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Heteroclinic solutions for perturbed second order systems

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Analisi matematica. — *Heteroclinic solutions for perturbed second order systems.* Nota (*) di Massimiliano Berti, presentata dal Corrisp. A. Ambrosetti.

Abstract. — The existence of infinitely many heteroclinic orbits implying a chaotic dynamics is proved for a class of perturbed second order Lagrangian systems possessing at least 2 hyperbolic equilibria.

Key words: Heteroclinic orbits; Homoclinic orbits; Chaotic dynamics.

Riassunto. — *Soluzioni eterocline per sistemi perturbati del secondo ordine*. Viene dimostrata l'esistenza di infinite orbite eterocline per una classe di sistemi lagrangiani del secondo ordine, perturbati, aventi almeno 2 equilibri iperbolici. La dinamica è caotica.

1. Introduction

In a recent paper [2] the existence of infinitely many homoclinic orbits implying a chaotic dynamics for perturbed second order Lagrangian systems possessing an hyperbolic equilibrium is proved by means of a variational approach. The aim of the present *Note* is to extend these results proving the existence of infinitely many heteroclinic orbits for perturbed Lagrangian systems possessing two or more hyperbolic equilibria.

Let consider second order systems of differential equations like:

$$\ddot{u} + \nabla V(u) + \varepsilon \nabla_u W(t, u) = 0 \tag{1.1}$$

with $u \in \mathbb{R}^n$. Suppose that the potential V has two isolated critical points x_1 and x_2. A heteroclinic solution u of (1.1) connecting x_1 to x_2 is a $C^2(\mathbb{R}, \mathbb{R}^n)$ function satisfying the conditions:

$$\lim_{t \to -\infty} u(t) = x_1, \qquad \lim_{t \to +\infty} u(t) = x_2 \qquad \text{and} \qquad \lim_{|t| \to +\infty} \dot{u}(t) = 0.$$

Assume than the unperturbed system ($\varepsilon = 0$) possesses a heteroclinic solution z_0 connecting x_1 and x_2. Under general assumptions we show that if the Poincaré functions:

$$\Gamma(\theta) = -\int_{\mathbb{R}} W(t, z_0(t+\theta))\,dt \qquad \text{and} \qquad \widetilde{\Gamma}(\theta) = -\int_{\mathbb{R}} W(t, z_0(-t-\theta))\,dt$$

have infinitely many minima or maxima sufficiently separated one each other then there exist infinitely many orbits u_ε winding in the phase space k times between x_1 and x_2. When k is odd u_ε is a heteroclinic solution connecting x_1 to x_2 when k is even u_ε turns out to be a homoclinic solution to x_i. A sufficient condition in which these results apply is when the perturbation W is almost-periodic in time and the Poincaré functions Γ and $\widetilde{\Gamma}$ are non-constant.

Moreover, using as in [2], estimates which do not depend on k, we obtain the existence of solutions of (1.1) which turns infinitely many times between x_1 and x_2. The ex-

(*) Pervenuta all'Accademia l'1 settembre 1997.

istence of these orbits implies a chaotic dynamics which can be described as in [2, 6] in terms of approximate and complete Bernoulli shifts structures.

Using the same approach it is possible to study the situation in which the system possesses p hyperbolic equilibria. If in the unperturbed system they are connected by a chain of heteroclinics, we prove the existence of infinitely many connecting orbits for the perturbed system.

We assume the reader familiar with the techniques introduced in [1, 2]. Since many computations and lemmas are the same as in [1, 2] many of them will be omitted and we will concentrate the attention on the lemmas which differs from [1, 2].

Notations. The notation C_i will be reserved to positive constants which have a fixed value. Moreover $o_L(1)$ (resp. $o_{L,\varepsilon}(1)$) will denote a quantity which tends to 0 as $L \to +\infty$ (resp. as $L \to +\infty$ and $\varepsilon \to 0$) independently of anything else. The expression «$a(z_1, \ldots, z_p) = O(b(z_1, \ldots, z_p))$» will mean that there is an absolute positive constant C such that for all $(z_1, \ldots, z_p)$, $|a(z_1, \ldots, z_p)| \leq C|b(z_1, \ldots, z_p)|$.

2. Existence of simple heteroclinic solutions

In this section we look for heteroclinic solutions z_ε of (1.1) connecting x_1 to x_2 near some $z_0(\cdot + \theta)$ as critical points of a suitable functional f_ε defined on a Hilbert space E with norm $\|\cdot\|$ induced by a scalar product $(\cdot, \cdot)$.

All our existence results will be obtained by means of a finite dimensional reduction looking for critical points of f_ε constrained to a finite dimensional manifold.

We prefix the following definition:

Definition 1. *A submanifold $M \subset E$ is called a natural constraint for the functional f if*

$$u \in M \quad \textit{and} \quad (f_{|M})'(u) = 0 \quad \textit{imply that} \; f'(u) = 0 .$$

Consider a family of $C^2(E, \mathbb{R})$ functionals $f_\varepsilon = f_0 + \varepsilon G$ satisfying the following assumptions:

- (h_1) f_0 has a d-dimensional manifold Z of critical points at level $b = f_0(Z)$;
- (h_2) For all $z \in Z$ the second derivative $f_0''(z)$ is Fredholm of index 0;
- (h_3) For all $z \in Z$, $\mathrm{Ker} f_0''(z) = T_z Z$.

