Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

# Rendiconti Lincei
# Matematica e Applicazioni

Anatolii S. Kalashnikov

# Instantaneous shrinking of the support for solutions to certain parabolic equations and systems

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9*, Vol. **8** (1997), n.4, p. 263–272.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1997_9_8_4_263_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1997.

*Rend. Mat. Acc. Lincei*
*s. 9, v. 8:263-272 (1997)*

**Equazioni a derivate parziali.** — *Instantaneous shrinking of the support for solutions to certain parabolic equations and systems.* Nota (*) di ANATOLII S. KALASHNIKOV, presentata dal Socio O. A. Oleinik.

ABSTRACT. — The paper contains conditions ensuring instantaneous shrinking of the support for solutions to semilinear parabolic equations with compactly supported coefficients of nonlinear terms and reaction-diffusion systems.

KEY WORDS: Semilinear parabolic equations; Reaction-diffusion systems; Compactly supported solutions.

RIASSUNTO. — *Contrazione istantanea del supporto per soluzioni di alcune equazioni e sistemi parabolici.* Questo lavoro contiene condizioni che garantiscono contrazione istantanea del supporto per soluzioni delle equazioni semi-lineari a coefficienti con supporti compatti e sistemi della classe «reazione-diffusione».

## 1. INTRODUCTION

Let $Q_T$ be the strip $\{(x,t)\,|\,x \in \boldsymbol{R},\ 0 < t \leqslant T\}$. For an arbitrary nonnegative function $w(x,t)$ continuous in $\overline{Q}_T$ we set

$$\zeta(t; w) = \sup\{\,|x|\,\,|w(x,t) > 0\}\,. \tag{1.1}$$

DEFINITION 1. Suppose that $w \in C(\overline{Q}_T)$, $w(x,t) \geqslant 0$, $\zeta(0; w) = +\infty$. We say that instantaneous shrinking of the support (briefly, ISS) occurs for $w$ if $\zeta(t; w) < +\infty$ for all $t$ in some half-interval $(0, \tau]$, where $0 < \tau \leqslant T$.

In [1] the ISS phenomenon has been described for some classes of spatially homogeneous and autonomous nonlinear second order parabolic equations. The results obtained in [1] yield the following proposition.

THEOREM 1 (see [1]). Assume that a function $u \in C^{2,1}_{x,t}(Q_T) \cap C(\overline{Q}_T) \cap L^\infty(Q_T)$ satisfies the equation

$$D_t u - D_x^2 u + |u|^p \operatorname{sign} u = 0\,, \qquad (x,t) \in Q_T \qquad (p > 0) \tag{1.2}$$

and the initial condition

$$u(x, 0) = f(x)\,, \qquad x \in \boldsymbol{R}\,, \tag{1.3}$$

where

$$f \in C(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})\,, \qquad f(x) > 0\,, \qquad \forall x \in \boldsymbol{R}\,, \qquad \lim_{|x| \to \infty} f(x) = 0\,. \tag{1.4}$$

Then ISS occurs for $u$ if and only if

$$p < 1\,. \tag{1.5}$$

This theorem has been subjected to various generalizations and modifications.

(*) Pervenuta all'Accademia il 25 giugno 1997.

Thus, in [2] the Cauchy problem has been considered for the equation

$$(1.6) \qquad \mathcal{a}u \equiv D_t u - D_x^2 u + g(x,t)|u|^p \operatorname{sign} u = 0\,, \qquad (x,t) \in Q_T \qquad (p > 0)$$

and its multidimensional counterpart under hypotheses (1.4), (1.5) and the following additional assumptions:

$$(1.7) \qquad f(x) \leqslant M(1 + |x|)^{-\alpha} \qquad (M > 0, \alpha > 0);$$

$$(1.8) \qquad g \in C(\overline{Q}_T) \cap L^\infty(Q_T);$$

$$(1.9) \qquad g(x,t) \geqslant g_0(1 + |x|)^{-\beta} \qquad (g_0 > 0, \beta \geqslant 0).$$

It has been proved that ISS occurs for the solution of problem (1.6), (1.3) if and only if $\alpha(1-p) > \beta$. In [3] similar results have been obtained for some equations with nonlinear leading terms.

