Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Semigroups and generators on convex domains with the hyperbolic metric

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **8** (1997), n.4, p. 231–250.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1997_9_8_4_231_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1997.

Rend. Mat. Acc. Lincei
s. 9, v. 8:231-250 (1997)

Analisi matematica. — *Semigroups and generators on convex domains with the hyperbolic metric.* Nota di SIMEON REICH e DAVID SHOIKHET, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — Let D be domain in a complex Banach space X, and let ϱ be a pseudometric assigned to D by a Schwarz-Pick system. In the first section of the paper we establish several criteria for a mapping $f\colon D \to X$ to be a generator of a ϱ-nonexpansive semigroup on D in terms of its nonlinear resolvent. In the second section we let $X = H$ be a complex Hilbert space, $D = B$ the open unit ball of H, and ϱ the hyperbolic metric on B. We introduce the notion of a ϱ-monotone mapping and obtain simple characterizations of generators of semigroups of holomorphic self-mappings of B.

KEY WORDS: Banach space; Generator; Holomorphic mapping; Hyperbolic metric; Monotone operator.

RIASSUNTO. — *Semigruppi e generatori su domini convessi con metrica iperbolica.* Sia D un dominio in uno spazio di Banach complesso X e sia ϱ una pseudometrica assegnata a D da un sistema di Schwarz-Pick. Nella prima parte del lavoro si stabiliscono alcuni criteri affinché una applicazione $f\colon D \to X$ sia un generatore di un semigruppo ϱ-non espansivo su D. Nella seconda parte si suppone che sia $X = H$, spazio di Hilbert complesso, che $D = B$ disco unitario aperto di H e che sia ϱ la metrica iperbolica su B. Si introduce la nozione di applicazione ϱ-monotona e si ottengono semplici caratterizzazioni di generatori di semigruppi di applicazioni olomorfe di B in sè.

INTRODUCTION

In the last thirty years the theory of monotone and accretive operators has been intensively developed by many mathematicians (see, for example, [8, 6]) with many applications to nonlinear analysis and optimization. This theory is closely connected with the generation theory of nonlinear one-parameter semigroups of nonexpansive mappings and with nonlinear evolution problems.

In a parallel development (and even earlier) the generation theory of one-parameter semigroups of holomorphic mappings in C^n has been an object of interest in the theory of Markov stochastic processes and, in particular, in the theory of branching processes (see, for example, [19, 28, 16, 21]). The central problem in the study of such processes is to locate the extinction probability which can be defined as the smallest common fixed point of a semigroup of holomorphic mappings or equivalently, as the smallest null point of its generator.

Later such semigroups appeared in other fields: one-dimensional complex analysis [7], finite-dimensional manifolds [2], the geometry of complex Banach spaces [5, 30], control theory and optimization [20], and Krein spaces [31-33]. For the finite dimensional case, M. Abate proved in [2] that each continuous semigroup of holomorphic mappings is everywhere differentiable with respect to its parameter, *i.e.*, is generated by a holomorphic mapping. In addition, he established a criterion for a holomorphic map-

(*) Nella seduta del 19 giugno 1997.

ping to be a generator of a one-parameter semigroup. (Such a problem is equivalent to the global solvability of a complex dynamical system). Earlier, for the one-dimensional case, similar facts were presented by E. Berkson and H. Porta in their study [7] of linear continuous semigroups of composition operators in Hardy spaces. It seems that the first deep study of semigroups of holomorphic mappings in the infinite dimensional case is due to E. Vesentini. See, for example, [31-33]. In [31] he investigates semigroups of those fractional-linear transformations on the open unit Hilbert ball B which are isometries with respect to the infinitesimal hyperbolic metric on B. The approach used there is based on the correspondence between such nonlinear semigroups and the strongly continuous semigroups of linear operators which leave invariant the indefinite metric on a Pontryagin space of defect 1. In [32, 33] this approach has been developed for general Pontryagin spaces and also for Krein spaces. Note that generally speaking such semigroups are not everywhere differentiable, and the generator of the corresponding linear semigroup is only densely defined. As a matter of fact, it turns out that the everywhere differentiability of a semigroup of holomorphic mappings on a bounded domain is equivalent to its continuity in the topology of local uniform convergence. Since, in the finite dimensional case, this topology is equivalent to the compact open topology, the study of complex dynamical systems generated by holomorphic mappings includes in this case the study of semigroups of holomorphic mappings which are pointwise continuous. On the other hand, holomorphic self-mappings of a domain D in a complex Banach space are nonexpansive with respect to any pseudometric ϱ assigned to D by a Schwarz-Pick system [18]. Therefore it is natural to inquire whether a theory analogous to the theory of monotone and accretive operators can be developed in the setting of those mappings which are nonexpansive with respect to such pseudometrics. We note in passing that the class of ϱ-nonexpansive mappings properly contains the class of holomorphic mappings (cf. [17, p. 118]). In particular, a question originating from the theory of norm nonexpansive mappings is whether $I - F$ is a generator whenever F is a holomorphic (or even ϱ-nonexpansive) self-mapping of a domain.

In this paper we answer these questions in the affirmative. More precisely, in the first section of the paper we study the generators of ϱ-nonexpansive semigroups on a domain D in a complex Banach space X. We first establish necessary (Theorem 1.1) and sufficient (Theorem 1.2) conditions for a mapping $f\colon D \to X$ to be a generator of such a semigroup. When D is bounded and convex, and f is uniformly continuous on each ϱ-ball, we then deduce (Theorem 1.3) a complete characterization of such generators in terms of their resolvents. We also obtain various results on the null point sets of these generators, and on the asymptotic behavior of their resolvents and of the semigroups they generate.

In the second section of the paper we let $X = H$ be a complex Hilbert space, $D = B$ the open unit ball of H, and ϱ the hyperbolic metric on B. We introduce the notion of a ϱ-monotone mapping $f\colon B \to H$ (Definition 2.1), and prove several characterizations of such mappings (Theorems 2.1 and 2.3). We obtain, in particular, simple characterizations (Theorem 2.2) of generators of continuous semigroups of holomorphic self-mappings of B. Finally, we briefly discuss the null point sets of ϱ-monotone mappings.

1. ϱ-GENERATORS ON CONVEX DOMAINS IN BANACH SPACES

Let D be a domain (open connected subset) in a complex Banach space X, and let ϱ be a pseudometric assigned to D by a Schwarz-Pick system [18, 15, 17, 12]. If ϱ generates the original topology on D, then the domain D is said to be ϱ-hyperbolic. If, in addition, (D, ϱ) is a complete metric space, we say that D is a complete ϱ-hyperbolic domain. It is well-known that if D is bounded, then it is ϱ-hyperbolic for each pseudometric ϱ assigned to it by a Schwarz-Pick system. One of the more interesting cases occurs when D is a convex domain in X. In this case all pseudometrics assigned to D by Schwarz-Pick systems coincide [13]. We call this common pseudometric the hyperbolic pseudometric of D. If D is bounded, then as noted above, it is, in fact, a metric, and D is a complete ϱ-hyperbolic domain.

Let (D, ϱ) be any metric space. A mapping $T\colon D \to D$ is said to be ϱ-nonexpansive if

$$\varrho(Tx, Ty) \leqslant \varrho(x, y)$$

for all x and y in D.

We denote by $\mathrm{RN}_\varrho(D)$ the class of all mappings $f\colon D \to X$ for which the resolvent $(I + rf)^{-1}$ is a well-defined ϱ-nonexpansive self-mapping of D for each positive r.

THEOREM 1.1. *Let D be a bounded convex domain in a complex Banach X, and let ϱ be its hyperbolic metric. Let $\{F(t)\colon 0 < t \leqslant T\}$ be a family of ϱ-nonexpansive self-mappings of D such that*

$$f(x) = \lim_{s \to 0+} (x - F(s)x)/s$$

exists for all $x \in D$, uniformly on each ϱ-ball, and $f\colon D \to X$ is continuous. Then $f \in \mathrm{RN}_\varrho(D)$.

PROOF. *Step 1.* Assume without loss of generality that $0 \in D$, and let p be the Minkowski functional of D. If $G\colon D \to D$ is a holomorphic self-mapping of D such that

$$\sup\{p(G(x))\colon x \in D\} \leqslant 1 - r$$

for some $r > 0$, then the proof of the Earle-Hamilton theorem [14] shows that

$$\varrho(G(x), G(y)) \leqslant d(d + r)^{-1}\varrho(x, y)$$

for all x and y in D, where $d > \sup\{p(x - y)\colon x, y \in D\}$.

