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On groups with many nearly maximal subgroups


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Teoria dei gruppi. — On groups with many nearly maximal subgroups. Nota (*) di Silvana Franciosi e Francesco de Giovanni, presentata dal Socio G. Zappa.

Abstract. — A subgroup $M$ of a group $G$ is nearly maximal if the index $|G : M|$ is infinite but every subgroup of $G$ properly containing $M$ has finite index, and the group $G$ is called nearly $IM$ if all its subgroups of infinite index are intersections of nearly maximal subgroups. It is proved that an infinite (generalized) soluble group is nearly $IM$ if and only if it is either cyclic or dihedral.

Key words: Nearly maximal subgroup; Nearly $IM$-group; Soluble group.

Riassunto. — Sui gruppi con molti sottogruppi massimali generalizzati. Un sottogruppo $M$ di un gruppo $G$ si dice «nearly maximal» se l’indice $|G : M|$ è infinito mentre ogni sottogruppo di $G$ che contenga propriamente $M$ ha indice finito in $G$, ed il gruppo $G$ si dice «nearly IM» se ogni suo sottogruppo di indice infinito è intersezione di sottogruppi «nearly maximal». Si prova che un gruppo risolubile (generalizzato) infinito è «nearly IM» se e solo se è ciclico oppure diedrale.

1. Introduction

A subgroup $M$ of an infinite group $G$ is said to be nearly maximal if it is a maximal element of the set of all subgroups of $G$ having infinite index, i.e. if the index $|G : M|$ is infinite but every subgroup of $G$ properly containing $M$ has finite index in $G$. The near Frattini subgroup of a group $G$ is defined to be the intersection of all nearly maximal subgroups of $G$, with the stipulation that it shall equal $G$ if $G$ has no nearly maximal subgroups. These concepts were introduced by Riles [4] in 1969, and more recently Lennox and Robinson [1] have shown that certain group theoretical properties of finitely generated soluble groups can be detected from the behaviour of nearly maximal subgroups.

A group $G$ is called an $IM$-group if all its subgroups are intersections of maximal subgroups. Soluble $IM$-groups have been completely characterized by Menegazzo [2]. Here we will consider the corresponding concept obtained replacing maximal by nearly maximal subgroups. A group $G$ is said to be nearly $IM$ if every subgroup of infinite index of $G$ is intersection of nearly maximal subgroups. In particular, if $G$ is a nearly $IM$-group, all infinite factor groups of $G$ have trivial near Frattini subgroup.

The aim of this article is to prove the following result, which shows that (generalized) soluble nearly $IM$-groups have a very restricted structure.

Theorem. Let $G$ be an infinite hyper-(abelian or finite) group. Then $G$ is nearly $IM$ if and only if it is either cyclic or dihedral.

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The consideration of Tarski groups shows that in the above theorem the hypothesis that the group \( G \) is hyper-abelian or finite cannot be omitted.

Most of our notation is standard and can be found in [5].

2. **Proof of the Theorem**

We shall prove that the Fitting subgroup of an arbitrary infinite nearly IM-group is abelian. This will be done using the following two lemmas.

**Lemma 1.** Let \( G \) be an infinite nearly IM-group. Then \( G \) does not contain finite non-trivial normal subgroups.

**Proof.** Let \( E \) be a finite normal subgroup of \( G \). If \( M \) is any nearly maximal subgroup of \( G \), the index \( |EM : M| \) is obviously finite, so that \( EM = M \) and \( E \) is contained in \( M \). On the other hand, the near Frattini subgroup of \( G \) is trivial, and hence \( E = 1 \). Therefore the group \( G \) does not contain finite non-trivial normal subgroups. \( \Box \)

**Lemma 2.** Let \( G \) be a nearly IM-group, and let \( N \) be a normal subgroup of infinite index of \( G \). If \( M \) is a nearly maximal subgroup of \( G \) which does not contain \( N \), then \( M \) is maximal in \( MN \).