In [2] and [4] the ISS phenomenon has been studied for certain semilinear parabolic systems. Specifically, the following proposition has been proved.

Theorem 2 (see [4]). Assume that a vector-valued function $(u_1, u_2) \in (C_{x,t}^{2,1}(Q_T) \cap \cap C(\overline{Q}_T) \cap L^\infty(Q_T))^2$ satisfies the system

$$(1.10) \qquad D_t u_i - D_x^2 u_i + |u_1|^{p_{i1}} |u_2|^{p_{i2}} \operatorname{sign} u_i = 0\,, \qquad (x,t) \in Q_T \qquad (i = 1, 2)$$

with the initial conditions

$$(1.11) \qquad u_i(x, 0) = f_i(x)\,, \qquad x \in \boldsymbol{R} \qquad (i = 1, 2),$$

and the following relations hold:

$$(1.12) \qquad f_i \in C(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R}); \qquad m_j(1 + |x|)^{-\alpha_i} \leqslant f_i(x) \leqslant M_i(1 + |x|)^{-\alpha_i},$$
$$M_i \geqslant m_i > 0\,, \qquad \alpha_i > 0 \qquad (i = 1, 2);$$

$$(1.13) \qquad p_{ij} \geqslant 0 \qquad (i, j = 1, 2);$$

$$(1.14) \qquad 0 < p_{11} < 1;$$

$$(1.15) \qquad \alpha_1(1 - p_{11}) > \alpha_2 p_{12};$$

$$(1.16) \qquad p_{22} \geqslant 1.$$

Then ISS occurs for $u_1$ and does not occur for $u_2$.

Moreover, in [4] the best possible upper estimate has been obtained for $\zeta(t; u_1)$. It has been also shown that ISS for $u_1$ may be absent if (1.15) is violated.

Prof. V. V. Zhikov has posed the following question: is it possible that ISS occurs for both $u_1$ and $u_2$ in the case where the inequalities

$$(1.17) \qquad 0 < p_{22} < 1$$

holds instead of (1.16)? We study this question in Section 3 of the present paper.

As a preliminary step, in Section 2 we consider problem (1.6), (1.3), where the function $g(x,t)$ possesses the ISS property. We obtain conditions ensuring the occurrence of ISS for the solution of the said problem and derive two-sided estimates for this solution. We apply the results of Section 2 to the study of problem (1.10), (1.11). Besides that, those results are of interest in themselves.

In order to simplify the exposition, we do not state our theorems in their maximal generality. Specifically, they can be easily extended to the case of arbitrarily many independent variables. Moreover, instead of (1.10), a similar system can be considered containing more than two equations.

The ISS phenomenon has been studied by a number of authors. Along with the above cited works [2-4], where one can find many further references, we call the reader's attention to the recent paper [5] containing a vast bibliographical list. As a supplement to it, we also refer to the articles [6-8].

## 2. Parabolic equations with ISS for coefficients

Consider equation (1.6) with initial data (1.3). Hypotheses (1.4), (1.5), (1.7), and (1.8) being preserved, we replace (1.9) by the following less restrictive assumption:

$$g(x,t) \geqslant g_0(1+|x|)^{-\beta}[1-ht(1+|x|)^{\mu}]_+^{\sigma} \qquad (g_0>0, h>0, \mu>0, \sigma\geqslant 0, \beta \geqslant 0). \tag{2.1}$$

Here we have used the notation $y_+ = \max\{y, 0\}$.

We set $Q_{\tau,\xi,\eta} = (\xi,\eta)\times(0,\tau]$, where $-\infty \leqslant \xi < \eta \leqslant +\infty$, $\tau>0$. In the sequel, we use the following Comparison Principle, which can be obtained, *e.g.*, as a special case of Theorem 4 in [3].