Step 2. Setting in Step 1

$$G(x) = \alpha x + (1 - \alpha)y$$

for a fixed y in D and $0 < \alpha < 1$, we obtain

$$\varrho(G(x), G(0)) = \varrho(\alpha x + (1 - \alpha)y, (1 - \alpha)y) \leqslant d(d + (1 - \alpha)(1 - p(y)))^{-1}\varrho(x, 0).$$

Thus we have the following estimate which will be useful in the sequel:

$$\varrho(\alpha x + (1-\alpha)y, 0) \leqslant d(d + (1-\alpha)(1-p(y)))^{-1}\varrho(x, 0) + \varrho((1-\alpha)y, 0).$$

Step 3. Set $f_t = (I - F_t)/t$. Since the equation $x + rf_t(x) = y$ is equivalent, for each $y \in D$, to the equation

$$x = r(r+t)^{-1}F_t(x) + t(r+t)^{-1}y,$$

and since the mapping defined by $S(x) = r(r+t)^{-1}F_t(x) + t(r+t)^{-1}y$ is the composition of the holomorphic mapping $G(x) = \alpha x + (1-\alpha)y$ (where $\alpha = r(r+t)^{-1}$) with the ϱ-nonexpansive mapping F_t, it follows from Step 1 and Banach's fixed point theorem that this equation has a unique solution $J_{r,t}(y) = x_t(y)$ and $J_{r,t}(y) = (I + rf_t)^{-1}: D \to D$ is ϱ-nonexpansive (see also [17]).

Step 4. We show now that

$$\limsup_{t\to 0+} \varrho(x_t, 0) \leqslant M < \infty.$$

This means that $\{x_t\}$ is contained in a ϱ-ball centered at the origin (which is strictly inside D [12]) when $t > 0$ is small enough.

Indeed, using Steps 1-3, we have the following inequalities:

$$\begin{aligned}
\varrho(x_t, 0) &= \varrho(r(r+t)^{-1}F_t(x_t) + t(r+t)^{-1}y, 0) \leqslant \\
&\leqslant d(d + t(r+t)^{-1}(1-p(y)))^{-1}\varrho(F_t(x_t), 0) + \varrho(t(r+t)^{-1}y, 0) \leqslant \\
&\leqslant (r+t)d(d(r+t) + t(1-p(y)))^{-1}[\varrho(F_t(x_t), F_t(0)) + \varrho(F_t(0), 0)] + \varrho(t(r+t)^{-1}y, 0) \leqslant \\
&\leqslant (r+t)d(d(r+t) + t(1-p(y)))^{-1}[\varrho(x_t, 0) + \varrho(F_t(0), 0)] + \varrho(t(r+t)^{-1}y, 0).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\limsup_{t\to 0+} \varrho(x_t, 0) \leqslant \\
&\leqslant \limsup_{t\to 0+}\left[\frac{(r+t)d}{t(1-p(y))}\varrho(F_t(0), 0) + \frac{(r+t)d + t(1-p(y))}{t(1-p(y))}\varrho\left(\frac{t}{r+t}y, 0\right)\right] \leqslant M < \infty,
\end{aligned}$$

because $\varrho(z, 0) \leqslant \operatorname{arctanh}(\|z\|/\operatorname{dist}(0, \partial D))$, whenever $\|z\| < \operatorname{dist}(0, \partial D)$ and $(\operatorname{arctanh} s)/s \to 1$, as $s \to 0+$ (see [15, 17]).

Step 5. Since r is constant in the following considerations we will omit the subscript r and write only $J_t = (I + rf_t)^{-1}$ for the resolvent $J_{r,t} = (I + rf_t)^{-1}$. We will show that for each $y \in D$ the net $\{J_t(y)\}$ strongly converges to some element $J(y) \in D$. Since (D, ϱ) is a complete ϱ-hyperbolic domain, it is enough to show that $\{J_t(y)\}$ is a Cauchy net. To see this, let $x_s = J_s(y)$ for $s > 0$ and set $y_{t,s} = (I + rf_t)(x_s)$. Since the net $\{f_t\}$ converges uniformly on the ϱ-ball $\widetilde{D} = \{x \in D \colon \varrho(0, x) < M\}$, for each $\varepsilon > 0$ there exists $t_0 > 0$ such that $\|f_s(x) - f_t(x)\| < \varepsilon/r$ for all $0 < s,\ t < t_0$ and $x \in \widetilde{D}$. It follows that for such s, t,

$$\|y - y_{t,s}\| = \|x_s + rf_s(x_s) - x_s - rf_t(x_s)\| < \varepsilon.$$

Since the ϱ-metric on D is locally equivalent to the norm and $J_t(y_{t,s}) = J_s(y)$, we now obtain

$$\varrho(J_s(y), J_t(y)) \leqslant \varrho(J_s(y), J_t(y_{t,s})) + \varrho(J_t(y_{t,s}), J_t(y)) \leqslant \varrho(y_{t,s}, y) < c\varepsilon$$

for some $c > 0$.

Step 6. For each $y \in D$, let $J(y)$ be the strong limit of the net $\{J_t(y)\}$ as $t \to 0+$. Then it follows by Step 3 that J is a nonexpansive self-mapping of D. Now we want to show that for a fixed $y \in D$ the element $x^* = J(y)$ is the unique solution of the equation $x + rf(x) = y$. To do this, denote $x_s = y - rf_s(x_s)$. By Step 5,

$$x^* = \lim_{s \to 0+} x_s$$

and we have

$$\|y - (x^* + rf(x^*))\| = \|x_s + rf_s(x_s) - x^* - rf(x^*)\| \leqslant$$
$$\leqslant \|x_s - x^*\| + r(\|f_s(x_s) - f(x_s)\| + \|f(x_s) - f(x^*)\|) \to 0$$

as $s \to 0+$ because f is continuous.

Hence $x^* + rf(x^*) = y$. Suppose now that there is another solution $\tilde{x}$ of the equation. Then setting $y_t = \tilde{x} + rf_t(\tilde{x})$ we have $y_t \to y$ as $t \to 0+$, and $J_t(y_t) = \tilde{x}$. Hence

$$\varrho(\tilde{x}, J_t(y)) = \varrho(J_t(y_t), J_t(y)) \leqslant \varrho(y_t, y).$$

But

$$\lim_{t \to 0+} J_t(y) = x^*.$$

Hence $x^* = \tilde{x}$ and this concludes our proof.

Now let X be a (real or complex) Banach space and let D be a domain in X with a metric ϱ.

We will say that a mapping $f\colon D \to X$ is an infinitesimal ϱ-generator if for some $T > 0$ there exists a one parameter semigroup $S = \{S_t\}_{t \in (0,T)}$ of self-mappings $S_t\colon D \to D$ $(S_{t+r}(x) = S_t(S_r(x)),\ x \in D,\ 0 < t + r < T)$ which are ϱ-nonexpansive for each $t \in (0, T)$: $\varrho(S_t(x), S_t(y)) \leqslant \varrho(x, y)$, $x, y \in D$, and such that

$$\lim_{t \to 0+} \frac{1}{t}\, \varrho(x - tf(x), S_t(x)) = 0$$

exists for all $x \in D$ uniformly on each ϱ-ball in D.

In this case we will also say that f is a ϱ-generator on $(0, T)$. If $T = \infty$ we will write $f \in \mathrm{GN}_\varrho(D)$.

So if, in particular, D is a bounded convex domain in a complex Banach space, and ϱ is its hyperbolic metric, then a continuous ϱ-generator on some interval $(0, T)$ belongs to the class $RN_\varrho(D)$ by Theorem 1.1. We will show that as a matter of fact, if $f\colon D \to X$ is uniformly continuous on each ϱ-ball in D, then the converse assertion also holds. Moreover, in this case the Cauchy problem

$$u'(t) + f(u(t)) = 0, \qquad u(0) = x, \qquad x \in D,$$

has a unique global solution on $R^+ = [0, \infty)$ for each $x \in D$.

This also holds for a somewhat more general case, when D is a domain in a Banach space X (real or complex) with a metric which has similar properties to those enjoyed by hyperbolic metrics. More precisely, we will establish the following assertion.

THEOREM 1.2. *Let D be a domain in a Banach space X, and let ϱ be a metric on D which satisfies the following conditions*:

(*i*) *(D, ϱ) is a complete metric space*;

(*ii*) *ϱ is locally Lipschitz continuous on D in the following sense*: *if $x \in D$ and $d > 0$ are such that $B_d(x) \subset D$, then there exists $L = L(d)$ such that $\varrho(x, y) \leqslant L\|x - y\|$, whenever $y \in B_d(x)$*;

(*iii*) *for each $x \in D$ and $R > 0$ the metric ball $\mathcal{B}_R(x) = \{y \in D: \varrho(x, y) < R\}$ lies strictly inside D*, i.e. $\operatorname{dist}(\mathcal{B}_R(x), \partial D) > 0$.

Let f be a mapping from D to X such that f is bounded and uniformly continuous on each subset strictly inside D, as a mapping from (D, ϱ) to X.