The following lemma, proved in [1, Lemmas 2, 4, Theorem 6], locally defines a natural constraint for f_ε near to Z.

Lemma 1. *There exist $\varepsilon_0 > 0$ and a C^1 function $w = w(z, \varepsilon) \in E$ such that:*

- *(i)* $w(z, 0) = 0$ *and* $\|w(z, \varepsilon)\| = O(\varepsilon)$;
- *(ii) The manifold defined locally as* $Z_\varepsilon = \{z + w(z, \varepsilon) \,|\, |\varepsilon| \leq \varepsilon_0\}$ *is a natural constraint for* f_ε;

- (*iii*) *The functional f_ε restricted to Z_ε is given by*:

$$f_{\varepsilon|Z_\varepsilon}(z) = f_\varepsilon(z + w(z, \varepsilon)) = f_0(z) + \varepsilon G(z) + o(\varepsilon) = b + \varepsilon G(z) + o(\varepsilon).$$

By Lemma 1 and Definition 1 it follows (see [1, Theorems 6-7]) that if G has a proper minimum or maximum in a point $\bar{z} \in Z$ the functional f_ε possesses a critical point $\tilde{z} + w(\tilde{z}, \varepsilon)$ near $\bar{z}$.

We will apply Lemma 1 to study the existence of heteroclinics for perturbed second order systems like (1.1). We assume that:

- (V_1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$, $V(x_1) = V(x_2) = 0$, $\nabla V(x_1) = \nabla V(x_2) = 0$, $D^2 V(x_1)$, $D^2 V(x_2)$ are negative definite matrix;

- (W_1) $W \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $W(t, x_1) = W(t, x_2) = 0$, $\nabla_u W(t, x_1) = \nabla_u V(t, x_2) = 0$, $D_u^2 W(t, x_i) \in L^\infty(\mathbb{R})$ and $D_u^2 W(t, \cdot)$ is continuous uniformly with respect to t.

Because of (V_1) the points x_1, x_2 are hyperbolic equilibria of the unperturbed system:

$$\ddot{u} + \nabla V(u) = 0. \tag{2.1}$$

We will assume:

- (V_2) There exists a heteroclinic solution z_0 of (2.1) connecting x_1 and x_2 such that the solutions $\phi \in E$ of the linearized equation: $\ddot{\phi} + D^2 V(z_0)\phi = 0$ form a 1-dimensional space.

Since the unperturbed system (2.1) is autonomous all the translated $z_\theta(\cdot) = z_0(\cdot + \theta)$ are still heteroclinic solutions of (1.1) connecting x_1 to x_2.

REMARK 1. *In the geometric language of the dynamical systems hypothesis* (V_2) *means that the heteroclinic z_0 is transversal on the energy level containing the equilibria x_i.*

Indeed since z_0 in a heteroclinic connecting x_1 and x_2 and equation (2.1) *is autonomous results that $\gamma_0 = (z_0, \dot{z}_0)(\mathbb{R}) \subseteq M^u(x_1) \cap M^s(x_2)$, where $M^u(x_1)$ is the unstable manifold of x_1 and $M^s(x_2)$ is the stable manifold to x_2. Hence for any $x \in \gamma_0$ there results that*:

$$T_x\gamma_0 \subseteq T_x M^u(x_1) \cap T_x M^s(x_2). \tag{2.2}$$

Since the linearized equation of hypothesis (V_2) *is the variational equation of* (2.1) *results that* $\dim[\ker f_0''(z_\theta - z_0)] = \dim[T_x M^u(x_1) \cap T_x M^s(x_2)]$. *Hence* ($V_2$) *exactly means that $T_x M^u(x_1) \cap T_x M^s(x_2)$ is* 1*-dimensional and from* (2.2) *that*:

$$T_x\gamma_0 = T_x M^u(x_1) \cap T_x M^s(x_2). \tag{2.3}$$

This also implies, calling $H_0 = \{(x, \dot{x}) \in \mathbb{R}^{2n} \mid (1/2)\dot{x}^2 + V(x) = V(x_i)\}$ the $(2n-1)$-dimensional energy level that $T_x M^u(x_1) + T_x M^s(x_2) = T_x H_0$. Indeed, since $M^u(x_1)$, $M^s(x_2) \subseteq H_0$, we clearly have for all $x \in \gamma_0$ that

$$T_x M^u(x_1) + T_x M^s(x_2) \subseteq T_x H_0. \tag{2.4}$$

Since $\dim(T_x M^u(x_1) \cap T_x M^s(x_2)) = 1$, *we have*:

$$\dim(T_x M^u(x_1) + T_x M^s(x_2)) =$$
$$= \dim T_x M^u(x_1) + \dim T_x M^s(x_2) - \dim(T_x M^u(x_1) \cap T_x M^s(x_2)) = 2n - 1,$$

from which we deduce that in (2.4) *equality holds, that is that* γ_0 *is transversal on* H_0.

