|$ and let A: $D(A) \subset X \mapsto X$ generate an analytic semigroup e^{tA} in X. For any $x \in X$, $\theta \in (0, 1)$, and $p \geq 1$ we set

$$|x|^p_{D_A(\theta,p)} = \int_0^1 \left\| \xi^{1-\theta} A e^{\xi A} x \right\|^p \frac{d\xi}{\xi},$$

and

$$|x|^p_{D_{A^2}(\theta,p)} = \int_0^1 \left\| \xi^{2(1-\theta)} A^2 e^{\xi A} x \right\|^p \frac{d\xi}{\xi}.$$

The interpolation spaces $D_A(\theta, p)$, $D_A(\theta + 1, p)$ and $D_{A^2}(\theta, p)$ are defined by

$$D_A(\theta, p) = \left\{x \in X : |x|_{D_A(\theta,p)} < +\infty\right\}, \qquad \|x\|_{D_A(\theta,p)} = \|x\| + |x|_{D_A(\theta,p)},$$

$$D_A(\theta + 1, p) = \left\{x \in D(A) : Ax \in D_A(\theta, p)\right\}, \qquad \|x\|_{D_A(\theta+1,p)} = \|x\| + \|Ax\|_{D_A(\theta,p)},$$

$$D_{A^2}(\theta, p) = \left\{x \in X : |x|_{D_{A^2}(\theta,p)} < +\infty\right\}, \qquad \|x\|_{D_{A^2}(\theta,p)} = \|x\| + |x|_{D_{A^2}(\theta,p)}.$$

There is some difference for $\theta = 0$. Assume for simplicity that $0 \in \rho(A)$, so that the seminorm

$$x \mapsto |x|_{D_A(0,p)} = \left(\int_0^1 \left\| \xi A e^{\xi A} x \right\|^p \frac{d\xi}{\xi} \right)^{1/p}$$

is in fact a norm. The space

$$X_0 = \left\{x \in X : |x|_{D_A(0,p)} < +\infty\right\},$$

endowed with the norm $|\cdot|_{D_A(0,p)}$, is not complete in general. So, $D_A(0, p)$ is defined as the completion of X_0 in the norm $|\cdot|_{D_A(0,p)}$. If $0 \in \rho(A)$, for all $\omega \in \mathbb{R}$ such that $A - \omega I$ is of negative type the spaces $D_{A-\omega I}(0, p)$ are equivalent, so we may choose $\omega = 1 + 2$ type of A and we set $D_A(0, p) = D_{A-\omega I}(0, p)$. The semigroup e^{tA} has a natural extension to $D_A(0, p)$, which will be still denoted e^{tA}, and it turns out to be a holomorphic semigroup. For a detailed treatment of the spaces $D_A(0, p)$ see [3].

In the proof of our result the next proposition will play a key role.

PROPOSITION 2.1. *For every* $p \geq 1$, $\theta \in [0, 3/2)$ *it holds*

$$D_A(\theta + 1/2, p) = D_{A^2}(\theta/2 + 1/4, p) , \tag{2.1}$$

with equivalence of the respective norms.

We will also need the Burkholder inequality, which does not hold in any Banach space. A class of Banach spaces in which it holds are the 2-uniformly convex spaces [7].

We denote by $L^p_\beta(0, T; X)$ the set of all adapted X-valued stochastic processes φ such that

$$\mathbb{E}\left(\int_0^T \|\varphi(t)\|^p \, dt\right) < +\infty .$$

$L^p_\beta(0, T; X)$ is a Banach space endowed with the norm

$$\|\varphi\|_{L^p_\beta(0, T; X)} = \left[\mathbb{E}\left(\int_0^T \|\varphi(t)\|^p \, dt\right)\right]^{1/p} .$$

From now on we assume that X and $p \geq 1$ are such that the Burkholder inequality holds, that is

$$\begin{cases} \exists C_p > 0 \quad \text{such that for all} \quad \varphi \in L^p_\beta(0, T; X) \\ \mathbb{E}\left(\left\|\int_0^T \varphi(s)\, d\beta(s)\right\|^p\right) \leq C_p \mathbb{E}\left[\left(\int_0^T \|\varphi(s)\|^2 ds\right)^{p/2}\right] . \end{cases} \tag{2.2}$$