Suppose also that for some $T > 0$, f satisfies the following condition: *for each $0 < r < T$ the mapping $J_r = (I + rf)^{-1}$ is a well-defined self-mapping of D and*

$$\varrho(J_r(x), J_r(y)) \leqslant \varrho(x, y) \quad \textit{for all } x \textit{ and } y \textit{ in } D\,.$$

Then f is a ϱ-generator of a ϱ-nonexpansive semigroup on $(0, T)$.

PROOF. *Step 1.* Take any $x \in D$ and $R > 0$. Let $d > 0$ be such that $B_d(z) \subset D$ for all $z \in \mathcal{B}_R(x)$. Denote $M = \sup\{\|f(y)\|: y \in \mathcal{B}_R(x)\}$ and $\tau = \min\{d/M, T\}$. Then for each $r \in (0, \tau)$ and all $z \in \mathcal{B}_R(x)$ we have $y = z + rf(z) \in B_d(z)$ and $J_r(y) = z$. Therefore, for such r we have

$$\varrho(J_r(z), z) \leqslant \varrho(z, y) \leqslant L\|y - z\| = LMr\,. \tag{1.1}$$

Step 2. For each $z \in D$, $r \in (0, T)$ and $k = 1, 2, 3, \ldots$ we have by the triangle inequality

$$\varrho(J_r^k(z), z) \leqslant \sum_{j=1}^{k} \varrho(J_r^j(z), J_r^{j-1}(z)) \leqslant k\varrho(J_r(z), z)\,. \tag{1.2}$$

In particular, if $z \in \mathcal{B}_R(x)$ and $0 < r < \tau$, it follows that

$$\varrho(J_r^k(z), z) \leqslant kLMr\,. \tag{1.3}$$

Step 3. Since f is uniformly continuous on $\mathcal{B}_R(x)$, for each $\varepsilon > 0$ we can find $\delta > 0$ such that $\|f(z) - f(y)\| < \varepsilon$, whenever $z, y \in \mathcal{B}_R(x)$ and $\varrho(z, y) < \delta$. Set $\mu = \min\{\tau, \delta/(ML)\}$. Then for all $r \in (0, \mu)$ and each $p = 1, 2, \ldots$ we obtain by (1.3)

$$\left\|f(z) - \frac{z - J_{r/p}^p(z)}{r}\right\| \leqslant \sum_{k=1}^{p} \frac{1}{r}\left\|\frac{r}{p} f(z) - J_{r/p}^{k-1}(z) + J_{r/p}^k(z)\right\| \leqslant \tag{1.4}$$

$$\leqslant \frac{1}{r} \sum_{k=1}^{p}\left\|\frac{r}{p} f(z) + J_{r/p}(J_{r/p}^{k-1}(z)) - J_{r/p}^{k-1}(z)\right\| = \frac{1}{r}\sum_{k=1}^{p} \frac{r}{p}\|f(z) - f(J_{r/p}^k(z))\| \leqslant \varepsilon\,,$$

whenever $z \in \mathcal{B}_R(z)$.

Step 4. Note that by (1.1) we also have $\varrho(J_r(z), z) < \delta$ for all $z \in \mathcal{B}_R(x)$ whenever $r \in (0, \mu)$. Together with (1.4) this implies

$$(1.5)\qquad \|J_r(z) - J^p_{r/p}(z)\| \leqslant \|z - J^p_{r/p}(z) - rf(z)\| + \|rf(z) - z + J_r(z)\| \leqslant$$
$$\leqslant \varepsilon r + r\|f(z) - f(J_r(z))\| \leqslant 2r\varepsilon,$$

for all $z \in \mathcal{B}_R(x)$, $r \in (0, \mu)$, $p = 1, 2, \ldots$.

Step 5. Fix now $0 < t < T$. We intend to show that there is $R > 0$ such that the sequence $\{z_q = J^{k(q)}_{t/q}(x),\ q, k(q) \in N,\ k(q) \leqslant q\}$ is contained in the metric ball $\mathcal{B}_R(x)$.

Indeed, choose $q_0 > T\|f(x)\|/\operatorname{dist}(x, \partial D)$. Then for $q > q_0$ the point $y_q = x + (t/q)f(x) \in B_{d_1}(x)$, where $d_1 < \operatorname{dist}(x, \partial D)$. Hence we have by (1.2)

$$\varrho(J^{k(q)}_{t/q}(x), x) \leqslant k(q)\,\varrho(J_{t/q}(x), x) =$$
$$= k(q)\,\varrho(J_{t/q}(x), J_{t/q}(y_q)) \leqslant k(q)\,\varrho(x, y_q) \leqslant k(q)\|x - y_q\|\cdot L_1 \leqslant TL_1\|f(x)\|,$$

where $L_1 = L_1(d_1)$ depends on d_1.

Setting $R = \max\{T\cdot\|f(x)\|L_1,\ \varrho(x, z_q),\ q = 1, \ldots, q_0\}$ we obtain the required conclusion.

Step 6. Choosing now d and $L = L(d)$ in Steps 1-4 corresponding to this R, we claim that the sequence $\{J^n_{t/n}(x)\}$ is a Cauchy sequence in (D, ϱ), *i.e.* $\varrho(J^n_{t/n}(x), J^m_{t/m}(x)) \to 0$, as $n, m \to \infty$. Indeed, taking any $\varepsilon > 0$, $z \in \mathcal{B}_R(x)$, and $n_0, m_0 > T/\mu$, with μ as in Step 3, we have by (1.5) (setting $r = t/n$ and $p = m$),

$$(1.6)\qquad \varrho(J_{t/n}(z), J^m_{t/(nm)}(z)) \leqslant$$
$$\leqslant L\|J_{t/n}(z) - J^m_{t/(nm)}(z)\| < 2LT\varepsilon/n \text{ for all } n > m_0, \qquad m = 1, 2, \ldots.$$

Similarly, setting in (1.5) $r = t/m$ and $p = n$, we get

$$(1.7)\qquad \varrho(J_{t/m}(z), J^n_{t/(nm)}(z)) < 2LT\varepsilon/m$$

for all $m > m_0$, $n = 1, 2, \ldots$. Now for such $m > m_0$ and $n > n_0$ we have by (1.6) and (1.7),

$$\varrho(J^n_{t/n}(x), J^m_{t/m}(x)) \leqslant \varrho(J_{t/n}(x), J^{nm}_{t/(nm)}(x)) + \varrho(J^{nm}_{t/(nm)}(x), J^m_{t/m}(x)) \leqslant$$
$$\leqslant \sum_{j=0}^{n-1} \varrho(J^{n-j}_{t/n}(J^{jm}_{t/(nm)}(x)), J^{n-j-1}_{t/n}(J^{(j+1)m}_{t/(nm)}(x))) +$$
$$+ \sum_{i=0}^{m-1} \varrho(J^{m-i}_{t/m}(J^{in}_{t/(nm)}(x)), J^{m-i-1}_{t/m}(J^{(i+1)n}_{t/(nm)}(x))) \leqslant$$
$$\leqslant \sum_{j=0}^{n-1} \varrho(J_{t/n}(J^{jm}_{t/(nm)}(x)), J^{(j+1)m}_{t/(nm)}(x)) + \sum_{i=0}^{m-1} \varrho(J_{t/m}(J^{in}_{t/(nm)}(x)), J^{(i+1)n}_{t/(nm)}(x)) =$$
$$= \sum_{j=0}^{n-1} \varrho(J_{t/n}(J^{jm}_{t/(nm)}(x)), J^m_{t/(nm)}(J^{jm}_{t/(nm)}(x))) +$$

$$+\sum_{i=0}^{m-1} \varrho(J_{t/m}(J^{in}_{t/(nm)}(x)), J^{n}_{t/(nm)}(J^{in}_{t/(nm)}(x))) \leqslant n(2Lt\varepsilon/n) + m(2Lt\varepsilon/m) = 4Lt\varepsilon\,,$$

as required.

Step 7. Since (D, ϱ) is a complete metric space, Step 6 means that for each $t \in (0, T)$ and for each $x \in D$, there exists the limit

$$\lim_{n\to\infty} J^n_{t/n}(x) = S_t(x)\,, \tag{1.8}$$

and for each $\varepsilon > 0$, $t^{-1}\varrho(J^n_{t/n}(x), S_t(x)) < 4L\varepsilon$ for $n = n(\varepsilon, T)$ large enough.

Now it follows by (1.4) (see Step 3) that for each $n > 0$

$$\lim_{t\to 0+} \frac{x - J^n_{t/n}(x)}{t} = f(x)\,,$$

which implies by assumption *(ii)* that

$$\lim_{t\to 0+} \frac{1}{t}\,\varrho(x - tf(x), J^n_{t/n}(x)) = 0$$

for each $n = 1, 2, \ldots$. Hence

$$\lim_{t\to 0+} \frac{1}{t}\,\varrho(x - tf(x), S_t(x)) = 0\,. \tag{1.9}$$

Step 8. The semigroup property of S_t, *i.e.* $S_{t+s}(x) = S_t(S_s(x))$, can be proved in a standard way (see, for example, [11]), using (1.8) and passing from rational $t \geqslant 0$, $s \geqslant 0$ to real numbers if we know that $S_t(x)$ is continuous in $t \in (0, T)$.