THEOREM 3. Assume that the functions $u(x,t)$, $v(x,t)$, $V(x,t)$ belong to

$$L^\infty(Q_{\tau,\xi,\eta}) \cap C(\overline{Q_{\tau,\xi,\eta}}) \cap C_x^1(Q_{\tau,\xi,\eta}) \cap C_{x,t}^{2,1}(Q_{\tau,\xi,\eta}\backslash S),$$

where $S$ is a smooth arc, and satisfy the inequalities $v \leqslant u \leqslant V$ in $\overline{Q_{\tau,\xi,\eta}}\backslash Q_{\tau,\xi,\eta}$, $\mathcal{A}v \leqslant \mathcal{A}u \leqslant \mathcal{A}V$ in $Q_{\tau,\xi,\eta}\backslash S$. Then $v(x,t)\leqslant u(x,t) \leqslant V(x,t)$ for all $(x,t)\in \overline{Q_{\tau,\xi,\eta}}$.

COROLLARY. If $f(x)\geqslant 0$, then the solution $u(x,t)$ of problem (1.6), (1.3) is nonnegative everywhere in $\overline{Q}_T$.

This statement follows from Theorem 3 if we set $\tau = T$, $\xi = -\infty$, $\eta = +\infty$, $v(x,t)\equiv 0$.

THEOREM 4. Let $u \in C_{x,t}^{2,1}(Q_T)\cap C(\overline{Q}_T)\cap L^\infty(Q_T)$ be the solution of problem (1.6), (1.3). Assume that conditions (1.4), (1.5), (1.7), (1.8), (2.1) are satisfied and the inequality

$$\alpha(1-p) > \beta + \mu \tag{2.2}$$

holds. Then ISS occurs for $u$. Moreover, we have

$$\zeta(t;u) \leqslant (ht)^{-1/\lambda}, \qquad \forall t\in(0,\tau] \tag{2.3}$$

and

$$u(x,t) \leqslant 2M(1+|x|)^{-\alpha}, \qquad \forall (x,t)\in \overline{Q}_\tau, \tag{2.4}$$

where $0 < \tau \leqslant T$ and

$$\mu < \lambda < \alpha(1-p) - \beta. \tag{2.5}$$

PROOF. Consider the function

$$V(x,t) = 2M(1+x)^{-\alpha}[1 - ht(1+x)^{\lambda}]_{+}^{\omega} \tag{2.6}$$

in the half-strip $\overline{Q_{\tau,\xi,+\infty}}$ (a similar function has been used for the proof of Theorem 6 in [3]). Let us show that there exist parameters

$$\xi > 0, \tag{2.7}$$

$$\omega > 2, \tag{2.8}$$

and $\tau \in (0, T]$ such that

$$u(x,t) \leqslant V(x,t), \quad \forall (x,t) \in \overline{Q_{\tau,\xi,+\infty}}. \tag{2.9}$$

Let $G = Q_{\tau,\xi,+\infty}$. From (2.6)-(2.8) it follows that $V \in C^{2,1}_{x,t}(G) \cap L^{\infty}(G)$. Set

$$Z(x,t) = 1 - ht(1+x)^{\lambda}, \quad G_{+} = \{(x,t) \in G \mid Z(x,t) > 0\}. \tag{2.10}$$

We obviously have

$$\mathcal{A}V(x,t) = 0, \quad \forall (x,t) \in G \backslash G_{+}. \tag{2.11}$$

For $(x,t) \in G_{+}$ we get

$$\begin{aligned} \mathcal{A}V(x,t) = {} & (2M)^{p}(1+x)^{-\alpha p} g(x,t) Z^{\omega p} - 2Mh\omega(1+x)^{\lambda-\alpha} Z^{\omega-1} - \\ & -2M(1+x)^{-\alpha-2} Z^{\omega-2}\{\alpha(\alpha+1)Z^{2} + \omega\lambda(2\alpha-\lambda+1)ht(1+x)^{\lambda}Z + \\ & + \omega(\omega-1)\lambda^{2}[ht(1+x)^{\lambda}]^{2}\}, \quad \forall (x,t) \in G_{+}. \end{aligned} \tag{2.12}$$

Note that

$$ht(1+x)^{\lambda} < 1, \quad Z(x,t) \leqslant 1, \quad \forall (x,t) \in G_{+}. \tag{2.13}$$

