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A Boundary Value Problem Connected with Response of Semi-space to a Short Laser Pulse

Memoria (*) **di Gaetano Fichera**

This article is the last scientific contribution of Gaetano Fichera. The paper – whose draft was found on Gaetano Fichera's desk after his untimely death – was prepared for publication by O. A. Oleinik, with the collaboration of M. P. Colautti and trasmitted to the Accademia by his wife (1).

The original text was in its final form with the only apparent exception of the last few lines and, in particular, of the proof of Theorem 6.III. But a short Note added by O. A. Oleinik indicates how the theorem, which concludes the article, can be obtained by the same techniques developed by the Author in the preceding sections of the paper.

Abstract. — In this paper a mixed boundary value problem for the fourth order hyperbolic equation with constant coefficients which is connected with response of semi-space to a short laser pulse and belongs to generalized Thermoelasticity is studied. This problem was considered by R. B. Hetnarski and J. Ignaczak, who established some important physical consequences. The present paper contains proof of the existence, uniqueness and continuous dependence of a solution on the datum, together with an effective method for numerical computation of a solution and the behaviour of solutions as $t \to \infty$.

Key words: Hyperbolic equations; Mixed boundary value problems; Tauber-type theorem; Asymptotics of solutions for large values of the time.

Riassunto. — *Un problema di valori al contorno connesso alla risposta di un semispazio ad un breve impulso laser*. In questo lavoro viene studiato, nell'ambito della Termoelasticità generalizzata, un problema di valori al contorno di tipo misto per un'equazione iperbolica del quarto ordine a coefficienti costanti. Tale problema è connesso alla risposta di un semispazio ad un breve impulso laser. Di questa tematica si sono occupati R. B. Hetnarski e J. Ignaczak, ai quali si debbono alcuni importanti risultati fisici. Per il detto problema, in questo lavoro si dimostra l'esistenza, l'unicità di una soluzione, nonché la dipendenza continua dai dati. Si indica inoltre un metodo effettivo di calcolo numerico e si studia il comportamento delle soluzioni per $t \to \infty$.

(*) Presentata nella seduta del 19 giugno 1997 dal Socio E. Vesentini, Direttore del Comitato Consultivo.

(1) All'alba del 30 maggio 1996, Gaetano Fichera era già al suo tavolo di lavoro, intento a mandare avanti il più possibile la stesura della presente Memoria. Desiderava lavorare almeno per un paio d'ore prima di recarsi all'Accademia dei Lincei per partecipare ad un Convegno in ricordo di Federigo Enriques, Convegno al quale non voleva mancare. Era di ottimo umore, perché sentiva che ormai stava per concludere la sua ricerca, dopo lunghi ed estenuanti mesi di accanito ed appassionante lavoro, nel corso dei quali momenti di delusione si erano alternati a momenti di entusiasmo. Ma a quel punto egli era tranquillo e particolarmente felice, perché tutto – come lui mi aveva detto – era ormai chiaro nella sua mente. Gli restavano da redigere solo poche pagine e la Bibliografia.

L'idea di cimentarsi con questa ricerca, particolarmente complessa e laboriosa, si era affacciata alla sua mente durante il Convegno *Thermal Stresses 1995*, tenutosi ad Hamamatsu in Giappone. Ascoltando le con-

R. B. Hetnarski and J. Ignaczak have studied in a recent paper a Boundary Value Problem connected with response of semi-space to a short laser pulse, in the framework of generalized Thermoelasticity [1, 2].

From their analysis, they derive important mechanical consequences which throw light on the physical problem under consideration.

The present paper is intended to propose a theory of this new Boundary Value Problem, with the following aims:

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ferenze e parlando con dei Colleghi, si era incuriosito per un affascinante problema di termoelasticità. Aveva subito intuito che alla base di quel problema, squisitamente fisico, dovevano esserci notevoli implicazioni matematiche che andavano messe in luce e studiate, onde fornire al problema stesso una solida e coerente base matematica.

Ne aveva discusso a lungo, sia in Giappone che, più tardi, per corrispondenza, con il Prof. B. Hetnarski della Rochester University, il quale già si era occupato molto della stessa tematica e che, proprio in quei giorni, era giunto a Roma. Gaetano doveva incontrarlo nel pomeriggio e contava di illustrargli i suoi risultati: era sicuro di aver dato, con la sua Memoria, una completa e rigorosa giustificazione matematica al problema in questione.

Ma il destino non ha permesso tutto questo. Gaetano si è sentito male poco prima dell'inizio del Convegno ai Lincei, è stato soccorso ed affidato a quegli stessi meravigliosi medici che già, in circostanza non molto lontana, erano riusciti a salvargli la vita. Purtroppo, questa volta, si trattava di caso ben più grave ed a nulla è valso l'impegno totale, affettuoso e generoso di tutta l'equipe medica: Gaetano Fichera non ha più ripreso conoscenza ed è spirato all'alba del 1° giugno.

Nei pochi minuti che mi furono concessi accanto a lui prima che venisse portato in sala operatoria, egli, che con il suo rapido intuito aveva ben capito di non avere ormai speranza di salvezza, ma che stranamente era ancora perfettamente lucido e coerente, riuscì a dirmi tante cose. Ebbene, tra quanto un Uomo come lui, sentendosi morire, poteva dire, ci fu anche una frase inaspettata: «...e pensare che non potrò mai più finire la mia Memoria... ci tenevo così tanto...».

Quel suo accorato rimpianto suggerì in me, in seguito, un fermo proposito: bisognava riuscire a far pubblicare quel suo ultimo lavoro, sia pure in forma incompiuta, affinché potesse essere il suo messaggio estremo, un testimone che egli passava a chi, dopo di lui, ne volesse riprendere lo studio e trarne eventuali nuovi spunti di ricerca.

In questa idea sono stata confortata dalla pronta adesione del Presidente dell'Accademia Prof. Sabatino Moscati e del Presidente della Classe di Scienze Prof. Giorgio Salvini, che tanto affettuosamente mi sono stati vicini in quelle tragiche giornate ed ai quali rivolgo la mia più profonda gratitudine.

Mi è altresì caro ringraziare la Prof. Olga Oleinik dell'Università di Mosca, con la quale Gaetano aveva avuto occasione di discutere ampiamente di questa sua ricerca, per avere prontamente acconsentito alla mia preghiera di voler rivedere la Memoria prima di un'eventuale pubblicazione. Ella, con abnegazione, ha lavorato per lunghi mesi, controllando ogni passo del lavoro, rifacendo i complicati calcoli e completando la dimostrazione del Teorema 6.III. È stata validamente e generosamente aiutata dalla Prof. Maria Pia Colautti, già allieva di Gaetano e per lui anche sorella carissima. Pure a lei il mio grazie e la mia riconoscenza.

Penso che la pubblicazione di questa Memoria possa, in certo qual modo, ritenersi, oltre che l'ultimo insegnamento, anche l'ultimo esempio dello stile di vita e di lavoro di Gaetano Fichera. Infatti, egli, che ormai da lungo tempo era molto cagionevole di salute, alle continue ed incalzanti insistenze perché volesse risparmiare le sue forze e prendersi un po' di riposo, usava immancabilmente rispondere citando una frase di Benedetto Croce: «...soprattutto che la morte non mi colga in stupido ozio...». Così è stato.

MATELDA COLAUTTI FICHERA

Roma, 8 febbraio 1997.

i) to state rigorously a definition of a function class where the problem should be set;

ii) to prove existence, uniqueness and continuous dependence of a solution on the datum;

iii) to give effective methods for numerical computation of a solution and to bound rigorously the approximation error;

iiii) to investigate the behavior of solutions at $t \to +\infty$ (2).

1. New statement of the BVP

Let us denote by Q the following domain of the x, t plane: $x > 0$, $t > 0$.

Hetnarski and Ignaczak consider the following BVP for the unknown real valued functions $\varphi(x,t)$, $\theta(x,t)$ in the domain Q:

$$\Gamma\varphi = f(x,t) \quad \text{in } Q, \tag{1.1}$$

$$\left(1 + t^0 \frac{\partial}{\partial t}\right)\theta = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\varphi \quad \text{in } Q, \tag{1.2}$$

$$\varphi_{t^k}(x,0) = 0 \quad (k = 0, 1, 2, 3), \qquad x > 0, \tag{1.3}$$

$$\theta(x,0) = \theta_t(x,0) = 0, \qquad x > 0, \tag{1.4}$$

$$\varphi(0,t) = \varphi_{x^2}(0,t) = \theta(0,t) = 0, \qquad t > 0, \tag{1.5}$$

where Γ is the partial differential operator

$$\Gamma = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2} - t_0\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right) - \varepsilon\frac{\partial^3}{\partial x^2\partial t}\left(1 + t^0\frac{\partial}{\partial t}\right) =$$

$$= \frac{\partial^4}{\partial x^4} + t_0\frac{\partial^4}{\partial t^4} - (1 + t_0 + \varepsilon t^0)\frac{\partial^4}{\partial x^2\partial t^2} + \frac{\partial^3}{\partial t^3} - (1+\varepsilon)\frac{\partial^3}{\partial x^2\partial t};$$

t_0, t^0, ε are constants such that $t^0 \geqslant t_0 > 1$, $\varepsilon > 0$, $(1 + t_0 + \varepsilon t^0)^2 > 4t_0$, $f(x,t)$ is a given function. Actually Hetnarski and Ignaczak assume

$$f(x,t) = -\left(1 + t^0\frac{\partial}{\partial t}\right)Y(t)\exp(-\alpha x)$$

and, in particular, $Y(t) = Y_0 t^n \exp(-bt^m)$, with Y_0 constant and b, m, α, n positive constants.

Let us observe that in BVP (1.1)-(1.5) only apparently we have two unknown

(2) The point *iiii*) was not indicated on the first page of the original manuscript and has been added by the revisers, since problem *iiii*) has been treated and solved by G. Fichera in the Section 6 of his paper. Probably, at the beginning of his research, he did not yet intend to study also problem *iiii*). Since he left unfinished the proof of his last theorem, O. A. Oleinik finished the proof on the basis of Lemmas proved in the manuscript.

functions. Actually let us consider the following two BVP's

$$\begin{cases} \Gamma\varphi = f(x,t) \quad \text{in } Q, & (1.6)\\ \varphi_{t^k}(x,0) = 0 \quad (k=0,1,2,3), \quad x>0, & (1.7)\\ \varphi(0,t) = \varphi_{x^2}(0,t) = 0, \quad t>0, & (1.8)\end{cases}$$

$$\begin{cases} \left(1 + t^0 \dfrac{\partial}{\partial t}\right)\theta = h(x,t) \quad \text{in } Q, & (1.9)\\ \theta(x,0) = 0, \quad x>0. & (1.10)\end{cases}$$

Suppose that $h \in C^0(\overline{Q})$. Problem (1.9), (1.10) has only one solution which is given by

$$\theta(x,t) = \frac{1}{t^0}\int_0^t \exp\left(\frac{\tau - t}{t^0}\right) h(x,\tau)\,d\tau; \tag{1.11}$$

θ is such that $\theta \in C^0(\overline{Q})$, $\theta_t \in C^0(\overline{Q})$. If

$$h(0,t) = 0, \quad t \geqslant 0, \tag{1.12}$$

$\theta(x,t)$ satisfies (1.5).