Thus we have to establish the continuity of $S_t(x)$.

To do this it is enough (using (1.8)) to prove the uniform continuity of J_r for r sufficiently small.

To this end, let us choose $0 < t < r < \min\{\tau, R/ML\}$ (see Step 1). Then by (1.1), $\varrho(J_r(x), x) < R$, and hence $\|f(J_r(x))\| \leqslant M$. In addition, $J_r(x) \in \mathcal{B}_d(x)$ and thus the segment $[x, J_r(x)] \subset D$.

By the «resolvent identity» we obtain

$$\varrho(J_r(x), J_t(x)) = \varrho\left(J_t\left(\frac{t}{r}x + \frac{r-t}{r}J_r(x)\right), J_t(x)\right) \leqslant \varrho\left(\frac{t}{r}x + \frac{r-t}{r}J_r(x), x\right) =$$

$$= \varrho(x, x - (r-t)\,f(J_r(x))) \to 0 \quad \text{as } r - t \to 0\,.$$

The theorem is proved.

Corollary 1.1. *Under the hypotheses of Theorem* 1.2, *if in addition, ϱ is locally equivalent to the norm of X, then the Cauchy problem*

$$\frac{du(t)}{dt} + f(u(t)) = 0\,, \qquad u(0) = x \in D \tag{1.10}$$

has a unique global solution $u(t)$ on $[0, T)$ defined by the exponential formula

$$u(t)(x) = S_t(x) = \lim_{n\to\infty}\left(I + \frac{t}{n}f\right)^{-n}(x)\,, \qquad x \in D\,. \tag{1.11}$$

Proof. Since ϱ is locally equivalent to the norm of X, setting $S_0(x) = x$ for each $x \in D$ we get from (1.9) that $S_t(x)$ is right differentiable at $t = 0$ with derivative $-f(x)$.

Using the semigroup property we have that $S_t(x)$ is right differentiable for all $t \in (0, T)$ and $(d^+ S_t(x))/dt = -f(S_t(x))$.

Since its right derivative is continuous it follows that $S_t(x)$ is also left differentiable (see, for example, [34]) and $(d^\pm S_t(x))/dt = -f(S_t(x))$. Thus $u(t)$ as defined by (1.11) satisfies (1.10).

The uniqueness follows from Theorem 3 in [10].

Remark 1.1. As we saw in the proof of Theorem 1.2 the local version of this corollary also holds. More precisely, let D be as above, and let ϱ be a metric on D which is locally equivalent to the norm of X. Suppose that $f\colon D \to X$ is a continuous mapping on D, such that $(I + rf)^{-1}\colon D \to D$ is well-defined and ϱ-nonexpansive on D. Then for each $x \in D$ there are a neighborhood $U_x \subset D$ and a positive number $\mu = \mu(x)$ such that

$$\lim_{n\to\infty} J^n_{t/n}(z) = S_t(z)$$

exists for all $z \in U_x$ and $t \in [0, \mu]$. This limit is the solution of the Cauchy problem (1.10) on the interval $[0, \mu)$.

Now we return to the case when D is a convex bounded domain in a complex Banach space. In this case the hyperbolic metric on D can be represented as the integrated form of the Carathéodory-Reiffen-Finsler infinitesimal metric (see, for example, [15, 17, 12]).

It follows directly from the definition of this metric that each subset $\widetilde{D}$ which is strictly inside D ($\widetilde{D} \subset\subset D$) is contained in some ϱ-ball in D. In addition, as we mentioned above, each ϱ-ball in D lies strictly inside D. Hence, if $f\colon D \to X$ is uniformly continuous on each ϱ-ball in D and for some $T > 0$ the mapping $J_r = (I + rf)^{-1}$, $0 < r \leqslant T$, is a well-defined ϱ-nonexpansive self-mapping of D, then it follows from the formula (1.4) and Theorem 1.1 (setting $F(s) = J_s$) that $f \in \mathrm{RN}_\varrho(D)$, *i.e.* for each $r \in [0, \infty)$ the mapping $J_r = (I + rf)^{-1}$ is a well-defined ϱ-nonexpansive self-mapping of D.

This conclusion implies in its turn that if the Cauchy problem (1.10) has a local (on some interval $[0, \delta)$) solution $u(t)(\cdot) \in D$ which is ϱ-nonexpansive with respect to the initial values in D, then it also has a unique global solution on $[0, \infty)$.

In addition, we have the following analog of the Hille-Yosida theorem.

Theorem 1.3. *Let D be a bounded convex domain in a complex Banach space X, and let ϱ be its hyperbolic metric. Suppose that $f\colon D \to X$ is bounded and uniformly continuous on each ϱ-ball in D. Then $f \in \mathrm{GN}_\varrho(D)$ if and only if $f \in \mathrm{RN}_\varrho(D)$.*

Since each holomorphic bounded mapping $f\colon D \to X$ is locally uniformly Lipschitzian on each $\widetilde{D} \subset\subset D$, we have the following direct consequence of Theorem 1.3.

Corollary 1.2. *Let D be a bounded convex domain in a complex Banach space X, and let $f\colon D \to X$ be a bounded holomorphic mapping. Then f generates a one-parameter semigroup on R^+ of holomorphic self-mappings of D if and only if for some $T > 0$ and for each $r \in (0, T]$ the mapping $J_r = (I + rf)^{-1}$ is a well-defined holomorphic self-mapping of D.*

We will write in this case $f \in \mathrm{GH}(D)$.

COROLLARY 1.3. *Let D be as above. Then the sets*
$A = \{f \in GN_\varrho(D)$: *$f$ is bounded and uniformly continuous on each subset strictly inside* $D\}$
and

$$\widetilde{A} = \{f \in GH(D): f \text{ is bounded on } D\}$$

are real cones.

PROOF. Let f and g belong to A (or $\widetilde{A}$), and let α and β be non-negative. Let $\{F_t\}$ and $\{G_t\}$ be the semigroups generated by f and g respectively. Define the family $\{H(t)\colon 0 \leqslant t < \infty\}$ by $H(t)x = F_{\alpha t}(G_{\beta t}(x))$. It can be shown that for each $x \in D$,

$$h(x) = \alpha f(x) + \beta g(x) = \lim_{s \to 0+} (x - H(s)x)/s,$$

uniformly on each ϱ-ball in D. Hence h belongs to A (or $\widetilde{A}$) by Theorems 1.1 and 1.3, and Corollary 1.2.

Now we turn to the null point set of a ϱ-generator. The uniqueness of the solution to the Cauchy problem for $f \in \mathrm{RN}_\varrho(D)$ in the setting of Corollary 1.1 implies that

$$\mathrm{Null}_D f = \bigcap_{t \geqslant 0} \mathrm{Fix}_D S_t$$

where $S = \{S_t\}$ is the semigroup generated by f.

Thus one can study the null point set of f as the common fixed point set of the commutative family $\{S_t\}$.

Indeed, if D is a bounded convex domain in C^n and $\{S_t\}$ is a semigroup of holomorphic mappings, then it is well known [3] that the common fixed point set of this commutative family is a holomorphic retract of D. This approach becomes less transparent if X is an infinite dimensional space, or if $\{S_t\}$ is a semigroup of ϱ-nonexpansive self-mappings of D which are not necessarily holomorphic. Nevertheless, if $f \in \mathrm{RN}_\varrho(D)$, then the resolvent $J_r = (I + rf)^{-1}$ is well-defined and for each $r > 0$,

$$\mathrm{Fix}_D J_r = \mathrm{Null}_D f. \tag{1.12}$$

This equality and the Mazet-Vigué theorem [26] (for example) immediately imply that the null point set of a holomorphic generator on a convex bounded domain in a complex reflexive Banach space is also a holomorphic retract of D. (This, in turn, implies that $\mathrm{Null}_D f$ is a connected submanifold of D [9]. Moreover, for a Hilbert space we have that $\mathrm{Null}_D f$ is affine [27]).

For the more general case when $f\colon D \to X$ is only a continuous mapping belonging to $\mathrm{RN}_\varrho(D)$ we have the following information.

Let D be a bounded convex domain in C^n. Then for each $x \in D$ the net $\{J_r(x)\}$, $r > 0$, has a limit point $a \in \overline{D}$, as $r \to \infty$. If this net is ϱ-bounded in (D, ϱ), then $a \in D$, and it follows from the equality

$$(x - J_r(x))/r = f(J_r(x)), \qquad r > 0, \tag{1.13}$$

that a is a null point of f.