Set

$$c = \alpha(\alpha+1) + \omega\lambda(2\alpha - \lambda + 1) + \omega(\omega-1)\lambda^{2}. \tag{2.14}$$

Using (2.1), (2.3), (2.10), (2.11), and (2.12), we obtain

$$\mathcal{A}V(x,t) \geqslant 2M(1+x)^{-\alpha p-\beta} Z^{\omega p+\beta}[(2M)^{p-1}g_{0} - \varphi(x,t)], \quad \forall (x,t) \in G, \tag{2.15}$$

where

$$\begin{aligned} \varphi(x,t) = {} & h\omega(1+x)^{\lambda+\beta-\alpha(1-p)} Z^{\omega(1-p)-1-\sigma} + \\ & + c(1+x)^{\beta-\alpha(1-p)-2} Z^{\omega(1-p)-2-\sigma}. \end{aligned} \tag{2.16}$$

By virtue of (2.5), we have

$$\beta - \alpha(1-p) - 2 < \lambda + \beta - \alpha(1-p) < 0. \tag{2.17}$$

Choose

$$\omega \geqslant (2+\sigma)/(1-p). \tag{2.18}$$

It is evident that (2.18) implies (2.6). Taking into account (2.17), (2.18), (2.13), and (2.7), from (2.16) we get

$$\varphi(x,t) \leqslant (h\omega + c)(1+\xi)^{\lambda+\beta-\alpha(1-p)} < (2M)^{p-1}g_{0}, \tag{2.19}$$

if

$$\xi \geqslant [(2M)^{1-p} g_0^{-1}(h\omega + c)]^{1/[\alpha(1-p)-\beta-\lambda]} . \tag{2.20}$$

Let us fix some $\omega$ and $\xi$ satisfying (2.18) and (2.20), respectively. Then (2.19) holds and from (2.15) we obtain

$$\mathcal{A}V(x, t) > 0 , \quad \forall (x, t) \in G_+ . \tag{2.21}$$

From (2.11), (2.21), and (1.6) it follows that

$$\mathcal{A}u(x, t) \leqslant \mathcal{A}V(x, t) , \quad \forall (x, t) \in G \tag{2.22}$$

for any choice of $\tau \in (0, T]$.

Using (2.6), (1.3), and (1.7), we get

$$u(x, 0) < V(x, 0) , \quad \forall x \in [0, +\infty) . \tag{2.23}$$

Hence, $\min\{V(x, 0) - u(x, 0) | 0 \leqslant x \leqslant \xi\} > 0$. Therefore, there exists $\tau_1 \in (0, T]$ such that

$$u(x, t) < V(x, t) , \quad \forall (x, t) \in \overline{Q_{\tau_1, 0, \xi}} . \tag{2.24}$$

By Theorem 3, inequalities (2.22)-(2.24) imply (2.9) if we take $\tau = \tau_1$.

Now, we combine (2.9) with (2.24) and thus arrive at the inequality

$$u(x, t) \leqslant V(x, t) , \quad \forall (x, t) \in \overline{Q_{\tau_1, 0, +\infty}} . \tag{2.25}$$

Similarly, we prove that

$$u(x, t) \leqslant V(x, t) , \quad \forall (x, t) \in \overline{Q_{\tau_2, -\infty, 0}} \tag{2.26}$$

for some $\tau_2 \in (0, T]$. From (2.25), (2.26), and (2.6) it follows that relations (2.3) and (2.4) hold for $\tau = \min\{\tau_1, \tau_2\}$. $\square$

THEOREM 5. *Let $u \in C_{x,t}^{2,1}(Q_T) \cap C(\overline{Q}_T) \cap L^\infty(Q_T)$ be the solution of problem (1.6), (1.3). Assume that conditions (1.4), (1.5), (1.8) are satisfied and the following inequalities hold:*

$$f(x) \geqslant m(1 + |x|)^{-\alpha} \quad (m > 0, \alpha > 0) ; \tag{2.27}$$

$$0 \leqslant g(x, t) \leqslant g_1(1 + |x|)^{-\beta} \quad (g_1 > 0, \beta \geqslant 0) . \tag{2.28}$$