Since

$$\theta_t(x,t) = \frac{1}{t^0}h(x,t) - \frac{1}{(t^0)^2}\int_0^t \exp\left(\frac{\tau - t}{t^0}\right) h(x,\tau)\,d\tau,$$

if

$$h(x,0) = 0 \quad x \geqslant 0, \tag{1.13}$$

function $\theta(x,t)$ satisfies the condition $\theta_t(x,0) = 0$, $x \geqslant 0$.

Suppose that $\varphi \in C^2(\overline{Q})$ is a solution of BVP (1.6)-(1.8). Assuming

$$h(x,t) = \varphi_{xx}(x,t) - \varphi_{tt}(x,t), \tag{1.14}$$

conditions (1.12), (1.13) are satisfied and (φ, θ), with θ given by (1.11), is a solution of (1.1)-(1.5).

Viceversa, if (φ, θ) is a solution of (1.1)-(1.5) and $\varphi \in C^2(\overline{Q})$, $\theta \in C^0(\overline{Q})$, $\theta_t \in C^0(\overline{Q})$, then φ is a solution of (1.6)-(1.8) and θ is a solution of (1.9), (1.10), with h given by (1.14).

We have shown that the only problem to be considered is (1.6)-(1.8). In fact, by assuming θ given by (1.11), with $h(x,t)$ given by (1.14), we get a solution of Problem (1.1)-(1.5), as considered by Hetnarski and Ignaczak.

In this paper *we shall consider problem* (1.6)-(1.8) *in the function class* $C^3(\overline{Q}) \cap H_4(E)$, where E is any bounded subdomain of Q.

The space $H_m(E)$ $(m \geqslant 1)$ is the Hilbert space obtained through functional completion from the space $C^m(\overline{E})$ of real valued functions endowed with the following norm

$$\|u\|_m^2 = \sum_{0 \leqslant \alpha+\beta \leqslant m} \iint_E \left| \frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta}\right|^2 dx\,dt.$$

On the function $f(x,t)$ the following hypothesis will be assumed from now on:

$\mathcal{H}_0$) *$f(x,t)$ is continuous in $\overline{Q}$ and has the following derivatives: $f_x(x,t)$, $f_t(x,t)$, $f_{xt}(x,t)$ which are continuous in $\overline{Q}$.*

2. The auxiliary mixed BVP $\mathcal{M}_{a,T}$

Let $R_{a,T}$ $(a>0,\ T>0)$ be the rectangle: $0<x<a$, $0<t<T$. We consider the following mixed BVP, which we denote by $\mathcal{M}_{a,T}$: *find a function $u(x,t)$ belonging to $C^3(\overline{R}_{a,T})\cap H_4(R_{a,T})$ such that*

$$\Gamma u = f(x,t) \quad \text{in } R_{a,T}, \tag{2.1}$$

$$u(0,t)=u_{x^2}(0,t)=u(a,t)=u_{x^2}(a,t)=0, \qquad 0\leqslant t\leqslant T, \tag{2.2}$$

$$u_{t^k}(x,0)=0 \quad (k=0,1,2,3), \qquad 0\leqslant x\leqslant a \ (^3). \tag{2.3}$$

Set $\lambda_h = h\pi a^{-1}$, $h=1,2,\dots$, and suppose that $u(x,t)$ is a solution of Problem $\mathcal{M}_{a,T}$. We have from (2.1)

$$\int_0^a (\Gamma u)\sin\lambda_h x\,dx = \int_0^a f(x,t)\sin\lambda_h x\,dx. \tag{2.4}$$

Set

$$u^h(t)=\int_0^a u(x,t)\sin\lambda_h x\,dx, \qquad f^h(t)=\int_0^a f(x,t)\sin\lambda_h x\,dx.$$

From (2.4) we have

$$t_0 u^h_{t^4} + u^h_{t^3} + (1+t_0+\varepsilon t^0)\lambda_h^2 u^h_{t^2} + (1+\varepsilon)\lambda_h^2 u^h_t + \lambda_h^4 u^h = f^h \tag{2.5}$$

and from (2.3)

$$u^h_{t^k}(0)=0 \quad (k=0,1,2,3). \tag{2.6}$$

Equations (2.5), (2.6) determine uniquely $u^h(t)$.

2.I. *If Problem $\mathcal{M}_{a,T}$ has a solution, it is unique and is given by*

$$u(x,t)=\frac{2}{a}\sum_{h=1}^{\infty} u^h(t)\sin\lambda_h x \qquad (\lambda_h = h\pi/a). \tag{2.7}$$

Let us now prove that (2.7) gives a solution of $\mathcal{M}_{a,T}$.

(3) The problem (2.1)-(2.3) is the mixed boundary value problem for the fourth order hyperbolic operator with the boundary conditions which satisfy the so called «Lopatinsky conditions» and has constant coefficients.

The mixed boundary value problem for hyperbolic equations was studied by various methods and in different classes of functions only during the last decades. The first result for the mixed problem for higher order hyperbolic equations with constant coefficients in a half space with boundary conditions satisfying the so called «uniform Lopatinsky conditions» belong to S. Agmon [3]; for variable coefficients the problem was solved by R. Sakamoto [4, 5] (see also survey [6]).

In Fichera's paper a new method is suggested for the construction of a solution of the mixed boundary value problem (2.1)-(2.3) and for its investigation (O. A. Oleinik).

Let us consider the algebraic equation

$$t_0\omega_h^4 + (1 + t_0 + \varepsilon t^0)\lambda_h^2\omega_h^2 + \lambda_h^4 = 0 \tag{2.8}$$

and set $\nu = 1 + t_0 + \varepsilon t^0$,

$$\sigma_1 = \{[\nu - (\nu^2 - 4t_0)^{1/2}]/2t_0\}^{1/2}, \qquad \sigma_2 = \{[\nu + (\nu^2 - 4t_0)^{1/2}]/2t_0\}^{1/2}.$$

Equation (2.8) has the following roots:

$$\omega_h^1 = i\lambda_h\sigma_1, \qquad \omega_h^2 = -i\lambda_h\sigma_1, \qquad \omega_h^3 = i\lambda_h\sigma_2, \qquad \omega_h^4 = -i\lambda_h\sigma_2.$$

Let us now introduce the following function:

$$K^h(s) = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega_h^1 & \omega_h^2 & \omega_h^3 & \omega_h^4 \\ (\omega_h^1)^2 & (\omega_h^2)^2 & (\omega_h^3)^2 & (\omega_h^4)^2 \\ (\omega_h^1)^{-1}e^{\omega_h^1 s} & (\omega_h^2)^{-1}e^{\omega_h^2 s} & (\omega_h^3)^{-1}e^{\omega_h^3 s} & (\omega_h^4)^{-1}e^{\omega_h^4 s} \end{vmatrix}}{t_0 W(\omega_h^1, \omega_h^2, \omega_h^3, \omega_h^4)}$$

where $W(\omega_h^1, \omega_h^2, \omega_h^3, \omega_h^4)$ is the Vandermonde determinant of ω_h^1, ω_h^2, ω_h^3, ω_h^4:

$$W(\omega_h^1, \omega_h^2, \omega_h^3, \omega_h^4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega_h^1 & \omega_h^2 & \omega_h^3 & \omega_h^4 \\ (\omega_h^1)^2 & (\omega_h^2)^2 & (\omega_h^3)^2 & (\omega_h^4)^2 \\ (\omega_h^1)^3 & (\omega_h^2)^3 & (\omega_h^3)^3 & (\omega_h^4)^3 \end{vmatrix}$$

The function $K_s^h(s)$ [$K_{s^n}^h(s)$ here and in what follows means the derivative of $K^h(s)$ of order n with respect to s] is the solution of the differential equation

$$t_0 y''''(s) + \nu\lambda_h^2 y''(s) + \lambda_h^4 y(s) = 0$$

satisfying the initial conditions

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 0, \qquad y'''(0) = t_0^{-1}.$$

From (2.5), (2.6) we deduce

$$u^h(t) = \int_0^t K_t^h(t-\tau)[-u_{\tau^3}^h(\tau) - (1+\varepsilon)\lambda_h^2 u_\tau^h(\tau) + f^h(\tau)]\,d\tau. \tag{2.9}$$

Set

$$\mu = t_0[\min(2\sigma_1, \sigma_2 - \sigma_1)]^3.$$

Since

$$\begin{aligned} t_0 K_{s^n}^h(s) = {} & -(\omega_h^1)^{n-1}e^{\omega_h^1 s}/[(\omega_h^4 - \omega_h^1)(\omega_h^3 - \omega_h^1)(\omega_h^2 - \omega_h^1)] + \\ & + (\omega_h^2)^{n-1}e^{\omega_h^2 s}/[(\omega_h^4 - \omega_h^2)(\omega_h^3 - \omega_h^2)(\omega_h^2 - \omega_h^1)] - \\ & - (\omega_h^3)^{n-1}e^{\omega_h^3 s}/[(\omega_h^4 - \omega_h^3)(\omega_h^3 - \omega_h^2)(\omega_h^3 - \omega_h^1)] + \\ & + (\omega_h^4)^{n-1}e^{\omega_h^4 s}/[(\omega_h^4 - \omega_h^3)(\omega_h^4 - \omega_h^2)(\omega_h^4 - \omega_h^1)], \end{aligned}$$

we have

$$(2.10) \qquad |K^h_{s^n}(s)| \leq (4\sigma_2^{n-1}/\mu)\lambda_h^{n-4} \qquad (n = 0, 1, 2, \ldots).$$

From (2.9) we have

$$(2.11) \quad u^h(t) = -\int_0^t K^h_{t^4}(t-\tau)u^h(\tau)\,d\tau - \\ -(1+\varepsilon)\lambda_h^2\int_0^t K^h_{t^2}(t-\tau)u^h(\tau)\,d\tau + \int_0^t K^h_t(t-\tau)f^h(\tau)\,d\tau.$$

On the other hand

$$(2.12) \quad \int_0^t K^h_{t^m}(t-\tau)f^h(\tau)\,d\tau = \\ = K^h_{t^{m-1}}(t)f^h(0) - K^h_{t^{m-1}}(0)f^h(t) + \int_0^t K^h_{t^{m-1}}(t-\tau)f^h_\tau(\tau)\,d\tau \qquad (m = 1, 2, 3, \ldots).$$

Let $M_f(a, T)$ be such that one has in $\overline{R}_{a,T}$

$$(2.13) \quad \begin{cases} |f(a,t)| + |f(0,t)| + \int_0^a |f_x(x,t)|\,dx \leq M_f(a,T), \\ |f_t(a,t)| + |f_t(0,t)| + \int_0^a |f_{xt}(x,t)|\,dx \leq M_f(a,T). \end{cases}$$

From (2.10), (2.12), (2.13) we have

$$(2.14) \quad \left|\int_0^t K^h_{t^m}(t-\tau)f^h(\tau)\,d\tau\right| \leq (4\sigma_2^{m-2}/\mu)(2+T)M_f(a,T)\lambda_h^{m-6}$$

and from (2.11)

$$|u^h(t)| \leq \frac{4(2+T)M_f(a,T)}{\sigma_2\mu}\frac{1}{\lambda_h^5} + \frac{4}{\mu}[\sigma_2^3 + (1+\varepsilon)\sigma_2]\int_0^t |u^h(\tau)|\,d\tau.$$