Conversely, if f has a null point $a \in D$, then for each $x \in D$ we have by (1.12)

$$\varrho(J_r(x), a) = \varrho(J_r(x), J_r(a)) \leqslant \varrho(x, a).$$

Hence for each $x \in D$ the net $\{J_r(x)\}$ is ϱ-bounded in (D, ϱ). In addition, it follows that $\mathrm{Null}_D f$ is a ϱ-nonexpansive retract of D, and therefore by Theorem 5 in [25] it is a metrically convex subset of (D, ϱ).

If we now assume that f has no null point in D, but f is continuous on $\overline{D}$, then each limit point a of $\{J_r(x)\}$ belongs to the boundary of D and is a null point of f in $\overline{D}$ by (1.13).

If, in addition, we assume that D is a strongly convex C^2 domain in C^n, and $f \in \mathrm{GH}(D)$, then we can show, using a result in [25], that for each $x \in D$ such a point $a \in \partial D$ is unique. In other words, we claim that in this case for each $x \in D$, the net $\{J_r(x)\}$ converges as $r \to \infty$ to a point $a \in \partial D$ which is independent of x.

Indeed, for a more general case, when D is a convex bounded domain in a Banach space X, and $s > 0$, consider the mapping $J_s = (I + sf)^{-1}: D \to D$ for $f \in \mathrm{RN}_\varrho(D)$. Then for each $r > 0$, the mapping $G_r = (I + r(I - J_s))^{-1}$ is a well-defined ϱ-nonexpansive self-mapping of D. For each $x \in D$ this mapping can be defined as the unique solution of the equation

$$G_r(x) = r(r+1)^{-1} J_s(G_r(x)) + (r+1)^{-1} x. \tag{1.14}$$

On the other hand, since $I - J_s = sf(J_s)$ we have

$$[I + rsf(J_s)]\, G_r(x) = G_r(x) + rsf(J_s(G_r(x))) = x, \qquad x \in D.$$

Hence it follows by (1.14) that

$$J_s(G_r(x)) + (1 + r) sf(J_s(G_r(x))) = x, \qquad x \in D.$$

This means that

$$J_{(r+1)s} = J_s(G_r) = J_s(I + r(I - J_s))^{-1}. \tag{1.15}$$

Now let D be a strongly convex C^2 bounded domain in C^n, and let $f \in \mathrm{GH}(D)$. Then for each fixed $s > 0$, J_s $(= (I + sf)^{-1})$ is a holomorphic self-mapping of D. If f has no null point in D, then J_s has no fixed point in D, and it follows by Theorem 4 in [25] that for each $x \in D$ the net $\{G_r(x)\}$, defined by (1.14), converges to the same point $a \in \partial D$, as r tends to infinity.

Moreover, the same formula (1.14) and the boundedness of D imply that

$$J_s(G_r(x)) - G_r(x) \to 0, \qquad \text{as } r \to \infty.$$

Together with (1.15) we get that for each $x \in D$, $\lim_{r \to \infty} J_r(x) = a$ and we are done.

We summarize all this information in the following statement.

COROLLARY 1.4. *Let D be a bounded convex domain in C^n, and let $f \in \mathrm{GN}_\varrho(D)$. Then*

I. *The following hypotheses are equivalent*:

(*i*) $\mathrm{Null}_D f \left(= \bigcap_{t \geqslant 0} \mathrm{Fix}\, F_t \right) \neq \emptyset$, *where $\{F_t\}$, $t \geqslant 0$, is the semigroup generated by f*;

(*ii*) *For some* $x \in D$ *the net* $\{J_r(x)\}$, $J_r = (I + rf)^{-1}$, *is strictly inside* D;

(*iii*) *For each* $x \in D$ *the net* $\{J_r(x)\}$ *is strictly inside* D.

II. If $\mathrm{Null}_D f \neq \emptyset$, *then it is a metrically convex* ϱ*-nonexpansive retract of* D.

III. If f *has a continuous extension to* $\overline{D}$, *then it has a null point in* $\overline{D}$.

IV. If, in addition, D *is a strongly convex* C^2 *domain, and* $f \in \mathrm{Hol}(D, X)$ *has no null point in* D, *then there exists a unique point* $a \in \partial D$ *such that for each* $x \in D$ *the net* $\{J_r(x)\}$ *converges to* a, *as* $r \to \infty$.

Such a point a will be called a «sink» null point of f in ∂D.

In this connection, we note that in general one cannot study the null points of f on ∂D by using the semigroup generated by f. See the example in Section 2 of [22], which shows that a generator f may have two null points in $\overline{D}$ (one of them on ∂D), while the semigroup S generated by f has a unique stationary point in $\overline{D}$ (which is the interior null point of f).

Nevertheless, as we will see below, in our situation, if f has no null point in D, then its «sink» null-point on ∂D is an asymptotic limit of the semigroup generated by f.

This fact implies in its turn the following observation. If D is a strongly convex bounded C^2 domain in C^n and $\{S(t)\}$, $t > 0$, is a semigroup of ϱ-nonexpansive self-mappings of D such that for each $t > 0$, $S(t)$ has a fixed point in D, then it follows by Theorem 7 in [25] (see also [1]) that this semigroup has a common fixed point (stationary point) in D as a commutative family of holomorphic mappings.

As a matter of fact, for a continuous semigroup of ϱ-nonexpansive self-mappings it is enough to require the existence of an interior fixed point only for one $t_0 > 0$ to provide the existence of such a point for the whole semigroup. We will consider a somewhat more general situation.

We will say that a domain D in a Banach space X (real or complex) satisfies the iteration property with respect to a class $\mathfrak{M}$ of self-mappings of D if the following hypothesis holds:

(*i*) If $F \in \mathfrak{M}$ has no fixed point in D, then for each $x \in D$ the sequence $\{F^n(x)\}$ converges strongly to a point on the boundary of D.

We say that D has the common fixed point property with respect to $\mathfrak{M}$ if the following condition holds:

(*ii*) If $\{F_s\}_{s \in \mathcal{A}}$ is a net of commuting mappings in $\mathfrak{M}$ such that for each $s \in \mathcal{A}$, F_s has a fixed point in D, then $\bigcap_{s \in \mathcal{A}} \mathrm{Fix}_D F_s \neq \emptyset$.

PROPOSITION 1.1. *Let* X *be a real or complex Banach space, and let* D *be a domain in* X. *Suppose that* $\mathfrak{M}$ *is a class of self-mappings of* D *such that both hypotheses* (*i*) *and* (*ii*) *are satisfied. Let* $f\colon D \to X$ *be an infinitesimal generator of a one-parameter continuous semigroup* $\{S(t)\colon t \geqslant 0\} \subset \mathfrak{M}$ *such that* $\{S(t)\}$ *is locally uniformly Lipschitzian* *on D. If f has no null point in D, then for each $x \in D$ the net $\{S(t)(x)\}$ strongly converges to a point b on the boundary of D.*

PROOF. First we note that there is an interval $(0, \mu)$ such that for each $t \in (0, \mu)$, $S(t)$ has no null point in D. Indeed, if we suppose that there is a sequence $t_n \to 0$ such that for each n the mapping $S(t_n)$ has a fixed point in D, then by (ii) there is a point $x \in D$ such that $S(t_n)(x) = x$ for all n and hence $f(x) = 0$. This is a contradiction.

So, for each m large enough $(m > 1/\mu)$ all the mappings $S(1/m)$ have no fixed point in D. This means that for each $x \in D$ the sequence $S(n/m)(x)(= S^n(1/m)(x))$ converges strongly as $n \to \infty$ to a point b_m on the boundary of D. But it follows from the semigroup property that $S^{nm}(1/m)(x) = S^n(1)(x) \to b_m$, as $n \to \infty$, and hence $b_m = b$ does not depend on m.

We are now able to show that the semigroup $\{S(t)\}$ strongly converges to b as t tends to infinity. In fact, for a given $\varepsilon > 0$ and $x \in D$ we can choose $\delta > 0$ such that $\|S(t)(x) - S(t)(y)\| < \varepsilon/2$ for all $t > 0$, whenever $y \in D$ and $\|y - x\| < \delta$. For such δ we take $m \in N$ so large that $m^{-1} \in (0, \mu)$ and $\|S(h)(x) - x\| < \delta$ for all $h \in [0, 1/m)$.

Finally, for such m and $t > 0$, setting $n = [tm]$, we have $\|S(n/m)(x) - b\| = = \|S^n(1/m)(x) - b\| < \varepsilon/2$ for $t > 0$ big enough. Since $h = t - n/m \in [0, 1/m)$, we get for such $t > 0$,

$$\|S(t)(x) - b\| = \|S(n/m)(S(h)(x)) - b\| \leqslant$$
$$\leqslant \|S(n/m)(S(h)(x)) - S(n/m)(x)\| + \|S(n/m)(x) - b\| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and we are done.