**Proof.** Let \( H \) be a subgroup of \( MN \) properly containing \( M \). Then \( H \) has finite index in \( G \), and so also the index \( |N : H \cap N| \) is finite. Moreover, \( H \cap N \) is a normal subgroup of \( H \), and so it has only finitely many conjugates in \( G \). It follows that the core \( K \) of \( H \cap N \) in \( G \) has finite index in \( N \), and \( N/K \) is a finite normal subgroup of the infinite group \( G/K \). On the other hand, \( G/K \) is nearly IM, and hence \( N = K \) by Lemma 1. Thus \( N \) is contained in \( H \), and so \( H = MN \). Therefore \( M \) is a maximal subgroup of \( MN \). \( \Box \)

**Proposition 3.** Let \( G \) be an infinite nearly IM-group. Then the Fitting subgroup of \( G \) is abelian.

**Proof.** Let \( N \) be any nilpotent normal subgroup of \( G \), and suppose first that \( N \) has finite index in \( G \). If \( a \) is a non-trivial element of \( Z(N) \), the normal closure \( A = \langle a \rangle^G \) is a finitely generated abelian group, and hence it contains a proper characteristic subgroup \( B \) such that \( A/B \) is finite. Since the factor group \( G/B \) is nearly IM, it follows from Lemma 1 that \( G/B \) is finite. Then also the group \( N/Z(N) \) is finite, and so the commutator subgroup \( N' \) of \( N \) is finite, and another application of Lemma 1 yields that \( N' = 1 \) and \( N \) is abelian. Suppose now that the index \( |G : N| \) is infinite, and assume that \( N' \neq 1 \), so that there exists a nearly maximal subgroup \( M \) of \( G \) which does not contain \( N' \). Then

\[
M < MN' \leq MN,
\]

and \( M \) is a maximal subgroup of \( MN \) by Lemma 2, so that \( MN' = MN \). Thus \( N = N'(M \cap N) \), and hence \( N = M \cap N \) is contained in \( M \). This contradiction shows
that also in this case $N$ is abelian. Therefore all nilpotent normal subgroups of $G$ are abelian, and so also the Fitting subgroup of $G$ is abelian. □

**Lemma 4.** Let $G$ be a group and let $M$ be a nearly maximal subgroup of $G$. If $A$ is a finitely generated abelian normal subgroup of $G$ such that $A/(A \cap M)$ is infinite, then $A/(A \cap M)$ is torsion-free.

**Proof.** Let $T/(A \cap M)$ be the subgroup consisting of all elements of finite order of the group $A/(A \cap M)$. Then $T/(A \cap M)$ is finite and $T^M = T$, so that $M$ has finite index in $TM$ and hence $TM = M$. Thus $T$ is contained in $M$, and so $A \cap M = T$. Therefore $A/(A \cap M)$ is a torsion-free group. □

Our last lemma deals with finitely generated soluble-by-finite nearly $IM$-groups.

**Lemma 5.** Let $G$ be a finitely generated soluble-by-finite group. If $G$ is nearly $IM$, then it is abelian-by-finite.

**Proof.** Let $S$ be the soluble radical of $G$, and let $K$ be the smallest non-trivial term of the derived series of $S$. By induction on the derived length of $S$ the factor group $Q = G/K$ is abelian-by-finite, and hence also polycyclic-by-finite. In particular, the finitely generated group $G$ is abelian-by-polycyclic-by-finite, and so it satisfies the maximal condition on normal subgroups (see [5, Part I, Theorem 5.34]). Thus $K$ contains a maximal proper $G$-invariant subgroup $L$, and $K/L$ is a simple $Q$-module. It follows that $K/L$ is finite (see [6]), and hence the factor group $G/L$ must be finite by Lemma 1. Therefore $G$ is abelian-by-finite. □

**Proof of the Theorem.** Clearly both the infinite cyclic group and the infinite dihedral group are nearly $IM$.

Conversely, suppose that $G$ is nearly $IM$, and let

$$1 = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

be an ascending normal series of $G$ whose factors either are finite or abelian. Consider any ordinal $\alpha < \tau$ such that the subgroup $G_\alpha$ has infinite index in $G$. Then the factor group $G/G_\alpha$ does not contain finite non-trivial normal subgroups by Lemma 1, and hence $G_{\alpha+1}/G_\alpha$ is abelian. It follows that the group $G$ is hyperabelian-by-finite. Let $N$ be a hyperabelian normal subgroup of finite index of $G$, and assume that $N$ is not soluble. Then there exists an ascending chain