For Gronwall's inequality we get

$$(2.15) \qquad |u^h(t)| \leq \gamma_0(T)M_f(a,T)/\lambda_h^5$$

with

$$\gamma_0(T) = [4(2+T)/\sigma_2\mu]\exp\{(4/\mu)[\sigma_2^3 + (1+\varepsilon)\sigma_2]T\}.$$

From (2.11) we have

$$u^h_t(t) = -\frac{1}{t_0}u^h(t) - \int_0^t K^h_{t^5}(t-\tau)u^h(\tau)\,d\tau - \\ -(1+\varepsilon)\lambda_h^2\int_0^t K^h_{t^3}(t-\tau)u^h(\tau)\,d\tau + \int_0^t K^h_{t^2}(t-\tau)f^h(\tau)\,d\tau,$$

$$u_{t^2}^h(t) = -\frac{1}{t_0} u_t^h(t) - \int_0^t K_{t^6}^h(t-\tau)\, u^h(\tau)\, d\tau -$$

$$-(1+\varepsilon)\lambda_h^2 \int_0^t K_{t^4}^h(t-\tau)\, u^h(\tau)\, d\tau + \int_0^t K_{t^3}^h(t-\tau) f^h(\tau)\, d\tau\,,$$

$$u_{t^3}^h(t) = -\frac{1}{t_0} u_{t^2}^h(t) - \left[K_{t^6}^h(0) + \frac{1+\varepsilon}{t_0}\lambda_h^2\right] u^h(t) - \int_0^t K_{t^7}^h(t-\tau)\, u^h(\tau)\, d\tau -$$

$$-(1+\varepsilon)\lambda_h^2 \int_0^t K_{t^5}^h(t-\tau)\, u^h(\tau)\, d\tau + \int_0^t K_{t^4}^h(t-\tau) f^h(\tau)\, d\tau\,,$$

$$u_{t^4}^h(t) = -\frac{1}{t_0} u_{t^3}^h(t) - \left[K_{t^6}^h(0) + \frac{1+\varepsilon}{t_0}\lambda_h^2\right] u_t^h(t) - \int_0^t K_{t^8}^h(t-\tau)\, u^h(\tau)\, d\tau -$$

$$-(1+\varepsilon)\lambda_h^2 \int_0^t K_{t^6}^h(t-\tau)\, u^h(\tau)\, d\tau + \frac{1}{t_0} f^h(t) + \int_0^t K_{t^5}^h(t-\tau) f^h(\tau)\, d\tau\,.$$

Hence

$$|u_{t^s}^h(t)| \leq \gamma_s(a,T)\, M_f(a,T)/\lambda_h^{5-s} \qquad (s = 0, 1, 2, 3, 4) \tag{2.16}$$

with

$$\gamma_1(a,T) = \left[\frac{a}{\pi t_0} + \frac{4T\sigma_2^4}{\mu} + (1+\varepsilon)\frac{4T\sigma_2^2}{\mu}\right]\gamma_0(T) + \frac{4(2+T)}{\mu}\,,$$

$$\gamma_2(a,T) = \frac{a}{\pi t_0}\gamma_1(a,T) + \frac{4\sigma_2(2+T)}{\mu} + 4[T\sigma_2^5 + (1+\varepsilon)T\sigma_2^3]\frac{\gamma_0(T)}{\mu}\,,$$

$$\gamma_3(a,T) = \frac{a}{\pi t_0}\gamma_2(a,T) + \frac{4\sigma_2^2(2+T)}{\mu} +$$

$$+4\left[\frac{a\sigma_2^5}{\pi} + \frac{(1+\varepsilon)a\mu}{\pi t_0} + T\sigma_2^6 + (1+\varepsilon)T\sigma_2^4\right]\frac{\gamma_0(T)}{\mu}\,,$$

$$\gamma_4(a,T) = \frac{a}{\pi t_0}\gamma_3(a,T) + \frac{4\sigma_2^3(2+T)}{\mu} + \left[\frac{4\sigma_2^5}{\mu} + \frac{(1+\varepsilon)}{t_0}\right]\frac{a}{\pi}\gamma_1(a,T) +$$

$$+4\left[\frac{a\sigma_2^6}{\pi} + \frac{(1+\varepsilon)a\sigma_2^4}{\pi} + T\sigma_2^7 + (1+\varepsilon)T\sigma_2^5\right]\frac{\gamma_0(T)}{\mu}\,.$$

From (2.15) and (2.16) it follows that:

i) The series on the right hand side of (2.7) converges totally (4) in $\overline{R}_{a,T}$ with any differentiated series up to the order 3.

(4) A series of functions $\sum_{h=1}^{\infty} \varphi_h(x,t)$ is said to be *totally convergent* in a set E of the x, t-plane if $|\varphi_h(x,t)| \leq M_h$ for $(x,t) \in E$ and the numerical series $\sum_{h=1}^{\infty} M_h$ is convergent.

ii) The differentiated series of order 4 converge in the space $L^2(R_{a,T})$.

Hence

2.II. *Under the assumption* $\mathcal{H}_0$) *for* $f(x,t)$, *Problem* $\mathcal{M}_{a,T}$ *has a solution* $u(x,t)$.

From (2.7), (2.14), (2.15), (2.16) we deduce

$$(2.17) \qquad \max_{\overline{R}_{a,T}} \left| \frac{\partial^{p+q} u}{\partial x^p \partial t^q} \right| \leqslant \frac{2}{a}\left(\frac{a}{\pi}\right)^{5-(p+q)} \zeta[5-(p+q)]\,\gamma_q(a,T) M_f(a,T) \qquad \text{for } 0 \leqslant p+q \leqslant 3,$$

$$(2.18) \qquad \int_0^a |u_{x^p t^q}(x,t)|^2 dx \leqslant \frac{2a}{\pi^2}\zeta(2)[\gamma_q(a,T)]^2[M_f(a,T)]^2 \qquad \text{for } p+q=4, \quad 0 \leqslant t \leqslant T,$$

where ζ is *Riemann's zeta function*.

Inequalities (2.17), (2.18) prove the continuous dependence of $u(x,t)$ and of its derivatives on the datum $f(x,t)$.

3. The auxiliary Cauchy Problem $\mathcal{C}_{\alpha,\beta}$

Assume $0 \leqslant \alpha < \beta$. Consider the algebraic equation

$$\lambda^4 - \nu\lambda^2 + t_0 = 0 \qquad (\nu = 1 + t_0 + \varepsilon t^0)$$

which has the following real roots

$$\tilde{\sigma}_1 = \{[\nu - (\nu^2 - 4t_0)^{1/2}]/2\}^{1/2}, \qquad \tilde{\sigma}_2 = \{[\nu + (\nu^2 - 4t_0)^{1/2}]/2\}^{1/2},$$

$$\tilde{\sigma}_3 = -\{[\nu + (\nu^2 - 4t_0)^{1/2}]/2\}^{1/2}, \qquad \tilde{\sigma}_4 = -\{[\nu - (\nu^2 - 4t_0)^{1/2}]/2\}^{1/2}.$$

Let $D_{\alpha,\beta}$ be the isosceles triangle domain defined by the inequalities

$$0 < t < \tilde{\sigma}_1(x-\alpha) \text{ for } \alpha < x \leqslant \frac{\alpha+\beta}{2}, \qquad 0 < t < \tilde{\sigma}_4(x-\beta) \text{ for } \frac{\alpha+\beta}{2} \leqslant x < \beta.$$

We consider the following Cauchy Problem which we denote by $\mathcal{C}_{\alpha,\beta}$.

$\mathcal{C}_{\alpha,\beta}$: *find a function* $v(x,t)$ *belonging to* $C^3(\overline{D}_{\alpha,\beta}) \cap H_4(D_{\alpha,\beta})$ *such that*

$$(3.1) \qquad \Gamma v = f(x,t) \quad \text{in } D_{\alpha,\beta}$$

$$(3.2) \qquad v_{t^k}(x,0) = 0 \quad (k = 0, 1, 2, 3) \quad \alpha \leqslant x \leqslant \beta.$$

We associate to $\mathcal{C}_{\alpha,\beta}$ the following Cauchy Problem $\tilde{\mathcal{C}}_{\alpha,\beta}$ relative to a first order system:

$\widetilde{\mathcal{C}}_{\alpha,\beta}$: *find a vector* $w \equiv (w_1, w_2, w_3, w_4)$ *belonging to* $[C^0(\overline{D}_{\alpha,\beta})]^4 \cap [H_1(D_{\alpha,\beta})]^4$ *such that*

$$\frac{\partial w_1}{\partial x} - \frac{1}{2}\nu\frac{\partial w_3}{\partial x} - \frac{1}{2}\nu\frac{\partial w_2}{\partial t} + t_0\frac{\partial w_4}{\partial t} = (1+\varepsilon)w_2 - w_4 + f(x,t),$$

$$\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial t} = 0, \qquad \frac{\partial w_3}{\partial x} - \frac{\partial w_2}{\partial t} = 0, \qquad \frac{\partial w_4}{\partial x} - \frac{\partial w_3}{\partial t} = 0,$$

$$w_h(x, 0) = 0 \quad (h = 1, 2, 3, 4), \qquad \alpha \leqslant x \leqslant \beta .$$

It is evident that if v is a solution of $\mathcal{C}_{\alpha,\beta}$, and letting

$$w_1 = \frac{\partial^3 v}{\partial x^3}, \qquad w_2 = \frac{\partial^3 v}{\partial x^2\,\partial t}, \qquad w_3 = \frac{\partial^3 v}{\partial x\,\partial t^2}, \qquad w_4 = \frac{\partial^3 v}{\partial t^3},$$

we see that the vector $w \equiv (w_1, w_2, w_3, w_4)$ is a solution of $\widetilde{\mathcal{C}}_{\alpha,\beta}$.

On the other hand, if $w \equiv (w_1, w_2, w_3, w_4)$ is a solution of $\widetilde{\mathcal{C}}_{\alpha,\beta}$, the function

$$v(x,t) = -\frac{1}{3!}\sum_{h=0}^{3}\binom{3}{h}\int_{([\alpha+\beta]/2,\,0)}^{(x,t)} w_{h+1}(\xi,\tau)\,d[(x-\xi)^{3-h}(t-\tau)^h] \tag{3.3}$$

is a solution of $\mathcal{C}_{\alpha,\beta}$. Integration is taken over any smooth path joining $([\alpha+\beta]/2, 0)$ with (x, t) and contained in $\overline{D}_{\alpha,\beta}$.

For solving Problem $\widetilde{\mathcal{C}}_{\alpha,\beta}$ we shall use the classical method of characteristics (see [7, pp. 464-471] and [8]), by adapting this method to our particular problem and introducing some simplifications.