COROLLARY 1.5. *Under the conditions of Proposition* 1.1, *assume in addition that f is also locally Lipschitzian. Then the semigroup $\{S(t)\}$ has a stationary point in D if and only if for some $t_0 > 0$ the mapping $S(t_0)$ has a fixed point in D.*

Note now that by results in [1] (see also [25]) each strongly convex bounded C^2 domain D in C^n has the iteration and the common fixed point properties with respect to the class of holomorphic self-mappings of D. In addition, it is known (see [2]) that each one-parameter semigroup of holomorphic self-mappings on D is differentiable with respect to the parameter. Thus we are led to the following assertion.

COROLLARY 1.6. *Let D be a strongly convex bounded C^2 domain in C^n and let $\{S(t)$: $t \geqslant 0\}$ be a semigroup of holomorphic self-mappings of D. Then*

I. The following assertions are equivalent:

(a) The semigroup $\{S(t)\}$ has a stationary point in D.

(b) There exists $t_0 > 0$ such that $S(t_0)$ has a fixed point in D.

(c) There exists $x \in D$ and a sequence $t_n \to \infty$ such that $\{S(t_n)(x)\}$ is strictly inside D.

(d) For each $x \in D$ there is a sequence $t_n \to \infty$ such that $\{S(t_n)(x)\}$ is strictly inside D.

II. If $\{S(t)\}$ has no stationary point in D, then there exists a unique point $a \in \partial D$ such that:

(a) For all $x \in D$ the set $\{S(t)(x)\}$ strongly converges, as $t \to \infty$, to a.

(b) For all $x \in D$ the net $J_r(x) = (I - rS'(0))^{-1}(x)$ strongly converges as $r \to \infty$ to the same point a.

PROOF. To establish our assertion it is enough to identify in our situation the points a and b obtained in Corollaries 1.4 and 1.5. Setting $f = -S'(0)$ we consider again the resolvents $J_s = (I + sf)^{-1}$, $s > 0$, which are holomorphic self-mappings of D. Since f has no null points in D, J_s has no fixed point in D and as we saw above, the net $\{G_r(0)\}$ defined by the formula (1.14) converges to $a \in \partial D$ as $r \to \infty$ $(G_r(0) = r(r+1)^{-1}J_s(G_r(0)))$. It follows from the proof of Theorem 2.3 in [1] that for each $n \in N$ and for every $x \in D$ and $R > 0$, the following inclusion holds: $J_s^n(E_x(a, R)) \subset F_x(a, R)$, where $E_x(a, R)$ and $F_x(a, R)$ are the small and the big horospheres of center a, pole x, and radius R, defined by

$$E_x(a, R) = \left\{z \in D: \limsup_{w \to a} [\varrho(z, w) - \varrho(x, w)] < \frac{1}{2} \log R\right\},$$

$$F_x(a, R) = \left\{z \in D: \liminf_{w \to a} [\varrho(z, w) - \varrho(x, w)] < \frac{1}{2} \log R\right\},$$

where ϱ is the hyperbolic metric on D.

Therefore it follows by the exponential formula (1.11) that for all $t > 0$

$$S(t)(E_x(a, R)) \subset \overline{F_x(a, R)}.$$

Hence, if $b = \lim_{t \to \infty} S(t)x$, then $b = a$ and we are done.

2. ϱ-MONOTONICITY IN THE HILBERT BALL

Let now $X = H$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and let $D = B$ be the open unit ball in H. In this section we introduce the notion of a ϱ-monotone mapping in B and study in greater detail the class $\mathrm{RN}_\varrho(B)$.

To motivate Definition 2.1 below, we recall that a mapping $f: B \to H$ is said to be monotone on B if for each x and y in B,

$$\operatorname{Re}\langle x - y, f(x) - f(y)\rangle \geq 0. \tag{2.1}$$

It is easy to see that (2.1) is equivalent to the following condition: For each x and y in B,

$$\|x - y\| \leq \|x + rf(x) - (y + rf(y))\| \quad \text{for all } r \geq 0. \tag{2.2}$$

If, in addition, f satisfies the range condition:

$$(I + rf)(B) \supset B, \tag{RC}$$

then (2.2) implies that for each $r \geq 0$ the mapping $J_r = (I + rf)^{-1}$ is a single-valued and nonexpansive (with respect to the norm of H) self-mapping of B.

Since our intention is to study the class $\mathrm{RN}_\varrho(B)$ which consists of those mappings $f: B \to H$ the resolvents of which are ϱ-nonexpansive (where ϱ is the hyperbolic metric on B), it is natural to replace the norm in (2.2) with the ϱ-metric. The difficulty is that the expression $\varrho(x + rf(x), y + rf(y))$ may not be defined for all $r > 0$.

Nevertheless, it will become clear that it is sufficient to consider this expression only when r is small enough. Therefore we are led to the following definition.

DEFINITION 2.1. *A mapping $f: B \to H$ is called ϱ-monotone (with respect to the hyperbolic metric ϱ on B) if for each pair $x, y \in B$*

$$\varrho(x, y) \leqslant \varrho(x + rf(x), y + rf(y))$$

for all $r > 0$ such that $x + rf(x)$ and $y + rf(y)$ belong to B.

In order to characterize ϱ-monotone mappings, we first recall that in our case the hyperbolic metric ϱ is explicitly given by the formula $\varrho(x, y) = \operatorname{arctanh}(1 - \sigma(x, y))^{1/2}$ where $\sigma(x, y) = (1 - |x|^2)(1 - |y|^2)|1 - \langle x, y\rangle|^{-2}$. Note that $\varrho(x, y) \leqslant \varrho(u, v)$ if and only if $\sigma(x, y) \geqslant \sigma(u, v)$, and that $x = y$ if and only if $\sigma(x, y) = 1$.

If x, y, u and v are any four points in B and $\delta > 0$ is sufficiently small, then one can define a function $\psi: [0, \delta) \to [0, \infty)$ by

$$\psi(r) = \sigma(x + ru, y + rv)\,. \tag{2.3}$$

LEMMA 2.1. *The function ψ is differentiable at the origin and*

$$\psi'(0) =$$
$$= 2\sigma(x, y)\operatorname{Re}[(\langle u, y\rangle + \langle x, v\rangle)/(1 - \langle x, y\rangle) - \langle x, u\rangle/(1 - |x|^2) - \langle y, v\rangle/(1 - |y|^2)]\,.$$

PROOF. Indeed for $t \neq 0$ we have

$$\begin{aligned}\psi(t) = (1 - \|x\|^2 - 2t\operatorname{Re}\langle x, u\rangle - t^2\|u\|^2)&(1 - \|y\|^2 - 2t\operatorname{Re}\langle y, v\rangle - t^2\|v\|^2)\cdot\\
&\cdot|1 - \langle x, y\rangle - t\langle u, y\rangle - t\langle x, v\rangle - t^2\langle u, v\rangle|^{-2} =\\
&= (1 - \|x\|^2)(1 - \|y\|^2)(1 - 2t(\operatorname{Re}\langle x, u\rangle + t\|u\|^2)/(1 - \|x\|^2))\cdot\\
&\cdot(1 - 2t(\operatorname{Re}\langle y, v\rangle + t\|v\|^2)/(1 - \|y\|^2))|1 - \langle x, y\rangle|^{-2}\,|1 -\\
&- t(\langle u, y\rangle + \langle x, v\rangle + t\langle u, v\rangle)/(1 - \langle x, y\rangle)|^{-2} =\\
&= \sigma(x, y)(1 - 2t(\operatorname{Re}\langle x, u\rangle + t\|u\|^2)/(1 - \|x\|^2))\cdot\\
&\cdot(1 - 2t(\operatorname{Re}\langle y, v\rangle + t\|v\|^2)/(1 - \|y\|^2))/\\
&/(1 - 2t\operatorname{Re}((\langle u, y\rangle + \langle x, v\rangle + t\langle u, v\rangle)/(1 - \langle x, y\rangle) +\\
&\qquad + t^2|(\langle u, y\rangle + \langle x, v\rangle + t\langle u, v\rangle)/(1 - \langle x, y\rangle)|^2)\,.\end{aligned}$$

Thus

$$\psi'(0) = \lim_{t \to 0}\frac{1}{t}[\psi(t) - \sigma(x, y)] = -2\sigma(x, y)\cdot$$
$$\cdot[\operatorname{Re}\langle x, u\rangle/(1 - \|x\|^2) + \operatorname{Re}\langle y, v\rangle/(1 - \|y\|^2) - \operatorname{Re}((\langle u, y\rangle + \langle x, v\rangle)/(1 - \langle x, y\rangle))]$$

as claimed.

LEMMA 2.2. *For the function* ψ *defined by* (2.3), *the following are equivalent*:

(*i*) $\psi(r) \leqslant \psi(0)$, $0 \leqslant r \leqslant \delta$;

(*ii*) $\psi(r)$ *decreases on* $[0, \delta]$;

(*iii*) $\psi'(0) \leqslant 0$.