*Then there exist $\tau \in (0, T]$ and $H > 0$ such that*

$$u(x, t) \geqslant (m/4)(1 + |x|)^{-\alpha}[1 - Ht(1 + |x|)^{\alpha(1-p)-\beta}]_+^{1/(1-p)} , \quad \forall (x, t) \in \overline{Q}_\tau . \tag{2.29}$$

PROOF. By virtue of (1.3) and (2.27), there exist $\tau_1 \in (0, T]$ such that

$$u(0, t) \geqslant m/2 , \quad \forall t \in [0, \tau_1] . \tag{2.30}$$

Let $\tau \in (0, \tau_1]$. In the half-strip $\overline{Q_{\tau, 0, +\infty}}$ we compare $u(x, t)$ with the function

$$v(x, t) = (m/4)(1 + x)^{-\alpha}[1 - Ht(1 + x)^{\alpha(1-p)-\beta}]_+^{1/(1-p)} , \tag{2.31}$$

where $H > 0$ is to be chosen later.

From (1.3), (2.27), and (2.31) it follows that

$$v(x, 0) < u(x, 0) , \quad \forall x \in [0, +\infty) . \tag{2.32}$$

Moreover, by virtue of (2.30), we have

$$v(0,t) \leqslant m/4 < u(0,t), \quad \forall t \in [0, \tau_1]. \tag{2.33}$$

Let us introduce the following notation:

$$z(x,t) = 1 - Ht(1+x)^{\alpha(1-p)-\beta}; \tag{2.34}$$

$$\begin{cases} E = Q_{\tau,0,+\infty}; \quad S = \{(x,t) \in E \mid z(x,t) = 0\}; \\ E_+ = \{(x,t) \in E \mid z(x,t) > 0\}. \end{cases} \tag{2.35}$$

It is clear that $v \in L^\infty(E) \cap C^1(\overline{E}) \cap C^2(E \backslash S)$ and

$$\mathcal{C}v(x,t) = 0, \quad \forall (x,t) \in E \backslash (S \cup E_+). \tag{2.36}$$

For $(x,t) \in E_+$ we get

$$\begin{aligned} \mathcal{C}v(x,t) = {} & (m/4)(1+x)^{-\alpha p-\beta} z^{p/(1-p)} \cdot \\ & \cdot \{H(1-p)^{-1}[-1 + (\beta - \alpha(1-p))(\alpha(1+p)+\beta+1)t(1+x)^{-2}] + \\ & + (m/4)^{p-1}(1+x)^\beta g(x,t) - \alpha(\alpha+1)(1+x)^{\beta-\alpha(1-p)-2} - \\ & - H^2 p(1-p)^{-2}(\alpha(1-p)-\beta)^2 t^2 (1+x)^{\alpha(1-p)-\beta-2} z^{-1}\}, \quad \forall (x,t) \in E_+. \end{aligned} \tag{2.37}$$

Let us estimate the right-hand side of (2.37) from above. Rejecting two last terms in the curly brackets and taking into account (2.28), we obtain from (2.37):

$$\begin{aligned} \mathcal{C}v(x,t) \leqslant {} & (m/4)(1+x)^{-\alpha p-\beta} z^{p/(1-p)} \cdot \\ & \cdot \{H(1-p)^{-1}[-1 + \tau(\beta - \alpha(1-p))(\alpha(1+p)+\beta+1)] + g_1(m/4)^{p-1}\}, \\ & \forall (x,t) \in E_+. \end{aligned} \tag{2.38}$$