Let us consider the 4×4 matrices

$$A = ((a_{ik})) = \begin{pmatrix} 1 & 0 & -(1/2)\nu & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B = ((b_{ik})) = \begin{pmatrix} 0 & -(1/2)\nu & 0 & t_0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$C = ((c_{ik})) = \begin{pmatrix} 0 & 1+\varepsilon & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let F be the vector $(f, 0, 0, 0)$. We can rewrite the Problem $\widetilde{\mathcal{C}}_{\alpha,\beta}$ in the vector form

$$A\frac{\partial w}{\partial x} + B\frac{\partial w}{\partial t} = Cw + F \quad \text{in } D_{\alpha,\beta}, \tag{3.4}$$

$$w(x, 0) = 0 \quad \alpha \leqslant x \leqslant \beta . \tag{3.5}$$

Since $\det B = t_0 \neq 0$, we can consider the inverse matrix B^{-1} of B and deduce from (3.4) that

$$\frac{\partial w}{\partial t} + B^{-1}A\frac{\partial w}{\partial x} = B^{-1}Cw + B^{-1}F. \tag{3.6}$$

We have

$$\det(B - \lambda A) = \lambda^4 - \nu\lambda^2 + t_0 .$$

Hence the roots of the determinant equation (I = identity matrix) $\det(B^{-1}A - \varrho I) = \det B^{-1}\det(A - \varrho B) = 0$ are: $\varrho_h = \tilde{\sigma}_h^{-1}$ $(h = 1, 2, 3, 4)$.

Let $\xi^h \equiv (\xi_1^h, \xi_2^h, \xi_3^h, \xi_4^h)$ $(h = 1, 2, 3, 4)$ be four unit vectors such that

$$B^{-1}A\xi^h = \varrho_h \xi^h .$$

Set

$$B^{-1}A = ((q_{ik})),$$

$$\Omega = \begin{pmatrix} \xi_1^1 & \xi_1^2 & \xi_1^3 & \xi_1^4 \\ \xi_2^1 & \xi_2^2 & \xi_2^3 & \xi_2^4 \\ \xi_3^1 & \xi_3^2 & \xi_3^3 & \xi_3^4 \\ \xi_4^1 & \xi_4^2 & \xi_4^3 & \xi_4^4 \end{pmatrix}.$$

Since the four unit-vectors ξ^h are linearly independent, we have $|\det \Omega| \neq 0$. Let Ω^{-1} be the inverse matrix of Ω. Set $\Omega^{-1} = ((\tau_{ik}))$. We have

$$\sum_{k=1}^{4} \tau_{ik} \sum_{h=1}^{4} q_{kh}\xi_h^j = \sum_{k=1}^{4} \tau_{ik}\varrho_j \xi_k^j = \varrho_j \delta_{ij} ,$$

where $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$.

Using matrix notation

$$\Omega^{-1}B^{-1}A\Omega = \Delta ,$$

where Δ is the diagonal matrix

$$\begin{pmatrix} \varrho_1 & 0 & 0 & 0 \\ 0 & \varrho_2 & 0 & 0 \\ 0 & 0 & \varrho_3 & 0 \\ 0 & 0 & 0 & \varrho_4 \end{pmatrix}.$$

Set $z = \Omega^{-1}w$, *i.e.* $w = \Omega z$. From (3.6) we get

$$\frac{\partial z}{\partial t} + \Delta\frac{\partial z}{\partial x} = \Omega^{-1}B^{-1}C\Omega z + \Omega^{-1}B^{-1}F$$

and in scalar form

$$\frac{\partial z_i}{\partial t} + \varrho_i\frac{\partial z_i}{\partial x} = \sum_{k=1}^{4} h_{ik}z_k + g_i \qquad (i = 1, 2, 3, 4), \tag{3.7}$$

where

$$((h_{ik})) = \Omega^{-1}B^{-1}C\Omega , \qquad g = \Omega^{-1}B^{-1}F .$$

Since the function $z_i(x, t)$ belongs to $H_1(D_{\alpha,\beta})$, for almost every $x' \in (\alpha, \beta)$ the function $z_i(\varrho_i t + x', t)$ is an absolutely continuous function of t on the segment of the straight line $x = \varrho_i t + x'$ which is contained in $D_{\alpha,\beta}$. Let us consider the segment of the straight line $t = \tau$ contained in $\overline{D}_{\alpha,\beta}[0 \leq \tau < 2^{-1}\tilde{\sigma}_1(\beta - \alpha)]$. For almost every point (ξ, τ) of this segment we have from (3.7):

$$\frac{d}{dt} z_i[\varrho_i(t-\tau)+\xi, t] = \sum_{k=1}^{4} h_{ik} z_k[\varrho_i(t-\tau)+\xi, t] + g_i[\varrho_i(t-\tau)+\xi, t].$$

By integrating with respect to t from 0 to τ, since $z_i[-\varrho_i\tau+\xi, 0] = 0$, we get

$$z_i(\xi, \tau) = \sum_{k=1}^{4} h_{ik} \int_0^\tau z_k[\varrho_i(t-\tau)+\xi, t]\,dt + \int_0^\tau g_i[\varrho_i(t-\tau)+\xi, t]\,dt. \tag{3.8}$$

Because of continuity of the $z_i(\xi, \tau)$ in $\overline{D}_{\alpha,\beta}$, eq. (3.8) holds in every point (ξ, τ) of $\overline{D}_{\alpha,\beta}$.

Let us now consider the mapping

$$Z_i(\xi, \tau) = \sum_{k=1}^{4} h_{ik} \int_0^\tau z_k[\varrho_i(t-\tau)+\xi, t]\,dt + \int_0^\tau g_i[\varrho_i(t-\tau)+\xi, t]\,dt \qquad (i = 1, 2, 3, 4)$$

which we briefly denote by

$$Z = Lz + G, \tag{3.9}$$

where

$$G(\xi, \tau) \equiv \Bigg(\int_0^\tau g_1[\varrho_1(t-\tau)+\xi, t]\,dt, \int_0^\tau g_2[\varrho_2(t-\tau)+\xi, t]\,dt, \int_0^\tau g_3[\varrho_3(t-\tau)+\xi, t]\,dt, \int_0^\tau g_4[\varrho_4(t-\tau)+\xi, t]\,dt \Bigg). \tag{3.10}$$

(3.9) is a mapping from $[C^0(\overline{D}_{\alpha,\beta})]^4$ into itself.

Let us introduce in $[C^0(\overline{D}_{\alpha,\beta})]^4$ the norm

$$\|z\| = \left\{ \sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} [e^{-p\tau} |z_i(\xi, \tau)|^2] \right\}^{1/2}, \tag{3.11}$$

where p is any fixed positive number. Since

$$\exp\left(-p\frac{\tilde{\sigma}_1(\beta-\alpha)}{4}\right)\left[\sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} |z_i(\xi, \tau)|^2\right]^{1/2} \leq \|z\| \leq \left[\sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} |z_i(\xi, \tau)|^2\right]^{1/2},$$

this norm is equivalent to the usual C^0-norm in $[C^0(\overline{D}_{\alpha,\beta})]^4$.

Set

$$H = \left[\sum_{i,k}^{1,4} h_{ik}^2\right]^{1/2}, \qquad Z = Lz.$$

We have

$$\sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} [e^{-p\tau} |Z_i(\xi,\tau)|^2] =$$

$$= \sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} \left| \sum_{k=1}^{4} h_{ik} \int_0^{\tau} e^{-pt/2} z_k [\varrho_i(t-\tau)+\xi, t] e^{p(t-\tau)/2} dt \right|^2 \leqslant$$

$$\leqslant \sum_{i=1}^{4} \sum_{k=1}^{4} h_{ik}^2 \max_{\overline{D}_{\alpha,\beta}} \sum_{j=1}^{4} \left| \int_0^{\tau} e^{p(t-\tau)/2} z_j [\varrho_i(t-\tau)+\xi, t] e^{-pt/2} dt \right|^2 \leqslant$$

$$\leqslant \sum_{i,k}^{1,4} h_{ik}^2 \max_{\overline{D}_{\alpha,\beta}} \sum_{j=1}^{4} \int_0^{\tau} e^{p(t-\tau)} dt \int_0^{\tau} e^{-pt} |z_j[\varrho_i(t-\tau)+\xi, t]|^2 dt \leqslant H^2 \frac{1}{p} \tilde{\sigma}_1 \frac{\beta-\alpha}{2} \|z\|^2 .$$

Hence

$$\|Z\|^2 \leqslant H^2 \frac{\tilde{\sigma}_1(\beta-\alpha)}{2p} \|z\|^2 .$$

If we assume

$$p > H^2 \tilde{\sigma}_1 (\beta-\alpha)/2 \tag{3.12}$$

we see that the mapping (3.9) is a contraction of the space $[C^0(\overline{D}_{\alpha,\beta})]^4$, endowed with the norm (3.11), into itself. Hence there exists one and only one solution of the integral system (3.8) in the space $[C^0(\overline{D}_{\alpha,\beta})]^4$.

Set $z_n = \sum_{k=0}^{n} L^k G$, we have

$$z_n = L z_{n-1} + G . \tag{3.13}$$

On the other hand the series $\sum_{k=0}^{\infty} L^k G$ is totally convergent in $[C^0(\overline{D}_{\alpha,\beta})]^4$ since

$$\|L^k G\| \leqslant \delta^k \|G\| , \qquad \delta = H[\tilde{\sigma}_1(\beta-\alpha)/2p]^{1/2} . \tag{3.14}$$

From (3.13), for $n \to \infty$ we get $z = Lz + G$, *i.e.*

$$z = \sum_{k=0}^{\infty} L^k G$$

is the solution of (3.8) and, moreover, from (3.14) we have

$$\|z\| \leqslant [1/(1-\delta)] \|G\| .$$

If we denote by $((\beta_{ik}))$ the matrix B^{-1}, we have

$$\begin{cases} G_i(\xi,\tau) = \sum_{k,j}^{1,4} \int_0^{\tau} \tau_{ik} \beta_{kj} F_j [\varrho_i(t-\tau)+\xi, t] dt , \\ \dfrac{\partial G_i(\xi,\tau)}{\partial \xi} = \sum_{k,j}^{1,4} \int_0^{\tau} \tau_{ik} \beta_{kj} F_{jx} [\varrho_i(t-\tau)+\xi, t] dt . \end{cases} \tag{3.15}$$

The function $\partial G_i(\xi, \tau)/\partial\xi$ is continuous in $\overline{D}_{\alpha,\beta}$ because of hypothesis $\mathcal{H}_0$). We have

$$\frac{\partial}{\partial\xi} LG = L\frac{\partial G}{\partial\xi}$$

and in consequence

$$\frac{\partial}{\partial\xi} L^k G = L^k \frac{\partial G}{\partial\xi} .$$

Since the series $\sum_{k=0}^{\infty} L^k(\partial G/\partial\xi)$ is totally convergent in $\overline{D}_{\alpha,\beta}$ we deduce that $z(\xi, \tau)$ is continuously differentiable with respect to ξ in $\overline{D}_{\alpha,\beta}$. If we consider the straight line $\xi = \varrho_i(\lambda - \tilde{\tau}) + \tilde{\xi}$, $\tau = \lambda$ and assume that it crosses $D_{\alpha,\beta}$ we have from (3.8)

$$z_i[\varrho_i(\lambda - \tilde{\tau}) + \tilde{\xi}, \lambda] = \sum_{k=1}^{4} h_{ik} \int_0^\lambda z_k[\varrho_i(t - \tilde{\tau}) + \tilde{\xi}, t]\,dt + \int_0^\lambda g_i[\varrho_i(t - \tilde{\tau}) + \tilde{\xi}, t]\,dt$$

which implies

$$\frac{d}{d\lambda} z_i[\varrho_i(\lambda - \tilde{\tau}) + \tilde{\xi}, \lambda] = \sum_{k=1}^{4} h_{ik} z_k[\varrho_i(\lambda - \tilde{\tau}) + \tilde{\xi}, \lambda] + g_i[\varrho_i(\lambda - \tilde{\tau}) + \tilde{\xi}, \lambda]. \tag{3.16}$$

If l_i is the direction determined by the unit vector $(\varrho_i(1 + \varrho_i^2)^{-1/2}, (1 + \varrho_i^2)^{-1/2})$, equation (3.16) implies that in every point (ξ, τ) of $D_{\alpha,\beta}$ we have

$$\frac{\partial}{\partial l_i} z_i(\xi, \tau) = (1/(1 + \varrho_i^2)^{1/2})\left[\sum_{k=1}^{n} h_{ik} z_k(\xi, \tau) + g_i(\xi, \tau)\right].$$

Hence $z(\xi, \tau)$ is continuously differentiable with respect to both variables ξ and τ in $\overline{D}_{\alpha,\beta}$.