PROOF. It is clear that $(i) \Rightarrow (iii)$ and that $(ii) \Rightarrow (i)$. To show that $(iii) \Rightarrow (ii)$, let $t = 1 - r/\delta$ and note that

$$\psi(r) = \sigma(tx + (1-t)(x + \delta u), (1-t)y + t(y + \delta v)).$$

To show that $\psi(r)$ decreases on $[0, \delta]$ it suffices to show that the function φ: $[0, 1] \to [0, \infty)$ defined by

$$\varphi(t) := \varrho(tx + (1-t)(x + \delta u), (1-t)y + t(y + \delta v))^2$$

decreases on $[0, 1]$. But (*iii*) implies that $\varphi'(1) \leqslant 0$ and the result follows by Proposition 4.3 in [29].

Now we can present the following characterization of ϱ-monotone mappings. It is a direct consequence of Lemmata 2.1 and 2.2.

THEOREM 2.1. *Let* f: $B \to H$ *satisfy the range condition* (RC). *Then the following are equivalent*:

(*i*) f *is a* ϱ*-monotone mapping on* B;

(*ii*) *for each* $r \geqslant 0$ *the resolvent* $J_r = (I + rf)^{-1}$ *belongs to* $N_\varrho(B)$, i.e. J_r *is a single-valued* ϱ*-nonexpansive self-mapping of* B;

(*iii*) *for each* $x, y \in B$ *the following inequality holds*:

$$\mathrm{Re}\langle x, f(x)\rangle/(1 - \|x\|^2) + \mathrm{Re}\langle y, f(y)\rangle/(1 - \|y\|^2) \geqslant \geqslant \mathrm{Re}\,((\langle f(x), y\rangle + \langle x, f(y)\rangle)/(1 - \langle x, y\rangle)) \tag{2.4}$$

REMARK 2.1. Note that Lemmata 2.1 and 2.2 show that assertions (*i*) and (*iii*) are equivalent even without the range condition. As a matter of fact, in some cases the ϱ-monotonicity of f: $B \to H$ implies the range condition and therefore is equivalent to (*ii*), *i.e.* $f \in \mathrm{RN}_\varrho(B)$. This is, for example, the case when f is holomorphic and bounded on B. More precisely, we have the following result.

THEOREM 2.2. *Let* f: $B \to H$ *be a bounded holomorphic mapping. Then the following are equivalent*:

(*a*) $f \in \mathrm{RH}(B)$, i.e. *for each* $r \geqslant 0$ *the mapping* $J_r = (I + rf)^{-1}$ *is a well-defined holomorphic self-mapping on* B;

(*b*) f *is a* ϱ*-monotone on* B;

(*c*) *there exists a real number* m *such that*

$$\mathrm{Re}\langle f(x), x\rangle \geqslant m(1 - \|x\|^2) \tag{2.5}$$

for all $x \in B$.

PROOF. The implication $(a)\Rightarrow(b)$ follows from Theorem 2.1. This theorem also yields the implication $(b)\Rightarrow(c)$ (set $y=0$ in (2.4)). Thus we only need to prove the implication $(c)\Rightarrow(a)$.

Suppose (2.5) holds. We want to show that for each $r \geqslant 0$ and $y \in B$ the equation $x + rf(x) = y$ has a unique solution $x(y)$ which is holomorphic in $y \in B$. To this end consider the mapping $g(x) = x + rf(x) - y$. For every $x \in B$ with $\|x\| = s$, where $\|y\| < s < 1$ and s is close to 1, we have by (2.5),

$$\operatorname{Re}\langle g(x), x\rangle = \|x\|^2 + r\operatorname{Re}\langle f(x), x\rangle - \operatorname{Re}\langle y, x\rangle \geqslant s^2 + mr(1-s^2) - s\|y\| \geqslant \varepsilon > 0$$

for some $0 < \varepsilon < 1 - \|y\|$. Thus it follows from [4, Corollary 2] that the equation $g(x) = 0$ has a unique solution in B. This solution holomorphically depends on $y \in B$ (see [23, Corollary]). Our theorem is proved.

Our next result in this direction is the following one.

THEOREM 2.3. *Let B be the open unit ball in a separable complex Hilbert space H, and let $f\colon B \to H$ be a bounded continuous mapping. Then f is ϱ-monotone if and only if it belongs to* $\mathrm{RN}_\varrho(B)$.

PROOF. Clearly, as above it is enough to prove that if f is ϱ-monotone, then for every $z \in B$ and $r > 0$ there exists a solution $x \in B$ to

$$x + rf(x) = z\,. \tag{2.6}$$

Let $\{e_k\}_{k\geqslant 1}$ be an orthonormal basis of H. If $z \neq 0$ we take $e_1 = z/\|z\|$. For each $m \geqslant 1$, we denote $H_m = \operatorname{sp}\{e_k\}_{k=1}^m$, $B_m = H_m \cap B$, and $P_m\colon H \to H_m$ the orthogonal projection. We set $f_m = P_m \circ f$. Note that $f_m\colon B_m \to H_m$ is ϱ-monotone (thanks to 2.4).

We first show that for each m there exists a solution $u_m \in B_m$ of

$$u_m + rf_m(u_m) = z\,. \tag{2.7}$$

We denote by g_m the map $g_m(x) = x + rf_m(x) - z$ for $x \in B_m$. Using (2.4) with $y = 0$ we calculate (as in the proof of Theorem 2.2)

$$\operatorname{Re}\langle g_m(x), x\rangle = \|x\|^2 + r\operatorname{Re}\langle f_m(x), x\rangle - \operatorname{Re}\langle z, x\rangle \geqslant$$
$$\geqslant \|x\|(\|x\| - \|z\|) + r(1-\|x\|^2)\operatorname{Re}\langle x, f_m(0)\rangle\,.$$

Since $\|f_m(0)\| \leqslant \|f(0)\|$, there exists some $0 < s < 1$ (independent of m) such that $\operatorname{Re}\langle g_m(x), x\rangle \geqslant 0$ for all $x \in B_m$ with $\|x\| = s$.

Applying now a known fixed point theorem [24], we obtain the existence of a solution $u_m \in \overline{B_m(0,s)} \subset \overline{B(0,s)}$ of (2.7). Passing to a subsequence we may assume that $u_m \to u \in \overline{B(0,s)}$ weakly and that $\|u_m\| \to t$, so that $\|u\| \leqslant t \leqslant s$.

We may also assume that $f(u_m) \to \zeta \in H$ weakly. Next we claim that

$$\|u\| = t\,. \tag{2.8}$$

We first use (2.4) with $x = u$, $y = u_m$:

$$\operatorname{Re}\{\langle u, f(u)\rangle/(1-\|u\|^2) + \langle u_m, f(u_m)\rangle/(1-\|u_m\|^2)\} \geqslant \tag{2.9}$$
$$\geqslant \operatorname{Re}\{(\langle u, f(u_m)\rangle + \langle f(u), u_m\rangle)/(1-\langle u, u_m\rangle)\}\,.$$

Multiplying (2.7) by u_m and inserting the result in (2.9) we see that

$$\text{(2.10)} \quad \operatorname{Re}\{\langle u, rf(u)\rangle/(1-\|u\|^2) + (\langle u_m, z\rangle - \|u_m\|^2)/(1-\|u_m\|^2)\} \geqslant$$
$$\geqslant \operatorname{Re}\{(\langle u, rf(u_m)\rangle + \langle rf(u), u_m\rangle)/(1-\langle u, u_m\rangle)\}.$$

Passing to the limit in (2.10) we obtain

$$\text{(2.11)} \quad \operatorname{Re}\{\langle u, rf(u)\rangle/(1-\|u\|^2) + (\langle u, z\rangle - t^2)/(1-t^2)\} \geqslant$$
$$\geqslant \operatorname{Re}\{(\langle u, r\zeta\rangle + \langle rf(u), u\rangle)/(1-\|u\|^2)\}.$$

Note that from (2.7) it follows that

$$\text{(2.12)} \qquad \langle u, v\rangle + \langle r\zeta, v\rangle = \langle z, v\rangle, \qquad \forall v \in H.$$

Using this identity for $v = u$ on the right-hand side of (2.11) we get after simplification:

$$\operatorname{Re}\langle u, z\rangle\left(\frac{1}{1-t^2} - \frac{1}{1-\|u\|^2}\right) \geqslant \frac{t^2}{1-t^2} - \frac{\|u\|^2}{1-\|u\|^2}.$$

If $\|u\| < t$, then $\|u\|^2(1-\|u\|^2)^{-1} < t^2(1-t^2)^{-1}$ and we get $1 \leqslant \operatorname{Re}\langle u, z\rangle \leqslant s$, a contradiction. Hence (2.8) follows. From (2.8) we infer that $u_m \to u$ strongly. By the continuity of f, $f(u_m) \to f(u)$ and so $\zeta = f(u)$. Finally, from (2.12) it follows that $u + rf(u) = z$, *i.e.* u is a solution of (2.7).