Let $\tau_2 \in (0, T]$ be so small that the quantity entering the square brackets on the right-hand side of (2.38) exceeds $-1/2$ for $\tau \leqslant \tau_2$. Then from (2.38) it follows that

$$\begin{aligned} \mathcal{C}v(x,t) \leqslant {} & (m/4)(1+x)^{-\alpha p-\beta} z^{p/(1-p)} \cdot \\ & \cdot \{-[2(1-p)]^{-1}H + g_1(m/4)^{p-1}\} \leqslant 0, \quad \forall (x,t) \in E_+, \end{aligned} \tag{2.39}$$

if $\tau \leqslant \tau_2$ and

$$H \geqslant 2(1-p)g_1(m/4)^{-1}. \tag{2.40}$$

We choose some $H$ satisfying (2.40) and set $\tau = \min\{\tau_1, \tau_2\}$. Then relations (2.32), (2.33), (2.36), (2.38) hold, and the application of Theorem 3 yields the inequality

$$u(x,t) \geqslant v(x,t) \tag{2.41}$$

for all $(x,t) \in \overline{Q_{\tau,0,+\infty}}$. In the same way we prove that (2.41) is valid for all $(x,t) \in \overline{Q_{\tau,-\infty,0}}$. Thus, we have arrived at (2.29). □

COROLLARY. If the inequality

$$\alpha(1-p) \leqslant \beta \tag{2.42}$$

holds along with assumptions (1.4), (1.5), (1.8), (2.27), and (2.28), then ISS does not occur for $u$.

In fact, from (2.29) and (2.42) it follows that $u(x,t)>0$ for $x \in \boldsymbol{R}$ and $0 \leqslant t < 1/H$.

## 3. Parabolic systems

Now, consider system (1.10) with initial data (1.11).

Theorem 6. Let $(u_1, u_2) \in (C^{2,1}_{x,t}(Q_T) \cap C(\overline{Q}_T) \cap L^\infty(Q_T))^2$ be the solution of problem (1.10), (1.11). Assume that conditions (1.12)-(1.14) and (1.17) are satisfied. Then the following conclusions hold.

*A*) If we have

$$\alpha_1(1-p_{11}) > \alpha_2(1-p_{22}+p_{12}), \tag{3.1}$$

then ISS occurs for $u_1$. If in addition we have

$$\alpha_1 p_{21} \geqslant \alpha_2(1-p_{22}), \tag{3.2}$$

then ISS does not occur for $u_2$.

*B*) If we have

$$\alpha_2(1-p_{22}) > \alpha_1(1-p_{11}+p_{21}), \tag{3.3}$$

then ISS occurs for $u_2$. If in addition we have

$$\alpha_2 p_{12} \geqslant \alpha_1(1-p_{11}), \tag{3.4}$$

then ISS does not occur for $u_1$.

Proof. Set

$$g_1(x,t) = |u_2(x,t)|^{p_{12}}, \qquad g_2(x,t) = |u_1(x,t)|^{p_{21}}. \tag{3.5}$$

Then system (1.10) can be rewritten in the following manner:

$$\mathcal{A}_i u_i \equiv D_t u_i - D_x^2 u_i + g_i(x,t)|u_i|^{p_{ii}} \operatorname{sign} u_i = 0, \qquad (x,t) \in Q_T \qquad (i=1,2). \tag{3.6}$$

By the Corollary of Theorem 3, we get

$$u_i(x,t) \geqslant 0, \qquad \forall (x,t) \in \overline{Q}_T \qquad (i=1,2). \tag{3.7}$$

Let us prove Statement *A*). Suppose that (3.1) holds. The application of Theorem 5 with $\beta = 0$ to the second equation in (3.6) yields the inequality

$$u_2(x,t) \geqslant (m_2/4)(1+|x|)^{-\alpha_2}[1-Ht(1+|x|)^{\alpha_2(1-p_{22})}]_+^{1/(1-p_{22})}, \qquad \forall (x,t) \in \overline{Q}_{\tau_1},$$

where $\tau_1 \in (0,T]$, $H>0$. Hence,

$$g_1(x,t) \geqslant (m_2/4)^{p_{12}}(1+|x|)^{-\alpha_2 p_{12}}[1-Ht(1+|x|)^{\alpha_2(1-p_{22})}]_+^{p_{12}/(1-p_{22})}, \qquad \forall (x,t) \in \overline{Q}_{\tau_1}. \tag{3.8}$$

Now, we apply Theorem 4 to the first equation in (3.6), setting $p = p_{11}$, $\alpha = \alpha_1$, $\beta = \alpha_2 p_{12}$, $\mu = \alpha_2(1-p_{22})$, $\sigma = p_{12}/(1-p_{22})$, and taking into account (1.12), (1.14), (3.1), (3.7). It follows that ISS occurs for $u_1$ and the estimate

$$u_1(x,t) \leqslant 2M_1(1+|x|)^{-\alpha_1}, \qquad \forall (x,t) \in \overline{Q}_{\tau_2} \tag{3.9}$$

is valid with some $\tau_2 \in (0, \tau_1]$.