Since

$$\frac{\partial}{\partial l_i} z_i(\xi, \tau) = (\varrho_i/(1 + \varrho_i^2)^{1/2})\frac{\partial}{\partial\xi} z_i(\xi, \tau) + (1/(1 + \varrho_i^2)^{1/2})\frac{\partial}{\partial\tau} z_i(\xi, \tau)$$

we deduce that $z(x, t) \equiv (z_1(x, t), z_2(x, t), z_3(x, t), z_4(x, t))$ is a solution of system (3.6).

The following theorem has been proved.

3.I. *Problem* $\tilde{\mathcal{C}}_{\alpha,\beta}$ *has one and only one solution which at any point* (ξ, τ) *of* $\overline{D}_{\alpha,\beta}$ *is given by the series development*

$$w(\xi, \tau) = \sum_{k=0}^{\infty} \Omega L^k G,$$

where G *is the* 4*-vector whose components are given by* (3.10). *The vector* $w(\xi, \tau)$ *belongs to the space* $[C^0(\overline{D}_{\alpha,\beta})]^4$.

We have, keeping in mind that Ω is a matrix with unit vectors as columns,

$$\max_{\overline{D}_{\alpha,\beta}} \sum_{i=1}^{4} |w_i(\xi,\tau)| \leqslant \sum_{k=0}^{\infty} \max_{\overline{D}_{\alpha,\beta}} |L^k G| \leqslant \sum_{k=0}^{\infty} \left[\sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} |(L^k)_i G| \right] \leqslant$$

$$\leqslant \exp\left[p \frac{\widetilde{\sigma}_1(\beta-\alpha)}{4}\right] \sum_{k=0}^{\infty} \sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} [e^{-p\tau} |(L^k)_i G|^2]^{1/2} \leqslant$$

$$\leqslant \exp\left[p \frac{\widetilde{\sigma}_1(\beta-\alpha)}{4}\right] \sum_{k=0}^{\infty} \|L^k G\| \leqslant \exp\left[p \frac{\widetilde{\sigma}_1(\beta-\alpha)}{4}\right] \frac{1}{1-\delta} \|G\| =$$

$$= \frac{\sqrt{2p}}{\sqrt{2p} - H\sqrt{\widetilde{\sigma}_1(\beta-\alpha)}} \exp\left[p \frac{\widetilde{\sigma}_1(\beta-\alpha)}{4}\right] \|G\|$$

for any p satisfying (3.12).

On the other hand we have

$$\|G\|^2 = \sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} \left[e^{-p\tau} \left| \int_0^\tau g_i[\varrho_i(t-\tau)+\xi, t]\, dt \right|^2 \right] \leqslant$$

$$\leqslant \sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} \left| \int_0^\tau g_i[\varrho_i(t-\tau)+\xi, t]\, dt \right|^2 \leqslant$$

$$\leqslant \frac{\widetilde{\sigma}_1^2(\beta-\alpha)^2}{4} \sum_{i=1}^{4} \max_{\overline{D}_{\alpha,\beta}} |g_i(\xi,\tau)|^2 \leqslant C_1(\Omega, t^0, t_0, \varepsilon)\, \widetilde{\sigma}_1^2(\beta-\alpha)^2 \max_{\overline{D}_{\alpha,\beta}} |g(\xi,\tau)|^2 =$$

$$= C_1(\Omega, t^0, t_0, \varepsilon)\, \widetilde{\sigma}_1^2(\beta-\alpha)^2 \max_{\overline{D}_{\alpha,\beta}} |B^{-1}F|^2 \leqslant C_2(\Omega, t^0, t_0, \varepsilon)[\widetilde{\sigma}_1(\beta-\alpha) \max_{\overline{D}_{\alpha,\beta}} |f|]^2$$

where C_1 and C_2 are some constants.

Setting

$$\gamma(\alpha,\beta,p) = C_2[\widetilde{\sigma}_1(\beta-\alpha)(2p)^{1/2}]/[(2p)^{1/2} - H(\widetilde{\sigma}_1(\beta-\alpha))^{1/2}] \exp[p\widetilde{\sigma}_1(\beta-\alpha)/4]$$

we get

$$\max_{\overline{D}_{\alpha,\beta}} |w(\xi,\tau)| \leqslant \gamma(\alpha,\beta,p) \max_{\overline{D}_{\alpha,\beta}} |f(\xi,\tau)| . \tag{3.17}$$

By the same procedure

$$\max_{\overline{D}_{\alpha,\beta}} |w_\xi(\xi,\tau)| \leqslant \gamma(\alpha,\beta,p) \max_{\overline{D}_{\alpha,\beta}} |f_\xi(\xi,\tau)| . \tag{3.18}$$

From (3.6), (3.17), (3.18) we get

$$\max_{\overline{D}_{\alpha,\beta}} |w_\tau(\xi,\tau)| \leqslant [\gamma(\alpha,\beta,p)|B^{-1}C| + |B^{-1}|] \max_{\overline{D}_{\alpha,\beta}} |f(\xi,\tau)| + \\ + \gamma(\alpha,\beta,p)|B^{-1}A| \max_{\overline{D}_{\alpha,\beta}} |f_\xi(\xi,\tau)| \quad (3.19)$$

[5].

Inequalities (3.17), (3.18), (3.19) prove the continuous dependence of w and its derivatives on the datum of the Problem $\widetilde{\mathcal{C}}_{\alpha,\beta}$.

Let us consider a domain D of the x, t-plane which is *star-shaped* with respect to point (x_0, t_0) of D. This means that for each $(x,t) \in D$ all points of the segment: $\xi = x_0 + \lambda(x - x_0)$, $\tau = t_0 + \lambda(t - t_0)$, $0 \leqslant \lambda \leqslant 1$ belong to D. Suppose that D is bounded and let A and B be two positive numbers such that for each $(x,t) \in D$: $|x - x_0| \leqslant A$, $|t - t_0| \leqslant B$.

Let $\psi(x,t)$ be a function belonging to $C^n(\overline{D})$ and let Ψ_n be a positive constant such that

$$\max_{\overline{D}} \left| \frac{\partial^n \psi}{\partial x^{n-k} \partial t^k} \right| \leqslant \Psi_n \qquad (k = 0, 1, \ldots, n).$$

Suppose that ψ and any of its derivatives of order $\leqslant n - 1$ vanish in (x_0, t_0). We have the following lemma

3.II. *Under the above hypotheses on D and on $\psi(x,t)$, the following estimate holds*

$$\max_{\overline{D}} |\psi(x,t)| \leqslant \frac{2}{n!} (A + B)^n \Psi_n . \tag{3.20}$$

Set $\psi_h(x,t) = \psi_{x^{n-h}t^h}(x,t)$ $(h = 0, 1, \ldots, n)$. From a known theorem of elementary analysis we have

$$\psi(x,t) = \frac{-1}{n!} \sum_{h=0}^{n} \binom{n}{h} \int_{(x_0,t_0)}^{(x,t)} \psi_h(\xi,\tau)\, d[(x-\xi)^{n-h}(t-\tau)^h],$$

the integration being taken along any smooth path joining (x_0, t_0) with (x,t) and contained in D.

Let us assume for the integration path the segment with ends (x_0, t_0) and (x,t). Setting

$$\widetilde{\psi}_h(\lambda) = \psi_h[x_0 + \lambda(x - x_0), t_0 + \lambda(t - t_0)] \qquad (h = 0, 1, \ldots, n),$$

[5] If $\Lambda \equiv (\!(\lambda_{h,k})\!)$ is an $m \times m$ real matrix, we assume as usual

$$|\Lambda| = \max_x \left[\sum_{h=1}^{m} \left(\sum_{k=1}^{m} \lambda_{hk} x_k \right)^2 \right]^{1/2} \qquad \text{for } \sum_{k=1}^{m} x_k^2 = 1 .$$

It is evident that

$$|\Lambda| \leqslant \left(\sum_{h,k}^{1,m} \lambda_{hk}^2 \right)^{1/2}.$$

we have after elementary computation that

$$\psi(x,t) = \frac{-1}{(n-1)!} \sum_{h=0}^{n} \binom{n}{h} (x-x_0)^{n-h}(t-t_0)^h \int_0^1 \tilde{\psi}_h (1-\lambda)^{n-1} d\lambda .$$

From this the estimate (3.20) follows.

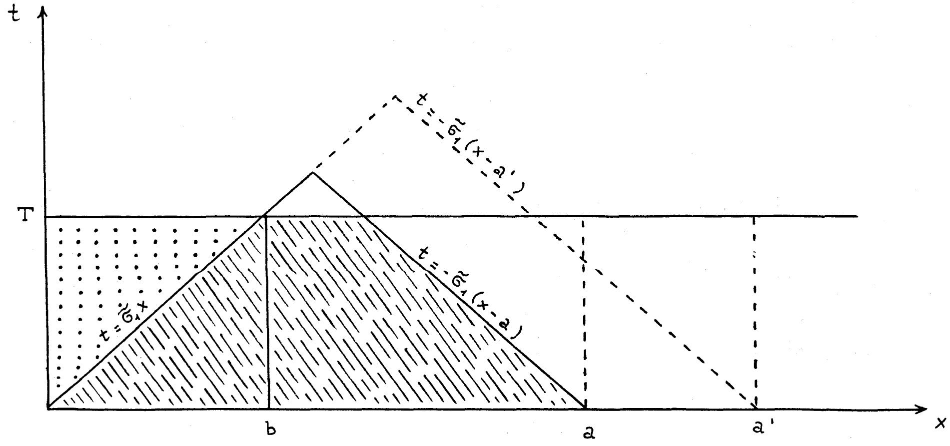
The triangle domain $D_{\alpha,\beta}$ and either the solution $v(x,t)$ of Problem $\mathcal{C}_{\alpha,\beta}$ or any of its partial derivatives of order $\leqslant 2$ satisfy the hypotheses of Lemma 3.II by assuming, for instance, like (x_0, t_0) the point $((\alpha+\beta)/2, 0)$. Hence Lemma 3.II holds assuming $n=3$, $n=2$, $n=1$, $A = 2^{-1}(\beta-\alpha)$, $B = 2^{-1}\tilde{\sigma}_1(\beta-\alpha)$. Ψ_n is given by the right hand side of (3.17). From this we deduce the continuous dependence of $v(x,t)$ and of its first and second partial derivatives on the datum $f(x,t)$ of Problem $\mathcal{C}_{\alpha,\beta}$.