Since x_1 and x_2 are hyperbolic equilibria the heteroclinic solution z_0 converges exponentially fast to x_1 and x_2 respectively as $t \to -\infty$ and as $t \to +\infty$; moreover also the derivative $\dot{z}_0$ converges exponentially fast to 0 as $|t| \to +\infty$.

We will work in the Sobolev space $E = H^1(\mathbb{R}, \mathbb{R}^n)$. We consider the following functional defined on E:

$$f_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}} \dot{v}^2 - \int_{\mathbb{R}} [V(z_0 + v) + \ddot{z}_0 v]\, dt - \varepsilon \int_{\mathbb{R}} W(t, z_0 + v)\, dt .$$

Because of (V_1) and (W_1) the functional f_ε is well-defined and smooth on E. A critical point v of f_ε is a C^2 solution of the following system of differential equations:

$$\ddot{v} + \nabla V(z_0 + v) + \ddot{z}_0 + \varepsilon \nabla_u W(t, z_0 + v) = 0 .$$

Hence $u = z_0 + v$ is a C^2 heteroclinic solution of (1.1).

The manifold:

$$Z = \{z_\theta - z_0,\ \theta \in \mathbb{R}\}$$

is a 1-dimensional critical manifold for f_0 at level

$$f_0(z_\theta - z_0) = b = -\int_{\mathbb{R}} V(z_0(t))\, dt .$$

The tangent space to Z in $z_\theta - z_0$ is given by $TZ_{(z_\theta - z_0)} = \mathrm{span}\langle \dot{z}_\theta \rangle$.

The following lemma shows that we can apply Lemma 1 to the functional f_ε:

LEMMA 2. *Assumptions* (h_1), (h_2), (h_3) *of Lemma* 1 *hold for* f_ε.

PROOF. (h_1) is obvious. For all $z_\theta - z_0 \in Z$ there results that $TZ_{(z_\theta - z_0)} \subseteq \ker f_0''(z_\theta - z_0)$; ($V_2$) exactly means that $\ker f_0''(z_\theta - z_0)$ is 1-dimensional and hence we deduce (h_3).

It remains to prove (h_2). The linear operator $f_0''(z_\theta - z_0)\colon E \to E$ is defined by:

$$(f_0''(z_\theta - z_0)y, w) = \int_{\mathbb{R}} \dot{y}\dot{w} - \int_{\mathbb{R}} D^2 V(z_\theta) yw .$$

Let consider a function $\gamma\colon \mathbb{R} \to M(n, \mathbb{R})$, the set of $n \times n$ matrices, with $\gamma(t)$ uniformly negative definite and such that $\lim_{t \to +\infty} \gamma(t) = D^2 V(x_2)$ and $\lim_{t \to -\infty} \gamma(t) = D^2 V(x_1)$. Hence we can write:

$$(f_0''(z_\theta - z_0)y, w) =$$
$$= \int_{\mathbb{R}} [\dot{y}\dot{w} - \gamma(t) yw] - \int_{\mathbb{R}} [D^2 V(z_\theta) - \gamma(t)] yw = (y, w)_{H^1} - (F''(z_\theta - z_0)y, w)$$

where

$$(y, w)_{H^1} = \int_{\mathbb{R}} [\dot{y}\dot{w} - \gamma(t) y w]$$

is a scalar product in E equivalent to the standard one. We now prove that $F''(z_\theta - z_0)$ is a compact operator. Indeed we have to show that $F''(z_\theta - z_0) y_n \to 0$ strongly on E whenever $y_n \rightharpoonup 0$. We have:

$$\|F''(z_\theta - z_0) y_n\| = \sup_{\|w\|=1} |(F''(z_\theta - z_0) y_n, w)| = \sup_{\|w\|=1} \left| \int_{\mathbb{R}} [D^2 V(z_\theta) - \gamma(t)] y_n w \right| .$$

Hence, by the Holder inequality, we get:

$$\|F''(z_\theta - z_0) y_n\| \leqslant \left(\int_{\mathbb{R}} |D^2 V(z_\theta) - \gamma(t)|^2 \, |y_n|^2 \right)^{1/2} .$$

The above integral tends to zero as $n \to \infty$ because $y_n \to 0$ in L^∞_{loc} and $\lim_{|t| \to \infty} D^2 V(z_\theta(t)) - \gamma(t) = 0$. Hence $f_0''(z_\theta - z_0)$ is an operator of the form *Id* + + *Compact* and then it is Fredholm of index 0. ∎

By Lemma 1-(*iii*) the expression of the functional f_ε on Z_ε is given by:

$$f_\varepsilon(z_\theta - z_0 + w_\varepsilon) = f_0(z_\theta - z_0) - \varepsilon \int_{\mathbb{R}} W(t, z_0 + (z_\theta - z_0))\, dt + o(\varepsilon) = b + \varepsilon \Gamma(\theta) + o(\varepsilon) \tag{2.5}$$

where

$$\Gamma(\theta) = - \int_{\mathbb{R}} W(t, z_0(t + \theta))\, dt$$

is the Poincaré function of the system.