Next, suppose that (3.2) holds as well. By virtue of (3.5) and (3.9), we get

$$g_2(x,t) \leqslant (2M_1)^{p_{21}}(1+|x|)^{-\alpha_1 p_{21}}, \qquad \forall (x,t) \in \overline{Q}_{\tau_2}.$$

Therefore, we can apply Theorem 5 to the second equation in (3.6) with $p = p_{22}$, $\alpha = \alpha_2$, and $\beta = \alpha_1 p_{21}$. In this case inequality (3.2) has the form (2.42), and the corollary of Theorem 5 enables us to conclude that ISS does not occur for $u_2$.

This completes the proof of Statement $A$). The proof of Statement $B$) is quite similar. □

DEFINITION 2. We say that total instantaneous shrinking of the supports (briefly, TISS) occurs in problem (1.10), (1.11) if ISS occurs for both components of the solution.

COROLLARY. A necessary condition for TISS to occur in problem (1.10), (1.11) is that the following relations hold:

*i*) either the inequality

$$\alpha_1(1-p_{11}) \leqslant \alpha_2(1-p_{22}+p_{12}), \tag{3.10}$$

or the system of inequalities

$$\alpha_1(1-p_{11}) > \alpha_2(1-p_{22}+p_{12}), \qquad \alpha_2(1-p_{22}) > \alpha_1 p_{21}; \tag{3.11}$$

*ii*) either the inequality

$$\alpha_2(1-p_{22}) \leqslant \alpha_1(1-p_{11}+p_{21}), \tag{3.12}$$

or the system of inequalities

$$\alpha_2(1-p_{22}) > \alpha_1(1-p_{11}+p_{21}), \qquad \alpha_1(1-p_{11}) > \alpha_2 p_{12}. \tag{3.13}$$

Theorem 6 does not supply us with any sufficient condition for TISS to occur, since the system of inequalities (3.1), (3.3) is incompatible for $\alpha_1 > 0$ and $\alpha_2 > 0$. In fact, (3.1) and (3.3) imply the inequality $\alpha_1 p_{21} + \alpha_2 p_{12} < 0$, which is false for any pair of positive numbers $(\alpha_1, \alpha_2)$, by virtue of (1.13). Below, we state some sufficient conditions for TISS to occur, under certain additional assumptions about the exponents $p_{ij}$.

First, we note that in the special case of the split system, *i.e.* in the case where $p_{12} = p_{21} = 0$, all the necessary conditions (3.10)-(3.13) derived above are satisfied for arbitrary $\alpha_1 > 0$ and $\alpha_2 > 0$. On the other hand, in this case, for any positive $\alpha_1$ and $\alpha_2$, TISS occurs in problem (1.10), (1.11), by virtue of Theorem 1 applied to each equation in (1.10). Thus, for the split system, conditions (3.10)-(3.13) are necessary and sufficient.

Now, let us consider the class of system (1.10) specified by the equalities

$$p_{21} = p_{11}, \qquad p_{12} = p_{22}. \tag{3.14}$$

THEOREM 7. Assume that conditions (1.12)-(1.14), (1.17), and (3.14) are satisfied. Then TISS occurs in problem (1.10), (1.11) if and only if the following relations hold:

$$p_{11} + p_{22} < 1 \tag{3.15}$$

and

$$f_1(x) \equiv f_2(x), \qquad \forall x \in \boldsymbol{R} \tag{3.16}$$

without any restrictions on the positive numbers $m_1 = m_2$, $M_1 = M_2$, and $\alpha_1 = \alpha_2$.