It must be observed that for developing the theory of Problems $\mathcal{C}_{\alpha,\beta}$ and $\mathcal{C}_{\alpha,\beta}$ the entire hypothesis $\mathcal{H}_0$) has not been used, but only the continuity of $f(x,t)$ and $f_x(x,t)$ in $\overline{D}_{\alpha,\beta}$.

4. Transplant operation. Existence and uniqueness theorems for Problem (1.6)-(1.8)

We are now going to perform a kind of surgical operation: *to transplant the solution of Problem $\mathcal{C}_{\alpha,\beta}$ into the solution of Problem $\mathcal{M}_{a,T}$*. This operation will permit us to obtain existence and uniqueness of solution of Problem (1.6)-(1.8). Moreover from this method we shall derive computational procedures for the solution.

Let us assume arbitrarily $a > 0$, $0 < T \leqslant 2^{-1}\tilde{\sigma}_1 a$ and consider the solution $u(x,t)$ of Problem $\mathcal{M}_{a,T}$. Let us consider the solution $v(x,t)$ of Problem $\mathcal{C}_{0,a}$ in the triangle $\overline{D}_{0,a}$. Because of the uniqueness theorem for Problem $\mathcal{C}_{\alpha,\beta}$ the function $u(x,t)$ must coincide with the function $v(x,t)$ in the closure $\overline{S}^0_{a,T}$ of the trapezium domain $S^0_{a,T}$ defined by the inequalities: $0 < x < a$, $0 < t < \min[\tilde{\sigma}_1 x, T, \tilde{\sigma}_1(a-x)]$ (see shaded area in the picture).



If we assume any $a' > a$ and consider the function $v'(x, t)$ solution of $\mathcal{C}_{0,a'}$ in $\overline{D}_{0,a'}$, since $v(x, t) \equiv v'(x, t)$ in $\overline{D}_{0,a}$, we see that *function $u(x, t)$ is independent of a* in the domain $S^0_{a,T}$. On the other hand, assuming $b = \tilde{\sigma}_1^{-1} T$, the function u is a solution in $R_{b,T}$ of BVP

$$\Gamma u = f(x, t) \quad \text{in } R_{b,T}, \tag{4.1}$$

$$u(0, t) = u_{x^2}(0, t) = 0, \tag{4.2}$$

$$u(b, t) = v(b, t), \qquad u_{x^2}(b, t) = v_{x^2}(b, t), \tag{4.3}$$

and belongs to $C^3(\overline{R}_{b,T}) \cap H_4(R_{b,T})$. Because of Theorem 2.I, Problem (4.1)-(4.3) has at most one solution u in the function class $C^3(\overline{R}_{b,T}) \cap H_4(R_{b,T})$, hence *$u(x, t)$ does not depend on a in the domain $R_{b,T}$*. In conclusion $u(x, t)$ does not depend on a in the trapezium domain $S_{a,T}$

$$0 < x < a, \qquad 0 < t < \min[T, \tilde{\sigma}_1(a - x)] \tag{4.4}$$

(dotted + shaded area in the picture).

If we assume $T' > T$ and consider the solution $u'(x, t)$ of Problem $\mathcal{M}_{a,T'}$, we have, for Theorem 2.I, $u'(x, t) \equiv u(x, t)$ in $\mathcal{M}_{a,T}$. Because of the arbitrariness of a and T we have proved the following theorem:

4.I. *There exists a solution $\varphi(x, t)$ of Problem* (1.6)-(1.8) *which belongs to the function class $C^3(\overline{Q}) \cap H_4(E)$ for any bounded domain $E \subset Q$. This solution is unique. The function $\varphi(x, t)$ coincides in any trapezium domain $S_{a,T}$ $(a > 0,\ 0 < T < 2^{-1}\tilde{\sigma}_1 a)$, defined by inequalities* (4.4) *with the solution of Problem $\mathcal{M}_{a,T}$ and in any triangle domain $D_{\alpha,\beta}$ $(0 \leqslant \alpha < \beta)$ with the solution of Problem $\mathcal{C}_{\alpha,\beta}$.*

It must be remarked that $\varphi(x, t)$ belongs to the class $C^4(\overline{A})$, where A is the angle: $x > 0,\ 0 < t < \tilde{\sigma}_1 x$. However $\varphi(x, t)$ does not belong, in general, to the class $C^4(\overline{Q})$ since this would imply $f(0, 0) = 0$, a hypothesis which was not assumed by us.

5. Computation procedures

Since the solution $\varphi(x, t)$ of Problem (1.6)-(1.8) is defined in the unbounded domain Q, it is reasonable to set the problem of the actual numerical computation of $\varphi(x, t)$ and of its derivatives in the following way: *given any bounded domain E contained in Q, to provide a numerical procedure for computing φ and its derivatives in the domain E.*

For *numerical procedure for computing a function* we mean a method which is able to construct a sequence which in a prescribed norm converges to the function and such that it is possible to give an explicit upper bound to the approximation error, estimating possibly its order of magnitude in terms of the order of the approximation. We will reach this goal if, on the basis of Theorem 4.I, we consider both the domains $S_{a,T}$ $(a > 0,\ 0 < T < 2^{-1}\tilde{\sigma}_1 a)$ and $D_{\alpha,\beta}$ $(0 \leqslant \alpha < \beta)$.

In $S_{a,T}$ we have $\varphi(x, t) \equiv u(x, t)$ where $u(x, t)$ is the solution of Problem $\mathcal{M}_{a,T}$. Let us assume for the approximation of order $n (n = 1, 2, \dots)$ of $\varphi(x, t)$ in $S_{a,T}$ the function

$$\varphi_n(x,t) = \frac{2}{a} \sum_{h=1}^{n} u_h(t) \sin \lambda_h t \qquad (\lambda_h = h\pi/a).$$

Proceeding as in the proofs of (2.17) and (2.18), we have for $0 \leqslant p + q \leqslant 3$

$$(5.1) \quad \max_{\bar{S}_{a,T}} \left| \frac{\partial^{p+q}}{\partial x^p \partial t^q} [\varphi(x,t) - \varphi_n(x,t)] \right| \leqslant$$

$$\leqslant \frac{2}{a} \left(\frac{a}{\pi} \right)^{5-(p+q)} \gamma_q(a,T) M_f(a,T) \sum_{h=n+1}^{\infty} \frac{1}{h^{5-(p+q)}}$$

and for $p + q = 4$

$$(5.2) \quad \left(\iint_{S_{a,T}} \left| \frac{\partial^{p+q}}{\partial x^p \partial t^q} [\varphi(x,t) - \varphi_n(x,t)] \right|^2 dx\, dt \right)^{1/2} \leqslant$$

$$\leqslant \frac{\sqrt{2a}}{\pi} \gamma_q(a,T) M_f(a,T) \left(\sum_{h=n+1}^{\infty} \frac{1}{h^2} \right)^{1/2}.$$

For $s > 1$ we have

$$\sum_{h=n+1}^{\infty} \frac{1}{h^s} \leqslant \int_{n+1}^{+\infty} \frac{dx}{(x-1)^s} = \frac{1}{s-1} \frac{1}{n^{s-1}}.$$

This permits us to make explicit estimates (5.1), (5.2) of the approximation error and to prove that, if D^s is any partial derivative of order s $(0 \leqslant s \leqslant 4)$ we have

$$\max_{\bar{S}_{a,T}} |D^s \varphi - D^s \varphi_n| = O\left(\frac{1}{n^{4-s}} \right),$$

$$\left(\iint_{S_{a,T}} |D^4 \varphi - D^4 \varphi_n|^2 dx\, dt \right)^{1/2} = O\left(\frac{1}{n^{1/2}} \right).$$

We know that in $D_{\alpha,\beta}$ $(0 \leqslant \alpha < \beta)$, $\varphi(x,t)$ coincides with $v(x,t)$, solution of $\mathcal{C}_{\alpha,\beta}$.

We assume the approximation of order n for the solution $w(\xi, \tau)$ of Problem $\widetilde{\mathcal{C}}_{\alpha,\beta}$, in $D_{\alpha,\beta}$ to be the function

$$w^n(\xi,\tau) = \sum_{k=0}^{n} \Omega L^k G.$$

By using the same approach employed for deriving (3.17), we see that

$$\max_{\bar{D}_{\alpha,\beta}} |w(\xi,\tau) - w^n(\xi,\tau)| \leqslant C_2 \exp\left[p \frac{\widetilde{\sigma}_1(\beta-\alpha)}{4} \right] \widetilde{\sigma}_1(\beta-\alpha) \frac{\delta^{n+1}}{1-\delta}$$

where

$$\delta = H[\widetilde{\sigma}_1(\beta-\alpha)/2p]^{1/2}, \qquad p > H^2 \widetilde{\sigma}_1(\beta-\alpha)/2.$$

Approximating $w_\xi(\xi,\tau)$ by $\sum_{k=0}^{n} \Omega L^k G_\xi$ and $w_\tau(\xi,\tau)$ through (3.6), we can explicitly estimate the approximation error by using the same procedure which led to (3.18) and (3.19). Approximation of the solution $v(x, t)$ of Problem $\mathcal{C}_{\alpha,\beta}$ and of its first and second derivatives are obtained by using Lemma 3.II. It must be remarked that all of the considered approximations have an explicit error estimate which is of the order of magnitude $O(\delta^{n+1})$.

6. Asymptotics for $t \to +\infty$

For investigating the behaviour of $\varphi(x, t)$ and its derivatives when $t \to +\infty$ it is necessary to state a Tauber-type theorem concerning Laplace transform.

6.I. *Let* $\widehat{\Phi}(z)$ $(z = \xi + i\eta)$ *be a function, which satisfies the following hypotheses:*

i) $\widehat{\Phi}$ *is holomorphic in the strip* $S(0, a)$: $0 < \xi < a$ *and is continuous in* $\bar{S}(0, a) - \{0\}$.

ii) A real constant μ *exists such that*

$$\lim_{z \to 0} z^{\mu}\, \widehat{\Phi}(z) = 0 .$$

iii) For each $\alpha > 0$ *the generalized integrals*

$$\int_{\alpha}^{+\infty} e^{it\eta}\, \widehat{\Phi}(i\eta)\, d\eta = \lim_{\beta \to +\infty} \int_{\alpha}^{\beta} e^{it\eta}\, \widehat{\Phi}(i\eta)\, d\eta \,, \tag{6.1}$$

$$\int_{-\infty}^{-\alpha} e^{it\eta}\, \widehat{\Phi}(i\eta)\, d\eta = \lim_{\beta \to +\infty} \int_{-\beta}^{-\alpha} e^{it\eta}\, \widehat{\Phi}(i\eta)\, d\eta \,, \tag{6.2}$$

exists for $t \geq T$ *(for some* $T \geq 0$*) and limits* (6.1), (6.2) *are uniform for* $t \geq T$.

iiii) For each $\xi \in [0, a]$

$$\lim_{|\eta| \to +\infty} \widehat{\Phi}(\xi + i\eta) = 0$$

uniformly for $\xi \in [0, a]$.