Since we are considering a reversible system, if z_0 is a heteroclinic solution of (2.1) from x_1 to x_2 the function $\tilde{z}_0(t) = z_0(-t)$ is still a solution of (2.1) which is a heteroclinic from x_2 to x_1.

Hence we can perform the same procedure for the heteroclinic $\tilde{z}_0$ dealing with the following functional:

$$\tilde{f}_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}} \dot{v}^2 - \int_{\mathbb{R}} [V(\tilde{z}_0 + v) + \ddot{\tilde{z}}_0 v] - \varepsilon \int_{\mathbb{R}} W(t, \tilde{z}_0 + v)\, dt .$$

As can be readily verified, from hypothesis (V_2) it follows that also the solutions $\phi \in E$ of the linearized equation: $\ddot{\phi} + D^2 V(\tilde{z}_0)\phi = 0$ form a 1-dimensional space. Hence one obtains for f_ε a formula like (2.5) with the new Poincaré function:

$$\tilde{\Gamma}(\theta) = - \int_{\mathbb{R}} W(t, \tilde{z}_\theta)\, dt .$$

In conclusion we find:

THEOREM 1. *Let* (V_1), (V_2) *and* (W_1) *hold. If* Γ (*resp.* $\tilde{\Gamma}$) *has a proper local* *minimum or maximum at some* $\overline{\theta}$ *then for* ε *small* (1.1) *has a heteroclinic solution* z_ε *connecting* x_1 *to* x_2 (*resp.* x_2 *to* x_1) *near* $z_0(\cdot + \overline{\theta})$ (*resp.* $\widetilde{z}_0(\cdot + \overline{\theta})$).

3. Existence of infinitely many heteroclinic solutions turning k times between x_1 and x_2

We now prove that it is possible to «glue» heteroclinic orbits z_{θ_i} and $\widetilde{z}_{\theta_j}$ in order to find orbits emanating at $t = -\infty$ from x_1 turning $k = 2l + 1$ times between x_1 and x_2 and arriving for $t \to +\infty$ to x_2 (heteroclinic orbit to x_1) or turning $k = 2l$ times and being asymptotic as $t \to +\infty$ to x_1 (homoclinic orbit). In the rest of the paper we will make the explicit computations for the heteroclinics only. In Remark 4 we will say how to obtain the homoclinic solutions; in the sequel hence k will always be an odd number, $k = 2l + 1$.

3.1. *The variational formulation and the «pseudo-critical» manifold.*

In order to get heteroclinic solutions turning $k = 2l + 1$ times between x_1 and x_2 we will study the behaviour of the functional f_ε near a suitable «pseudo-critical» manifold Z_L whose elements are the «candidate» *pseudo*-critical points z_θ^L near which we look for *true*-critical points of f_ε corresponding to heteroclinics turning k times between x_1 and x_2.

In the sequel we will always assume that $L > 2$ and the symbol θ will mean $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$. For any $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$ with $\min_i(\theta_{i+1} - \theta_i) > 3L$ we define a smooth family of functions depending on k parameters $z^L_{\theta_1, \ldots, \theta_k}$ such that for $i = 1, \ldots, l$

$$z_\theta^L = z^L_{\theta_1, \ldots, \theta_k} = \begin{cases} x_1 & \text{if } t \in (-\infty, -\theta_{(2l+1)} - L - 1], \\ z_{\theta_{(2i+1)}} & \text{if } t \in [-\theta_{(2i+1)} - L, -\theta_{(2i+1)} + L], \\ x_2 & \text{if } t \in [-\theta_{(2i+1)} + L + 1, -\theta_{2i} - L - 1], \\ \widetilde{z}_{\theta_{2i}} & \text{if } t \in [-\theta_{2i} - L, -\theta_{2i} + L], \\ x_1 & \text{if } t \in [-\theta_{2i} + L + 1, -\theta_{(2i-1)} - L - 1], \\ z_{\theta_1} & \text{if } t \in [-\theta_1 - L, -\theta_1 + L], \\ x_2 & \text{if } t \in [-\theta_1 + L + 1, +\infty). \end{cases}$$

In the complementary set

$$\bigcup_{j=1}^{2l+1} (-\theta_j - L - 1, -\theta_j - L) \bigcup_{j=1}^{2l+1} (-\theta_j + L, -\theta_j + L + 1)$$

such functions z_θ^L can be also taken so that are $C^\infty(\mathbb{R})$ and such that:

$$\sup_{i = 1, \ldots, l+1} \| \partial_{\theta_{(2i-1)}} z_\theta^L - \dot{z}_{\theta_{(2i-1)}} \| \to 0 \quad \text{and} \quad \sup_{i = 1, \ldots, l} \| \partial_{\theta_{2i}} z_\theta^L - \dot{\widetilde{z}}_{\theta_{2i}} \| \to 0 \quad \text{as } L \to \infty .$$

Clearly the family of functions z_θ^L has been chosen so that as $L \to +\infty$ they «splits» into the «sum» of $k = 2l + 1$ distinct 1-dimensional heteroclinics.