We are able now to state our main result,

THEOREM 2.2. *Let X be a Banach space, $p \geq 1$ be such that* (2.2) *holds. Let A: $D(A) \subset \subset X \mapsto X$ generate an analytic semigroup e^{tA} in X. Then for every $\theta \in [0, 1)$ and $\varphi \in \in L^p_\beta(0, T; D_A(\theta, p))$, $W_\beta(\varphi) \in L^p_\beta(0, T; D_A(\theta + 1/2, p))$, and there exists K independent of φ such that*

$$\| W_\beta(\varphi)\|_{L^p_\beta(0, T; D_A(\theta+1/2, p))} \leq K\|\varphi\|_{L^p_\beta(0, T; D_A(\theta, p))} .$$

PROOF. Recalling (2.1), we have only to estimate

$$J := \mathbb{E}\int_0^T dt \int_0^1 \left\|\xi^{3/2-\theta} A^2 e^{\xi A} \int_0^t e^{(t-s)A}\varphi(s)\, d\beta(s)\right\|^p \frac{d\xi}{\xi} = \tag{2.3}$$

$$= \mathbb{E}\int_0^T dt \int_0^1 \left\|\xi^{3/2-\theta} \int_0^t A^2 e^{(t-s+\xi)A}\varphi(s)\, d\beta(s)\right\|^p \frac{d\xi}{\xi} .$$

Using the Burkholder inequality (2.2) we get

$$J \leq C_p \mathbb{E} \int_0^T dt \int_0^1 \xi^{(3/2-\theta)p} \left(\int_0^t \| A^2 e^{(t-s+\xi)A} \varphi(s) \|^2 ds \right)^{p/2} \frac{d\xi}{\xi}. \tag{2.4}$$

Splitting $A^2 e^{(t-s+\xi)A} = Ae^{((t-s+\xi)/2)A} Ae^{((t-s+\xi)/2)A}$ and using the estimate $\|\sigma A e^{\sigma A}\|_{L(X)} \leq$ $\leq M$ for $0 < \sigma < (T+1)/2$ we get

$$J \leq C_p (2M)^p \, \mathbb{E} \int_0^T dt \int_0^1 \xi^{(3/2-\theta)p} \left(\int_0^t (t-s-\xi)^{-2} \| A e^{((t-s+\xi)/2)A} \varphi(s) \|^2 ds \right)^{p/2} \frac{d\xi}{\xi}, \tag{2.5}$$

so that by Hölder's inequality

$$J \leq c \mathbb{E} \int_0^T dt \int_0^1 \xi^{(3/2-\theta)p} \frac{1}{\xi^{p/2+1}} \int_0^t \| A e^{((t-s+\xi)/2)A} \varphi(s) \|^p ds \frac{d\xi}{\xi}. \tag{2.6}$$

Now, setting $\tau = \xi + t - s$ and exchanging integrals, we find

$$\begin{aligned}
J &\leq c \mathbb{E} \int_0^T ds \int_0^{T+1-s} \| A e^{\tau A/2} \varphi(s) \|^p d\tau \int_s^{\tau+s} (\tau - t + s)^{(1-\theta)p-2} dt = \\
&= c' \mathbb{E} \int_0^T ds \int_0^{T+1-s} \tau^{(1-\theta)p-1} \| A e^{\tau A/2} \varphi(s) \|^p d\tau \leq \\
&\leq c' \mathbb{E} \int_0^T ds \int_0^{T+1} \tau^{(1-\theta)p-1} \| A e^{\tau A/2} \varphi(s) \|^p d\tau = \\
&= c' \mathbb{E} \int_0^T ds \left(\int_0^{\min(2, T+1)} + \int_{\min(2, T+1)}^{T+1} \right) \tau^{(1-\theta)p-1} \| A e^{\tau A/2} \varphi(s) \|^p d\tau \leq \\
&\leq c'' \|\varphi\|_{L^p_\beta(0, T; D_A(\theta, p))}. \quad \square
\end{aligned}$$

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