Under these hypotheses the generalized integral

$$\Phi(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{t(a + i\eta)}\, \widehat{\Phi}(a + i\eta)\, d\eta$$

exists for $t \geq T$ *and one has*

$$\lim_{t \to +\infty} \frac{\Phi(t)}{t^{\mu - 1}} = 0 \qquad \text{for } \mu > 1 \,, \tag{6.3}$$

$$\lim_{t \to +\infty} \frac{\Phi(t)}{\log t} = 0 \qquad \text{for } \mu = 1 \,, \tag{6.4}$$

$$\lim_{t \to +\infty} \Phi(t) = 0 \qquad \text{for } \mu < 1 \,. \tag{6.5}$$

For arbitrarily given $\varepsilon > 0$, R_ε exists, $0 < R_\varepsilon < 1$, such that for $z = \varrho e^{i\theta}$, $0 < \varrho < R_\varepsilon$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, we have

$$|z^\mu \widehat{\Phi}(z)| < \varepsilon .$$

Let us assume t such that $t > R_\varepsilon^{-1}$ and denote by Γ_t the piece-wise smooth curve formed by the half straight-line: $\Gamma_t^- : \xi = 0$, $-\infty < \eta \leqslant -t^{-1}$, by the half circle Γ_t^0: $\varrho = |z| = t^{-1}$, $-\pi/2 \leqslant \theta \leqslant \pi/2$ and by the half straight-line $\Gamma_t^+ : \xi = 0$, $t^{-1} \leqslant \eta < +\infty$.

We assume as positive direction on Γ_t the one from $-\infty$ to $+\infty$. We have

$$(6.6) \qquad \left| \int_{+\Gamma_t^0} e^{tz} \widehat{\Phi}(z)\, dz \right| = \left| \int_{-\pi/2}^{\pi/2} e^{\cos\theta + i\sin\theta} \widehat{\Phi}(t^{-1}e^{i\theta})\, it^{-1}e^{i\theta} d\theta \right| \leqslant \varepsilon\pi e t^{\mu-1} ,$$

and for $t \geqslant T$

$$\left| \int_{+\Gamma_t^-} e^{tz} \widehat{\Phi}(z)\, dz \right| \leqslant \left| \int_{-\infty}^{-R_\varepsilon} e^{it\eta} \widehat{\Phi}(i\eta)\, d\eta \right| + \left| \int_{-R_\varepsilon}^{-t^{-1}} e^{it\eta} \widehat{\Phi}(i\eta)\, d\eta \right| .$$

By the Riemann-Lebesgue Lemma for the Fourier integral (see [7, p. 171]) T_ε exists $[T_\varepsilon > \max(T, R_\varepsilon^{-1})]$ such that for $t \geqslant T_\varepsilon$

$$\left| \int_{-\infty}^{-R_\varepsilon} e^{it\eta} \widehat{\Phi}(i\eta)\, d\eta \right| < \varepsilon .$$

Moreover

$$\left| \int_{-R_\varepsilon}^{-t^{-1}} e^{it\eta} \widehat{\Phi}(i\eta)\, d\eta \right| \leqslant \varepsilon \int_{-R_\varepsilon}^{-t^{-1}} \frac{d\eta}{|\eta|^\mu} \begin{cases} < \varepsilon t^{\mu-1}/(\mu-1), & \mu > 1, \\ < \varepsilon \log t, & \mu = 1, \\ < \varepsilon/(1-\mu), & \mu < 1. \end{cases}$$

Hence, for $t \geqslant T_\varepsilon$

$$(6.7) \qquad \left| \int_{+\Gamma_t^-} e^{tz} \widehat{\Phi}(z)\, dz \right| \begin{cases} < \varepsilon\mu/(\mu-1), & \mu > 1, \\ < \varepsilon(1 + \log t), & \mu = 1, \\ < \varepsilon(2-\mu)/(1-\mu), & \mu < 1. \end{cases}$$

The same estimates hold if we substitute Γ_t^+ for Γ_t^- in (6.7).

From (6.6) and from (6.7) (considered for Γ_t^- and for Γ_t^+) we obtain

$$(6.8) \qquad \left| \int_{+\Gamma_t} e^{tz} \widehat{\Phi}(z)\, dz \right| \begin{cases} < \varepsilon(2\mu/(\mu-1) + \pi e)t^{\mu-1}, & \mu > 1, \\ < \varepsilon(2 + \pi e + 2\log t), & \mu = 1, \\ < \varepsilon[2(2-\mu)/(1-\mu) + \pi e], & \mu < 1. \end{cases}$$

Let p be a positive number. Consider the part Γ_t^p of the curve Γ_t which is contained in the strip $|\eta| \leqslant p$ of the z plane. From the Cauchy integral theorem we have for any fixed t:

$$i\int_{-p}^{p} e^{t(a+i\eta)}\,\hat{\Phi}(a+i\eta)\,d\eta - \int_{0}^{a} e^{t(\xi+ip)}\,\hat{\Phi}(\xi+ip)\,d\xi -$$

$$- \int_{+\Gamma_t^p} e^{tz}\,\hat{\Phi}(z)\,dz + \int_{0}^{a} e^{t(\xi-ip)}\,\hat{\Phi}(\xi-ip)\,d\xi = 0\,.$$

Assuming $t > T$, from hypotheses *iii*) and *iiii*) we get

$$\lim_{p\to+\infty} i\int_{-p}^{p} e^{t(a+i\eta)}\,\hat{\Phi}(a+i\eta)\,d\eta = \int_{+\Gamma_t} e^{tz}\,\hat{\Phi}(z)\,dz\,.$$

Hence

$$\Phi(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{t(a+i\eta)}\,\hat{\Phi}(a+i\eta)\,d\eta = \frac{1}{2\pi i}\int_{+\Gamma_t} e^{tz}\,\hat{\Phi}(z)\,dz\,.$$

From the estimates (6.8) we deduce the proof of the theorem.

REMARKS. The theorem we have just considered is inspired by an analogous theorem due to G. Doetsch (see [7, pp. 494-496]) and uses the same technique for the proof. Our theorem has the advantage that μ is an arbitrary real number, while Doetsch assumes $\mu > 1$. On the other hand the asymptotics given by the theorem of Doetsch when $\mu > 1$ is sharper than ours. Actually assuming, instead of hypothesis *ii*), the following one

(6.9) $$\lim_{z\to 0} z^{\mu}\,\hat{\Phi}(z) = A \qquad (\mu > 1)\,,$$

Doetsch is able to prove that

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t^{\mu-1}} = \frac{A}{\Gamma(\mu)}\,.$$

Our theorem from the assumption (6.9), with $A \neq 0$, is only able to deduce that

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t^{\mu'-1}} = 0$$

for any $\mu' > \mu$.

Theorem 6.I could be stated by considering a strip $S(a_0, a)$: $a_0 < \xi < a_0 + a$ and by replacing hypothesis *ii*) by the following

$$\lim_{z\to z_0} (z - z_0)^{\mu}\,\hat{\Phi}(z) = 0\,,$$

where z_0 is any point of the straight-line $\xi = a_0$. This easy generalization could be obtained as in [7, p. 496].

For a fixed $t > 0$ assume $a = 2t\tilde{\sigma}_1^{-1}$.

From the theory developed in Section 4 we have that for $0 \leqslant x \leqslant a/2$, $0 \leqslant \tau \leqslant t$

the function $\varphi(x,\tau)$ is given by (2.7) *i.e.*

$$\varphi(x,\tau) = \frac{2}{a}\sum_{h=1}^{\infty} \sin\lambda_h x \int_0^a \varphi(\xi,\tau)\sin\lambda_h\xi\, d\xi\,,$$

$$\lambda_h = h\pi/a\,.$$

Moreover any partial derivative $\varphi_{x^p\tau^q}(x,\tau)$, $p+q \leqslant 3$, is given by the corresponding differentiated series and each of these series converges totally in $\overline{R}_{a/2,t} \equiv \equiv \overline{R}_{t/\tilde{\sigma}_1,t}$.

Assuming $\tau = t$, we have for $p+q\leqslant 3$

$$(6.10)\qquad \varphi_{x^pt^q}(x,t) = \frac{\tilde{\sigma}_1}{t}\sum_{h=1}^{\infty}\frac{d^p}{dx^p}\sin\left(\frac{h\pi\tilde{\sigma}_1}{2t}x\right)\int_0^{2t/\tilde{\sigma}_1}\varphi_{t^q}(\xi,t)\sin\left(\frac{h\pi\tilde{\sigma}_1}{2t}\xi\right)d\xi\,.$$

Assume arbitrarily $\alpha_0>0$. For $0\leqslant x\leqslant\alpha_0$, $t\geqslant\tilde{\sigma}_1\alpha_0$ (6.10) holds and for (2.16) we have

$$(6.11)\qquad \max_{R_{\alpha_0,t}}|\varphi_{x^p\tau^q}(x,\tau)| \leqslant \frac{2}{\pi}\left(\frac{2t}{\tilde{\sigma}_1\pi}\right)^{4-(p+q)}\zeta[5-(p+q)]\,\gamma_q\left(\frac{2t}{\tilde{\sigma}_1},t\right)M_f\left(\frac{2t}{\tilde{\sigma}_1},t\right).$$

Let us now assume, in addition to hypothesis $\mathcal{H}_0$), the following new hypothesis on f:

$\mathcal{H}_1$) *For every* $t\geqslant 0$ *the following integrals are finite*

$$\int_0^{+\infty}|f_x(x,t)|\,dx\,,\qquad \int_0^{+\infty}|f_{xt}(x,t)|\,dx$$

and the relevant functions of t *are bounded in* $(0,+\infty)$.

The functions $f(x,t)$, $f_t(x,t)$ *are bounded in* Q.

From $\mathcal{H}_1$) it follows that the function of t: $M_f(2t/\tilde{\sigma}_1,t)$ is bounded in $(0,+\infty)$. Looking at the definition of $\gamma_q(a,T)$ in Section 2 we see that some $\bar{t}>0$ exists such that for $t\geqslant\bar{t}$ and for $0\leqslant x\leqslant\alpha_0$ we have from (6.11)

$$(6.12)\qquad |\varphi_{x^pt^q}(x,t)| < e^{\xi_0 t}\,,$$

where ξ_0 is *any* fixed constant such that

$$(6.13)\qquad \xi_0 > \xi^* = (4/\mu)[\sigma_2^3 + (1+\varepsilon)\sigma_2]\ (^6)\,.$$

It must be observed that ξ^* *does not depend on* α_0.

It follows that for each $z=\xi+i\eta$ such that $\xi = \mathrm{Re}\,z > \xi_0$ the Laplace integral

$$\hat{\varphi}_{pq}(z,x) = \int_0^{+\infty} e^{-zt}\varphi_{x^pt^q}(x,t)\,dt \qquad (0\leqslant p+q\leqslant 3)$$

exists, for any $x\geqslant 0$, as the integral of an absolutely integrable function.

(6) For the definitions of ε, μ see Sections 1 and 2.

Let us now assume $p+q=4$.