PROOF. First, assume that relations (3.15) and (3.16) hold. Set

$$w(x, t) = u_1(x, t) - u_2(x, t). \tag{3.17}$$

Taking into account (1.10), (1.11), (3.7), (3.14), (3.16), and (3.17), we get

$$D_t w(x, t) - D_x^2 w(x, t) = 0, \qquad \forall (x, t) \in Q_T, \tag{3.18}$$

$$w(x, 0) = 0, \qquad \forall x \in \boldsymbol{R}. \tag{3.19}$$

Using the Maximum Principle for the heat equation, we obtain from (3.17)-(3.19):

$$u_1(x, t) = u_2(x, t), \qquad \forall (x, t) \in \overline{Q}_T. \tag{3.20}$$

We introduce the notation:

$$u(x, t) = u_i(x, t) \quad (i = 1, 2), \qquad p = p_{11} + p_{22}. \tag{3.21}$$

Formulas (1.10)-(1.12), (3.14), (3.20), and (3.21) imply that the function $u(x, t)$ satisfies (1.2)-(1.4). By virtue of (3.15) inequality (1.5) is also valid. Therefore, Theorem 1 is applicable to our case, and it follows that ISS occurs for $u$, *i.e.* TISS occurs in problem (1.10), (1.11).

Now, assume that TISS occurs in problem (1.10), (1.11). Then for any $t = \varepsilon > 0$ small enough there exists $x_\varepsilon$ such that

$$w(x, \varepsilon) = 0, \qquad \forall x \geqslant x_\varepsilon. \tag{3.22}$$

But it is well known that all solutions of the heat equation (3.18) are analytic with respect to $x$. Therefore, from (3.22) and (3.18) it follows that

$$w(x, \varepsilon) = 0, \qquad \forall x \in \boldsymbol{R}. \tag{3.23}$$

Since $\varepsilon > 0$ is arbitrary and $w(x, t)$ is continuous in $\overline{Q}_T$, identity (3.23) imply (3.19). Thus, we have proved (3.16). Moreover, we have already observed that from (3.17)-(3.19) the validity of (3.20) follows. Preserving notation (3.21) and taking into account relations (1.10)-(1.12), (3.14), (3.20), we again establish that $u(x, t)$ satisfies (1.2)-(1.4). By assumption, ISS occurs for $u$. Keeping this in mind and using Theorem 1, we arrive at inequality (3.15). □

## ACKNOWLEDGEMENTS

The research described in this paper has been supported in part by the Russian Foundation for Basic Research.

## REFERENCES

[1] L. C. EVANS - B. F. KNERR, *Instantaneous shrinking of the support of non-negative solutions to certain non-linear parabolic equations and variational inequalities*. Illinois J. Math., 23, 1979, 153-166.

[2] A. S. Kalashnikov, *On the conditions for instantaneous compactification of supports of solutions to semilinear parabolic equations and systems*. Matematicheskie Zametki, 47, 1990, 74-80.

[3] R. Kersner - F. Nicolosi, *The nonlinear heat equation with absorption: effects of variable coefficients*. J. Math. Anal. Appl., 170, 1992, 551-566.

[4] A. S. Kalashnikov, *On the behavior of the solutions of the Cauchy problem for parabolic systems with nonlinear dissipation near the initial hyperplane*. Trudy Seminara imeni I. G. Petrovskogo, 16, 1992, 106-113.

[5] R. Kersner - A. Shishkov, *Instantaneous shrinking of the support of energy solutions*. J. Math. Anal. Appl., 198, 1996, 729-750.

[6] A. S. Kalashnikov, *Nonlinear phenomena in nonstationary processes described by asymptotically linear equations*. Differentsial'nye Uravneniya, 29, 1993, 381-391.

[7] A. S. Kalashnikov, *Nonlinear mathematical models of thermal conduction involving a small parameter in exponents*. Differentsial'nye Uravneniya, 30, 1994, 1039-1049.

[8] U. G. Abdullaev, *On sharp local estimates for the supports of solutions to nonlinear parabolic equations*. Matematicheskii Sbornik, 186, 1995, 3-24.

Department of Mechanics and Mathematics
Moscow State University
119899 Mosca (Russia)