As can be readily verified the manifold:

$$Z_L = \{z_\theta^L - z_0 \mid \theta \in \mathbb{R}^k \min_{(\theta_{i+1} - \theta_i)} > 3L\}$$

is a k-dimensional «pseudo-critical» manifold for f_0, that is:

$$\sup_{z_\theta^L \in Z_L} \|f_0'(z_\theta^L - z_0)\| \to 0 \quad \text{as } L \to +\infty . \tag{3.1}$$

The tangent space of Z_L at $z_\theta^L - z_0$ is given by $TZ_{(z_\theta^L - z_0)} = \operatorname{span}\langle \partial_{\theta_1}, z_\theta^L, \ldots, \partial_{\theta_{(2l+1)}} z_\theta^L \rangle$.

For our purposes, differently from Lemma 1, we need here to define a natural constraint $Z_{L,\varepsilon}$ for f_ε close to Z_L in a global fashion; this is possible because from (V_1) and (W_1) f_0'' and G'' are bounded on bounded subsets of E. It is possible to prove (see [2, Lemmas 3, 4]) the following lemma:

LEMMA 3. *There exist* $\varepsilon_1, \delta_1, L_1, C_1 > 0$ *such that* $\forall L > L_1$ *there is a unique* C^1 *function*

$$w(L, \varepsilon, \theta) = w_L(\theta) + \overline{w}_{L,\varepsilon}(\theta): (-\varepsilon_1, \varepsilon_1) \times$$
$$\times \{(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k \mid \min_i (\theta_{i+1} - \theta_i) > 3L\} \to \{v \in E \mid \|v\| < \delta_1\}$$

such that:

- $\overline{w}_{L,0}(\theta_1, \ldots, \theta_k) = 0$ *and* $\|\overline{w}_{L,\varepsilon}\| \le C_1 |\varepsilon|$;
- $\sup_{\{\theta \mid \min_i(\theta_{i+1} - \theta_i) > 3L\}} \|w_L(\theta)\| \to 0$ *as* $L \to +\infty$;
- $w_L(\theta), \overline{w}_{L,\varepsilon}(\theta) \in TZ_{L(z_\theta^L - z_0)}$;
- $f_0'(z_\theta^L - z_0 + w_L(\theta)) \in TZ_{L(z_\theta^L - z_0)}$.

Moreover, defining,

$$Z_{L,\varepsilon} = \{z_\theta^L - z_0 + w(L, \varepsilon, \theta) \mid \min_i (\theta_{i+1} - \theta_i) > 3L\} .$$

$Z_{L,\varepsilon}$ *is a natural constraint for* f_ε.

By Lemma 3 we are led, in order to find heteroclinic solutions turning k times between x_1 and x_2, to look for critical points of the functional f_ε restricted to the k-dimensional manifold $Z_{L,\varepsilon}$. The expression of the functional f_ε restricted to $Z_{L,\varepsilon}$ is given by the following lemma:

LEMMA 4. *Let* $k = 2l + 1$, *for* $L > L_1$ *and* $|\varepsilon| < \varepsilon_1$, $f_{\varepsilon | Z_{L,\varepsilon}}$ *has the following form*:

$$f_\varepsilon(z_\theta - z_0 + w(L, \varepsilon, \theta)) = \tag{3.2}$$
$$= (k-1)a + kb + \varepsilon\left(\sum_{i=1}^{i=l+1} \Gamma(\theta_{(2i-1)}) + \sum_{i=1}^{i=l} \tilde{\Gamma}(\theta_{2i})\right) + o_L(1) + O(\varepsilon^2)$$

where $a = (1/2)\int_{\mathbb{R}} \dot{z}_0^2$ *is a constant.*

PROOF. Let $L > L_1$, $|\varepsilon| < \varepsilon_1$ and $\min_i(\theta_{i+1} - \theta_i) > 3L$. Since $\overline{w}_{L,\varepsilon} \in TZ_{L(z_\theta - z_0)}$, by Lemma 3, $(f_0'(z_\theta^L - z_0 + w_L(\theta)), \overline{w}_{L,\varepsilon}) = 0$; by Lemma 3 $\|\overline{w}_{L,\varepsilon}\| \leq C_1|\varepsilon|$. Moreover, since by (V_1), (W_1) f_0'' and G' are bounded on bounded subsets of E we can write:

$$f_\varepsilon(z_\theta^L - z_0 + w_L(\theta) + \overline{w}_{L,\varepsilon}(\theta)) = f_0(z_\theta^L - z_0 + w_L + \overline{w}_{L,\varepsilon}) + \\ + \varepsilon G(z_\theta^L - z_0 + w_L + \overline{w}_{L,\varepsilon}) = f_0(z_\theta^L - z_0 + w_L) + (f_0'(z_\theta^L - z_0 + w_L), \overline{w}_{L,\varepsilon}) + \\ + O(\|\overline{w}_{L,\varepsilon}\|^2) + \varepsilon G(z_\theta^L - z_0 + w_L) + \varepsilon O(\|\overline{w}_{L,\varepsilon}\|) = \\ = f_0(z_\theta^L - z_0 + w_L) + \varepsilon G(z_\theta^L - z_0 + w_L) + O(\varepsilon^2).$$