$$K_1 < K_2 < \ldots < K_n < K_{n+1} < \ldots$$

of soluble normal subgroups of $N$ such that the subgroup

$$K = \bigcup_{n \in \mathbb{N}} K_n$$

is not soluble (see [3, Lemma 5]). Moreover, since $N$ has finite index in $G$, the subgroups $K_n$ can be chosen to be normal in $G$. The Fitting subgroup $F$ of $G$ is abelian by Proposition 3, and hence we may also suppose that $K_1$ contains $F$. As $N$ is not soluble, the index $|G : F|$ is infinite, and the subgroup $F$ is not cyclic. Let $a$ be
any non-trivial element of $F$, and let $\mathcal{L}$ be the set of all nearly maximal subgroups of $G$ containing $a$ but not $F$. If $M$ is any element of $\mathcal{L}$, we have

$$M < MF \leq MK_1 \leq MK_2 \leq \ldots \leq MK_n \leq MK_{n+1} \leq \ldots,$$

and $M$ is a maximal subgroup of each $MK_n$ by Lemma 2. It follows that $MK_n = MF$ for all $n$, and hence $MK = MF$, so that in particular $M \cap F$ is a normal subgroup of $MK$. Moreover,

$$\langle a \rangle = F \cap \left( \bigcap_{M \in \mathcal{L}} M \right) = \bigcap_{M \in \mathcal{L}} (M \cap F),$$

and so $\langle a \rangle$ is a normal subgroup of $K$. Therefore $K$ acts on $F$ as a group of power automorphisms, and hence $K' \leq C_N(F) = F$, so that $K'$ is abelian and $K$ is soluble. This contradiction proves that $N$ is soluble, and so $G$ is soluble-by-finite.

Assume now that the theorem is false, and choose a counterexample $G$ whose soluble radical $S$ has minimal derived length. If $K$ is the smallest non-trivial term of the derived series of $S$, it follows that the factor group $G/K$ is finitely generated. Suppose first that $G/K$ is finite, and let $a$ be a non-trivial element of $K$. Then the normal closure $\langle a \rangle^G$ is a finitely generated torsion-free abelian group, so that $G/\langle (a)^G \rangle^2$ contains a finite non-trivial normal subgroup, and hence it is finite by Lemma 1. It follows that $G$ is finitely generated. Assume now that $G$ is not finitely generated, so that the factor group $G/K$ is infinite, and let $E$ be a finitely generated subgroup of $G$ such that $G = EK$. Then the index $|G:E|$ is infinite, and so $E$ is contained in a nearly maximal subgroup $M$ of $G$. Clearly $G = MK$ and $M \cap K$ is a normal subgroup of $G$. Since $M/(M \cap K)$ is a finitely generated nearly maximal subgroup of $G/(M \cap K)$, the group $G/(M \cap K)$ is also finitely generated. On the other hand, $G/(M \cap K)$ is the semidirect product of its infinite subgroups $M/(M \cap K)$ and $K/(M \cap K)$, and so it is a counterexample to the theorem. This argument shows that without loss of generality it can be assumed that the group $G$ is finitely generated. Then $G$ is abelian-by-finite by Lemma 5. In particular, the Fitting subgroup $A$ of $G$ is a finitely generated torsion-free abelian group. Suppose that $A$ is not cyclic, so that it contains a subgroup $B$ such that $A/B$ is infinite and the subgroup $T/B$ of all elements of finite order of $A/B$ is not trivial. Let $X$ be any nearly maximal subgroup of $G$ containing $B$. Then the index $|A : A \cap X|$ is infinite, and so $A/(A \cap X)$ is torsion-free by Lemma 4. It follows that $T$ is contained in $X$, a contradiction, since $B$ is intersection of nearly maximal subgroups of $G$. Therefore $A$ is infinite cyclic. Clearly $A$ is contained in the centre of the normal subgroup $C = C_G(A)$ of $G$, and so $C/Z(C)$ is finite. Then also $C'$ is finite, and hence $C' = 1$ and $A = C$, so that $G/A$ has order at most 2. If $G$ is not cyclic, it contains a non-trivial subgroup $H$ of infinite index (see [5, Part 1, Theorem 4.33]). Then $H \cap A = 1$ and $H$ has order 2, so that $G$ is infinite dihedral. The theorem is proved.

Note finally that the class of nearly IM-groups is not subgroup closed. To see this, let $T_p$ and $T_q$ be a Tarski $p$-group and a Tarski $q$-group, respectively (where $p$ and $q$ are distinct primes), and let $P$ be a subgroup of order $p$ of $T_p$. Then the direct product $G = T_p \times T_q$ is nearly IM, but its subgroup $H = PT_q$ does not have the same property.
References


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