REMARK 2.2. Note that the results in Section 1 imply that all the conditions of Theorem 2.2 are equivalent to the property of f to be a generator of a one-parameter continuous semigroup of holomorphic self-mappings of B. Theorems 1.1 and 2.1 also imply that all continuous generators of semigroups of ϱ-nonexpansive self-mappings are ϱ-monotone. In the other direction we need additional restrictions. Namely, if $f: B \to H$ is uniformly continuous on each subset strictly inside B and ϱ-monotone on B, and if H is separable, then f is the generator of a one-parameter semigroup of ϱ-nonexpansive self-mappings of B.

Now we return to the case of general complex Hilbert spaces and touch upon the description of null point sets of ϱ-monotone mappings. If $f \in \mathrm{RN}_\varrho(B)$ and $\mathrm{Null}_B f \neq \emptyset$, then this set is a ϱ-nonexpansive retract of B because $\mathrm{Null}_B f = \mathrm{Fix}_B J_r$ for each $r > 0$ (see [17]).

If $\mathrm{Null}_B f = \emptyset$, then we claim that there exists a point $e \in \partial B$ such that a for each $x \in B$, $J_r(x)$ strongly converges to e, as $r \to \infty$. Indeed, let $x \in B$ and suppose that for some sequence $r_n \to \infty$, $\{z_n = J_{r_n}(x)\}$ is ϱ-bounded.

It follows from the equality $x - J_{r_n}(x) = r_n f(J_{r_n}(x))$ that $f(z_n) \to 0$ as $n \to \infty$. Therefore the sequence $\{y_n = z_n + f(z_n)\}$ is also ϱ-bounded. But $y_n - J_1(y_n) = f(z_n) \to 0$, and it follows by Theorem 23.1 in [17] that $J_1: B \to B$ has a fixed point in B which is a null point of f. Thus $\|J_r(x)\| \to 1$ as $r \to \infty$. Now fix $r_0 > 0$ and consider the mapping $J_{r_0}: B \to B$. Since J_r has no fixed points in B, it is known [17] that there exists a point $e \in \partial B$ (a «sink» point) such that all the ellipsoids

$$E(e, K) = \{x \in B: |1 - \langle x, e\rangle|^2/(1-\|x\|^2) < K, K > 0\},$$

are invariant under J_{r_0}. But it follows from the resolvent identity

$$J_r x = J_{r_0}((r_0/r)x + (1 - r_0/r)J_r(x)),$$

$r > r_0$, that $E(e, k)$ is also invariant under J_r for all $r > r_0$. Now if for a fixed $x \in B$ we choose $K > 0$ such that $x \in E(e, K)$, then we get $|1 - \langle J_r(x), e\rangle| \to 0$, as $r \to \infty$. Hence, $J_r(x)$ converges strongly to e as $r \to \infty$.

Now we can easily prove the following assertion.

THEOREM 2.4. *Let B be the open unit ball in a complex Hilbert space H, and let $f\colon B \to H$ belong to $\mathrm{RN}_\varrho(B)$. If f has a continuous extension to $\overline{B}$, then it has a null point in $\overline{B}$.*

PROOF. If f has no null point in B, then as we saw above there exists a «sink» point $e \in \partial B$ such that $J_r(x)$ strongly converges to e, as $r \to \infty$. But since $f(J_r(x))$ converges to zero, as $r \to \infty$, it follows by continuity that $f(e) = 0$ and we are done.

ACKNOWLEDGEMENTS

The first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund-the Argentinian Research Fund.

Both authors are very grateful to Itai Shafrir for many helpful discussions. Theorem 2.3, in particular, is due to him.

REFERENCES

[1] M. ABATE, *Horospheres and iterates of holomorphic maps*. Math. Z., 198, 1988, 225-238.

[2] M. ABATE, *The infinitesimal generators of semigroups of holomorphic maps*. Ann. Mat. Pura Appl., 161, 1992, 167-180.

[3] M. ABATE - J. P. VIGUÉ, *Common fixed points in hyperbolic Riemann surfaces and convex domains*. Proc. Amer. Math. Soc., 112, 1991, 503-512.

[4] L. AIZENBERG - S. REICH - D. SHOIKHET, *One-sided estimaes for the existence of null points of holomorphic mappings in Banach spaces*. J. Math. Anal. Appl., 203, 1996, 38-54.

[5] J. ARAZY, *An application of infinite dimensional holomorphy to the geometry of Banach spaces*. Lecture Notes in Math., vol. 1267, Springer, Berlin 1987, 122-150.

[6] V. BARBU, *Nonlinear Semigroups and Differential Equations in Banach Spaces*. Noordhoff, Leyden 1976.

[7] E. BERKSON - H. PORTA, *Semigroups of analytic functions and composition operators*. Michigan Math. J., 25, 1978, 101-115.

[8] H. BRÉZIS, *Opérateurs Maximaux Monotones*. North Holland, Amsterdam 1973.

[9] H. CARTAN, *Sur les rétractions d'une variété*. C. R. Acad. Sci. Paris, 303, 1986, 715-716.

[10] P. CHERNOFF - J. E. MARSDEN, *On continuity and smoothness of group actions*. Bull. Amer. Math. Soc., 76, 1970, 1044-1049.

[11] M. G. CRANDALL - T. M. LIGGETT, *Generation of semigroups of nonlinear transformations on general Banach spaces*. Amer. J. Math., 93, 1971, 265-298.

[12] S. DINEEN, *The Schwarz Lemma*. Clarendon Press, Oxford 1989.

[13] S. DINEEN - P. M. TIMONEY - J. P. VIGUÉ, *Pseudodistances invariantes sur les domaines d'un espace localement convexe*. Ann. Scuola Norm. Sup. Pisa, 12, 1985, 515-529.

[14] C. J. EARLE - R. S. HAMILTON, *A fixed point theorem for holomorphic mappings*. Proc. Symp. Pure Math., vol. 16, Amer. Math. Soc., Providence, RI, 1970, 61-65.

[15] T. Franzoni - E. Vesentini, *Holomorphic Maps and Invariant Distances*. North Holland, Amsterdam 1980.
[16] I. I. Gikhman - A. V. Skorokhod, *Theory of Random Processes*. Nauka, Moscow 1973.
[17] K. Goebel - S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*. Dekker, New York-Basel 1984.
[18] L. A. Harris, *Schwarz-Pick systems of pseudometrics for domains in normed linear spaces*. Advances in Holomorphy, North Holland, Amsterdam 1979, 345-406.
[19] T. E. Harris, *The Theory of Branching Processes*. Springer, Berlin 1963.
[20] U. Helmke - B. Moore, *Optimization and Dynamical Systems*. Springer, London 1994.
[21] M. E. Jacobson, *Computation of extinction probabilities for the Bellman-Harris branching process*. Math. Biosciences, 77, 1985, 173-177.
[22] V. Khatskevich - S. Reich - D. Shoikhet, *Ergodic type theorems for nonlinear semigroups with holomorphic generators*. Recent Developments in Evolution Equations, Pitman Research Notes in Math., vol. 324, 1995, 191-200.
[23] V. Khatskevich - S. Reich - D. Shoikhet, *Global implicit function and fixed point theorems for holomorphic mappings and semigroups*. Complex Variables, 28, 1996, 347-356.
[24] M. A. Krasnoselski - P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*. Springer, Berlin 1984.
[25] T. Kuczumow - A. Stachura, *Iterates of holomorphic and k_D-nonexpansive mappings in convex domains in C^n*. Advances Math., 81, 1990, 90-98.
[26] P. Mazet - J. P. Vigué, *Points fixes d'une application holomorphe d'un domaine borné dans lui-même*. Acta Math., 166, 1991, 1-26.
[27] W. Rudin, *The fixed point sets of some holomorphic maps*. Bull. Malaysian Math. Soc., 1, 1978, 25-28.
[28] B. A. Sevastyanov, *Branching Processes*. Nauka, Moscow 1971.
[29] I. Shafrir, *Coaccretive operators and firmly nonexpansive mappings in the Hilbert ball*. Nonlinear Analysis, 18, 1992, 637-648.
[30] H. Upmeier, *Jordan Algebras in Analysis, Operator Theory and Quantum Mechanics*. CBMS-NSF Regional Conference Series in Math., AMS, Providence 1987.
[31] E. Vesentini, *Semigroups of holomorphic isometries*. Advances Math., 65, 1987, 272-306.
[32] E. Vesentini, *Krein spaces and holomorphic isometries of Cartan domains*. In: S. Coen (ed.), *Geometry and Complex Variables*. Dekker, New York 1991, 409-413.
[33] E. Vesentini, *Semigroups of holomorphic isometries*. In: S. Coen (ed.), *Complex Potential Theory*. Kluwer, Dordrecht 1994, 475-548.
[34] K. Yosida, *Functional Analysis*. Springer, Berlin 1968.

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