Now, by Lemma 3 $\|w_L(\theta)\| = o_L(1)$, hence:

$$f_0(z_\theta^L - z_0 + w_L(\theta)) = f_0(z_\theta^L - z_0) + o_L(1) = \\ = \int_{\mathbb{R}} \frac{1}{2}(\dot{z}_\theta^L - \dot{z}_0)^2 - \int_{\mathbb{R}} [V(z_\theta^L) + \ddot{z}_0(z_\theta^L - z_0)] + o_L(1)$$

which is equal, by an integration by parts, to:

$$\int_{\mathbb{R}} \left[\frac{1}{2}(\dot{z}_\theta^L)^2 - V(z_\theta^L)\right] - \frac{1}{2}\int_{\mathbb{R}} \dot{z}_0^2 + o_L(1).$$

By (V_1) and the «splitting» properties of z_θ^L as $L \to \infty$ we have that the above term is equal to:

$$\sum_i \int_{\mathbb{R}} \left[\frac{1}{2}(\dot{z}_{\theta_i})^2 - V(z_{\theta_i})\right] + o_L(1) - \frac{1}{2}\int_{\mathbb{R}} \dot{z}_0^2 = \\ = k\left(\frac{1}{2}\int_{\mathbb{R}} \dot{z}_0^2 + b\right) + o_L(1) - \frac{1}{2}\int_{\mathbb{R}} \dot{z}_0^2 = (k-1)a + kb + o_L(1),$$

where $a = \frac{1}{2}\int_{\mathbb{R}} \dot{z}_0^2$. In the same way, because of (W_1), and the properties of z_θ^L as $L \to +\infty$ we have:

$$G(z_\theta^L - z_0^L + w_L(\theta)) = \left(\sum_{i=1}^{i=l+1} \Gamma(\theta_{(2i-1)}) + \sum_{i=1}^{i=l} \tilde{\Gamma}(\theta_{2i})\right) + o_L(1).$$

This concludes the proof of Lemma 4. ∎

By the above lemma the existence of minima for Γ and $\tilde{\Gamma}$ sufficiently far one each other ensures the existence of critical points of f_ε restricted to $Z_{L,\varepsilon}$ and hence implies the existence of heteroclinics turning k times between x_1 and x_2. To be more precise we make the following hypotheses on Γ and $\tilde{\Gamma}$:

CONDITION 1. *There are $\eta > 0$ and a sequence $(U_n = (c_n, d_n))_{n \in \mathbb{Z}}$, $(\tilde{U}_n = (\tilde{c}_n, \tilde{d}_n))_{n \in \mathbb{Z}}$ of bounded open intervals of $\mathbb{R}$ which satisfy*:

(*i*) *$\Gamma_{|U_n}$, $\tilde{\Gamma}_{|\tilde{U}_n}$ attain its minimum resp. at some $a_n \in (c_n, d_n)$, $\tilde{a}_n \in (\tilde{c}_n, \tilde{d}_n)$ and $\Gamma_{|\{c_n, d_n\}} \geq \Gamma(a_n) + \eta$, resp. $\tilde{\Gamma}_{|\{\tilde{c}_n, \tilde{d}_n\}} \geq \tilde{\Gamma}(\tilde{a}_n) + \eta$;*

(*ii*) *$c_n, \tilde{c}_n \to +\infty$ as $n \to +\infty$ and $d_n, \tilde{d}_n \to -\infty$ as $n \to -\infty$.*

Condition 1 is satisfied for example if Γ and $\widetilde{\Gamma}$ are non-constant periodic, quasi-periodic or almost-periodic functions. (See [2, Section 2.4]).

Hence it is possible to prove the following:

THEOREM 2. *Let* (V_1), (V_2), (W_1) *and condition* (1) *hold. For all* $k = 2l + 1$, *for* $\varepsilon \neq 0$ *small enough there exists* L_ε *such that if* $\min_{i=1,\dots,l} (\widetilde{c}_{j_{2i}} - d_{j_{(2i-1)}}) > L_\varepsilon$ *and* $\min_{i=1,\dots,l} (c_{j_{(2i+1)}} - \widetilde{d}_{j_{2i}}) > L_\varepsilon$ *then equation* (1.1) *has a heteroclinic solution* u_ε *located near some* $z^{L_\varepsilon}_{\theta_1,\dots,\theta_k}$ *with* $\theta_{j_{(2i-1)}} \in U_{j_{(2i-1)}}$ *for* $i = 1, \dots, l+1$ *and* $\theta_{j_{2i}} \in \widetilde{U}_{j_{2i}}$ *for* $i = 1, \dots, l$.

As a consequence of Theorem 2 we have the following corollary:

COROLLARY 1. *For all* $k = 2l + 1$ *there exists* $\bar{\varepsilon} > 0$ *such that* $\forall \varepsilon \in (-\bar{\varepsilon}, 0) \cup (0, \bar{\varepsilon})$ *equation* (1.1) *has infinitely many heteroclinics winding* $k = 2l + 1$ *times between* x_1 *and* x_2.

REMARK 2. *Note that* $L_\varepsilon \to +\infty$ *as* $\varepsilon \to 0$. *Using the exponential decay property of* $\dot{z}_0$, *deriving from the fact that* x_1 *and* x_2 *are hyperbolic points the distance can be estimated as* $L_\varepsilon = -K \ln |\varepsilon|$ *for some positive constant* K, *see Lemma* 6.

REMARK 3. *It is also possible to obtain heteroclinic solutions of* (1.1) u_ε *turning* k *times between* x_1 *and* x_2 *located near* z^L_θ *where* θ_i *are all maxima of* Γ *and* $\widetilde{\Gamma}$. *Moreover with considerations like section* (3.3) *in* [2] *streghtening Condition* 1 *we can also find heteroclinics where* θ_i *are either near minima either near maxima of* Γ *and* $\widetilde{\Gamma}$.

REMARK 4. *For proving the existence of homoclinic solutions to* x_i *turning* $k = 2l$ *times between* x_1 *and* x_2 *it is enough to repeat the same arguments for the following functional defined on* $H^1(\mathbb{R})$:

$$\bar{f}_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}} \dot{v}^2 - \int_{\mathbb{R}} V(x_i + v) - \varepsilon \int_{\mathbb{R}} W(t, x_i + v)\, dt .$$

A critical point v *of* $\bar{f}_\varepsilon$ *gives rise to a solution* $u = x_i + v$ *of* (1.1) *homoclinic to* x_i.

However the constants L_ε and $\bar{\varepsilon}$ given by Theorem 2 and Corollary 1 can depend on k so that Theorem 2 cannot be directly used to obtain the existence of solutions turning infinitely many times between x_1 and x_2. The bound that we obtain for $\|u_\varepsilon - z^{L_\varepsilon}_\theta\|$ is not independent of k. We will show, following [2], in the next section how to derive estimates independent of k by using a different norm. We shall find solutions u_ε close to z^L_θ only in L^∞-norm but not in H^1-norm. See also [6].

3.2. *Existence of heteroclinic solutions turning infinitely many times between* x_1 *and* x_2.

In this section we show how to modify the previous lemmas in order to obtain constants ε_1 and L_ε independent of k.

For any $\theta_1 < \ldots < \theta_k$ we will consider the norm on E:

$$|u|_\theta^2 = \max_{i=1,\ldots,k} \int_{I_i} |u^2| + |\dot{u}|^2$$

where

$$I_1 = ((-\theta_1 - \theta_2)/2, +\infty), \qquad I_i = ((-\theta_{i+1} - \theta_i)/2, (-\theta_i - \theta_{i-1})/2)$$

and

$$I_k = (-\infty, (-\theta_k - \theta_{k-1})/2).$$

In the sequel $\|\cdot\|$ will still denote the H^1-norm.

Since for every $u \in E$ we have

$$|u|_\theta^2 \leq \|u\|^2 \leq k|u|_\theta^2,$$

the norm $|\cdot|_\theta$ is equivalent to the H^1-norm for fixed k. Moreover the following uniform bound can be easily proved: $\forall k \in \mathbb{N}$, $\forall(\theta_1, \ldots, \theta_k)$ with $\min_i(\theta_i - \theta_{i-1}) > 1$:

$$\|u\|_\infty \leq 2|u|_\theta.$$

With the above norm the estimates become independent of k.

A modified version of Lemma 3 (see [2, Lemmas 13-15]) in which the constants can be taken independent of k can be proved.

LEMMA 5. *There exist* $\varepsilon_2, \delta_2, L_2, C_2, C_3 > 0$ *such that* $\forall k$, $\forall L > L_2$ *there is a unique* C^1 *function*

$$w(L, \varepsilon, \theta) = w_L(\theta) + \overline{w}_{L,\varepsilon}(\theta): (-\varepsilon_2, \varepsilon_2) \times$$
$$\times \{(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k \mid \min_i(\theta_{i+1} - \theta_i) > 3L\} \to \{v \in E \mid \|v\| \leq \delta_2\}$$

such that:

- $\overline{w}_{L,0}(\theta) = 0$ *and* $|\overline{w}_{L,\varepsilon}|_\theta \leq C_2|\varepsilon|$;
- $|w_L(\theta)|_\theta = O(\exp(-C_3 L))$;
- $w_L(\theta), \overline{w}_{L,\varepsilon}(\theta) \in TZ_{L(z_\theta^L - z_0)}$;
- $f_0'(z_\theta^L - z_0 + w_L(\theta)) \in TZ_{L(z_\theta^L - z_0)}$.

Moreover, defining,

$$Z_{L,\varepsilon} = \{z_\theta^L - z_0 + w(L, \varepsilon, \theta) \mid \min_i(\theta_{i+1} - \theta_i) > 3L\}$$

$Z_{L,\varepsilon}$ *is a natural constraint for* f_ε.

By Lemma 5 we are led, in order to find heteroclinic solutions turning k times between x_1 and x_2 and then solutions turning infinitely many times between x_1 and x_2, to look for the critical points of the functional f_ε restricted to the k-dimensional manifold $Z_{L,\varepsilon}$. The next lemma provides a suitable expression of the functional f_ε restricted to $Z_{L,\varepsilon}$ (see [2, Lemma 18]):

LEMMA 6. $\forall|\varepsilon| < \varepsilon_2$, $\forall L > L_2$, $\forall k$, *for all* $(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$ *with* $\min_i(\theta_{i+1} - \theta_i) >$

$> 3L$ *there results:*

$$(3.3)\quad f_\varepsilon(z_\theta - z_0 + w_L(\theta) + \overline{w}_{L,\varepsilon}(\theta)) =$$
$$= (k-1)a + kb + \varepsilon\left(\sum_{i=1}^{i=l+1} \Gamma(\theta_{(2i-1)}) + \sum_{i=1}^{i=l} \widetilde{\Gamma}(\theta_{2i})\right) + \beta(L, \varepsilon, \theta)$$

where β *has the following property: there is a positive constant* C_4 *such that, if* θ_i' *satisfies* $\theta_i' - \theta_{i-1} > 3L$ *and* $\theta_{i+1} - \theta_i' > 3L$ *then*

$$(3.4)\quad |\beta(L, \varepsilon, \theta_1, \ldots, \theta_{i-1}, \theta_i', \theta_{i+1}, \ldots, \theta_k) - \beta(L, \varepsilon, \theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_k)| =$$
$$= O(\exp(-C_4 L)) + \varepsilon o_{\varepsilon, L}(1).$$

By Lemma 6 it is possible to prove (see [2, Theorem 3]) that:

THEOREM 3. *Let condition* (V_1), (V_2), (W_1) *and condition* (1) *hold. Then there exists a positive constant* C_5 *such that*: $\forall \omega > 0$ *there exists* $\varepsilon_3 > 0$ *such that for all* $k = 2l + 1$, $\forall \varepsilon \in (-\varepsilon_3, \varepsilon_3)$, $\varepsilon \neq 0$ *if* $\min_{i=1,\ldots,l} (\widetilde{c}_{j_{2i}} - d_{j_{(2i-1)}}) > L_\varepsilon = -C_5 \ln|\varepsilon|$ *and* $\min_{i=1,\ldots,l} (c_{j_{(2i+1)}} - \widetilde{d}_{j_{2i}}) > L_\varepsilon = -C_5 \ln|\varepsilon|$ *then there are* $\theta_{j_{(2i-1)}} \in U_{j_{(2i-1)}}$ *for* $i = 1, \ldots, l+1$, $\theta_{j_{2i}} \in \widetilde{U}_{j_{2i}}$ *for* $i = 1, \ldots, l$, *and a heteroclinic solution* u_ε *of* (1.1) *which satisfies*:

$$\|u_\varepsilon - z^{L_\varepsilon}_{\theta_{i_1}, \ldots, \theta_{i_k}}\|_{L^\infty(\mathbb{R})} \leq \omega .$$

Since L_ε does not depend on k by standard arguments (see [2, 6]) it is possible to get from the above theorem the existence of solutions turning infinitely many times between x_1 and x_2 according to the following theorem:

THEOREM 4. *Let condition* (V_1), (V_2), (W_1) *and condition* (1) *hold. Then there exist a positive constant* C_5 *such that*: $\forall \omega > 0$ *there exists* $\varepsilon_3 > 0$ *such that* $\forall \varepsilon \in (-\varepsilon_3, \varepsilon_3)$, $\varepsilon \neq 0$ *for any sequence of intervals with* $\min_{i=1,\ldots} (\widetilde{c}_{j_{2i}} - d_{j_{(2i-1)}}) > L_\varepsilon = -C_6 \ln|\varepsilon|$ *and* $\min_{i=1,\ldots} (c_{j_{(2i+1)}} - \widetilde{d}_{j_{2i}}) > L_\varepsilon = -C_6 \ln|\varepsilon|$ *then there are* $\theta_{j_{(2i-1)}} \in U_{j_{(2i-1)}}$ *and* $\theta_{j_{2i}} \in \widetilde{U}_{j_{2i}}$ *for* $i = 1, \ldots$ *and a solution* u_ε *of* (1.1) *which satisfies*:

$$\|u_\varepsilon - z^{L_\varepsilon}_\theta\|_{L^\infty(\mathbb{R})} \leq \omega .$$

The existence of solutions as in Theorem 4 implies a chaotic dynamic which can be described, if the perturbation is periodic, in terms of Bernoulli shift structures. If the Poincaré functions Γ and $\widetilde{\Gamma}$, possess non-degenerate critical points then a uniqueness result for the solutions given by Theorem 4 (see [2, Section 3.4]) can be proved and the dynamics of (1.1) possesses a complete Bernoulli shift structure.

Before ending we remark that the above theorems can be extended to the following situation. Assume that the potential V possesses other critical points $x_3, \ldots, x_N$ such that $V(x_i) = W(t, x_i) = 0$, $\nabla_u V(x_i) = \nabla_u W(t, x_i) = 0$ and $D^2 V(x_i)$ are negative definite matrices. Moreover assume that the unperturbed system possesses transversal heteroclinic orbits $z_{0(i,j)}$ connecting the hyperbolic equilibria x_i and x_j. Hence we can modify the above arguments in such a way that we manage to connect in the perturbed system the hyperbolic equilibria x_i as we wish. For this pourpose it is enough to build a manifold of «quasi-heteroclinic» solutions and to find a true critical point near to this manifold.

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