By using the same procedure for deriving (2.17) and by assuming $a=2t\tilde{\sigma}_1^{-1}$, we have for *any* $\alpha_0>0$ and for $t \geqslant \tilde{\sigma}_1 \alpha_0$

$$\int_0^{\alpha_0} |\varphi_{x^p t^q}(x,t)|^2 dx \leqslant \frac{4t}{\pi^2 \tilde{\sigma}_1} \zeta(2) \left[\gamma_q\left(\frac{2t}{\tilde{\sigma}_1}, t\right)\right]^2 \left[M_f\left(\frac{2t}{\tilde{\sigma}_1}, t\right)\right]^2 .$$

A positive $\bar{t}$ exists such that for $t \geqslant \bar{t}$

$$\int_0^{\alpha_0} |\varphi_{x^p t^q}(x,t)|^2 dx < e^{2\xi_0 t}$$

when ξ_0 satisfies (6.13).

For any complex z such that $\operatorname{Re} z > \xi_0$, we have

$$\int_0^{+\infty} dt \int_0^{\alpha_0} |e^{-zt} \varphi_{x^p t^q}(x,t)|\, dx \leqslant \iint_{R_{\alpha_0,\bar{t}}} |e^{-zt} \varphi_{x^p t^q}(x,t)|^2 dx\, dt +$$

$$+\alpha_0 \int_{\bar{t}}^{+\infty} e^{2(\xi_0 - \operatorname{Re} z)t} dt < +\infty .$$

Let z' be such that: $\operatorname{Re} z = \operatorname{Re} z' + \xi'$ with $\operatorname{Re} z' > \xi_0$, $\xi' > 0$. We have

$$(6.14) \qquad \int_0^{+\infty} dt \int_0^{\alpha_0} |e^{-zt} \varphi_{x^p t^q}(x,t)|\, dx \leqslant$$

$$\leqslant \left(\int_0^{+\infty} dt \int_0^{\alpha_0} |e^{-z't} \varphi_{x^p t^q}(x,t)|^2 dx\right)^{1/2} \left(\alpha_0 \int_0^{+\infty} e^{-2\xi' t} dt\right)^{1/2} .$$

It follows from classical results of Lebesgue integral theory that the Laplace transform $\hat{\varphi}_{pq}$ exists, even in the case $p+q=4$ for almost each $x \geqslant 0$, as the integral of an absolutely integrable function; $\hat{\varphi}_{pq}$ has an abscissa of convergence $\leqslant \xi^*$.

We have for $0 \leqslant p+q \leqslant 4$ and for $x>0$

$$(6.15) \qquad \hat{\varphi}_{pq}(z,x) = z^q \hat{\varphi}_{p0}(z,x) .$$

This is easily seen by considering eqs. (1.7) and the fact that $\varphi(x,t) \in C^3(\bar{A}) \cap \cap H_4(A)$ (see Section 4).

Let us now prove that for $p+q \leqslant 4$

$$(6.16) \qquad \frac{\partial}{\partial x} \hat{\varphi}_{p-1\,q}(z,x) = \hat{\varphi}_{pq}(z,x)$$

where the partial derivative $\partial/\partial x$ in the left hand side must be understood in the weak sense if $p+q=4$. To this end it is sufficient to prove that if $\psi(x)$ is any C^∞ function of x which has a bounded support in the interval (α,β) $(0<\alpha<\beta)$, we have

$$(6.17) \qquad \int_\alpha^\beta \psi'(x) \hat{\varphi}_{p-1\,q}(z,x)\, dx = -\int_\alpha^\beta \psi(x) \hat{\varphi}_{pq}(z,x)\, dx .$$

This proves (6.16) in the weak sense. However in the case $p+q\leqslant 3$, since $\widehat{\varphi}_{p-1\,q}(z,x)$, $\widehat{\varphi}(z,x)$ are continuous functions of x, (6.16) has a classical meaning.

It is evident that the integrals in (6.17) exist, as integrals of absolutely integrable functions, since for $p+q\leqslant 3$, $\widehat{\varphi}_{p-1\,q}(z,x)$ and $\widehat{\varphi}_{pq}(z,x)$ are continuous functions of x and in the case $p+q=4$, $\widehat{\varphi}_{pq}(z,x)$ is absolutely integrable in any bounded interval of the x-axis, as follows from (6.14).

We have

$$\int_\alpha^\beta \psi(x)\,\widehat{\varphi}_{pq}(z,x)\,dx=\int_\alpha^\beta \psi(x)\,dx\int_0^{+\infty} e^{-zt}\varphi_{x^pt^q}(x,t)\,dt=-\int_\alpha^\beta \psi'(x)\,\widehat{\varphi}_{p-1\,q}(z,x)\,dx\,.$$

Set

$$\widehat{\varphi}(z,x)=\widehat{\varphi}_{00}(z,x)\,,\qquad \widehat{f}(z,x)=\int_0^{+\infty} e^{-zt}f(x,t)\,dt\,.$$

From (1.1), (6.15), (6.16) we deduce

$$\widehat{\varphi}_{x^4}(z,x)-[\nu t^2+(1+\varepsilon)z]\,\widehat{\varphi}_{x^2}(z,x)+(t_0z^4+z^3)\,\widehat{\varphi}(z,x)=\widehat{f}(z,x)\,. \tag{6.18}$$

Set

$$\omega_1(z)=2^{-1/2}\{\nu z^2+(1+\varepsilon)z+[(\nu z^2+(1+\varepsilon)z)^2-4t_0z^4-4z^3]^{1/2}\}^{1/2}\,,$$

$$\omega_2(z)=2^{-1/2}\{\nu z^2+(1+\varepsilon)z-[(\nu z^2+(1+\varepsilon)z)^2-4t_0z^4-4z^3]^{1/2}\}^{1/2}\,.$$

From now on, when considering an analytic function $[\gamma(z)]^{1/2}$ where $\gamma(z)$ is real and positive for $z=\xi+i0$, $\xi>0$, we suppose that the branch of this function we are considering is the one that on the positive real axis ξ coincides with the positive square root of $\gamma(\xi)$.

From (6.18) we get

$$\widehat{\varphi}(z,x)=c_1(z)e^{\omega_1(z)x}+c_2(z)e^{\omega_2(z)x}+c_3(z)e^{-\omega_1(z)x}+c_4(z)e^{-\omega_2(z)x}+ \tag{6.18*}$$

$$+A_1(z)\int_0^x[e^{\omega_1(z)(x-s)}-e^{-\omega_1(z)(x-s)}]\widehat{f}(z,s)ds+A_2(z)\int_0^x[e^{\omega_2(z)(x-s)}-e^{-\omega_2(z)(x-s)}]\widehat{f}(z,s)ds$$

where

$$A_1(z)=\{2\omega_1(z)[\omega_1(z)+\omega_2(z)]\,[\omega_1(z)-\omega_2(z)]\}^{-1}, \tag{6.19}$$

$$A_2(z)=\{2\omega_2(z)[\omega_1(z)+\omega_2(z)]\,[\omega_2(z)-\omega_1(z)]\}^{-1}. \tag{6.20}$$

The functions $\omega_1(z)$, $\omega_2(z)$, $A_1(z)$, $A_2(z)$ are algebraic functions of z.

Set

$$\alpha(z)=\nu z+1+\varepsilon\,,\qquad \beta(z)=\{[\alpha(z)]^2-4t_0z^2-4z\}^{1/2}\,.$$

We have

$$\omega_1(z)=2^{-1/2}z^{1/2}[\alpha(z)+\beta(z)]^{1/2}\,,\qquad \omega_2(z)=2^{-1/2}z^{1/2}[\alpha(z)-\beta(z)]^{1/2}\,,$$

$$A_1(z)=1/[2^{1/2}z^{3/2}[\alpha(z)+\beta(z)]^{1/2}\beta(z)],\quad A_2(z)=1/[2z^{3/2}[\alpha(z)-\beta(z)]^{1/2}\beta(z)].$$

Since the functions $\beta(z)$ and $[\alpha(z)]^2 - [\beta(z)]^2$ vanish only at points of the half-plane $\mathrm{Re}\, z < 0$, we see that the algebraic functions $\omega_1(z), \omega_2(z), A_1(z), A_2(z)$ have no critical points in the half-plane $\mathrm{Re}\, z > 0$, hence the branches of these functions considered by us are holomorphic in the half-plane $\mathrm{Re}\, z > 0$.

By elementary arguments it is easily proved that in a neighborhood I_0 of $z = 0$ we have

$$\omega_1(z) = z^{1/2}\omega_1^0(z), \qquad \omega_2(z) = z\omega_2^0(z),$$

$$A_1(z) = z^{-3/2}A_1^0(z), \qquad A_2(z) = z^{-2}A_2^0(z),$$

where $\omega_1^0(z)$, $\omega_2^0(z)$, $A_1^0(z)$, $A_2^0(z)$ are holomorphic in I_0 and

$$\omega_1^{(0)}(0) = (1+\varepsilon)^{1/2}, \qquad \omega_2^{(0)}(0) = \left(\nu - \frac{1}{1+\varepsilon}\right)^{-1},$$

$$A_1^0(0) = 1/2(1+\varepsilon)^{3/2}, \quad A_2^0(0) = -\left[2\left(\nu - \frac{1}{1+\varepsilon}\right)^{1/2}(1+\varepsilon)\right]^{-1}.$$

From (1.8) we deduce $\hat{\varphi}(z, 0) = 0$, $\hat{\varphi}_{x^2}(z, 0) = 0$. Hence

$$c_1(z) + c_2(z) + c_3(z) + c_4(z) \equiv 0$$

$$[\omega_1(z)]^2[c_1(z) + c_3(z)] + [\omega_2(z)]^2[c_2(z) + c_4(z)] \equiv 0,$$

which implies $c_1(z) \equiv -c_3(z)$, $c_2(z) \equiv -c_4(z)$,

(6.21) $$\hat{\varphi}(z, x) = c_1(z)[e^{\omega_1(z)x} - e^{-\omega_1(z)x}] + c_2(z)[e^{\omega_2(z)x} - e^{-\omega_2(z)x}] +$$

$$+A_1(z)\int_0^x [e^{\omega_1(z)(x-s)} - e^{-\omega_1(z)(x-s)}]\hat{f}(z, s)\, ds + A_2(z)\int_0^x [e^{\omega_2(z)(x-s)} - e^{-\omega_2(z)(x-s)}]\hat{f}(z, s)\, ds.$$

From (6.21) by differentiating with respect to x and solving the two equations, which we obtain, with respect to $c_1(z)$ and $c_2(z)$, we see that $c_1(z)$ and $c_2(z)$ are analytic functions of z in the same domain where $\hat{\varphi}(z, x)$ is analytic.

Assume $x > 0$. From (6.10), (6.11), for $\alpha_0 = x$, $t = \tilde{\sigma}_1 x$ and $N_f = \sup_{(0, +\infty)} M_f(2t/\tilde{\sigma}_1, t)$, we deduce for $0 < \tau \leqslant \tilde{\sigma}_1 x$:

$$|\varphi(x, \tau)| \leqslant \frac{8N_f}{\pi}\zeta(5)\left(\frac{2t}{\tilde{\sigma}_1\pi}\right)^4 \frac{2+t}{\sigma_2\mu} \exp\left\{\frac{4}{\mu}[\sigma_2^3 + (1+\varepsilon)\sigma_2]t\right\},$$

$$|\varphi_x(x, \tau)| \leqslant \frac{8N_f}{\pi}\zeta(4)\left(\frac{2t}{\tilde{\sigma}_1\pi}\right)^3 \frac{2+t}{\sigma_2\mu} \exp\left\{\frac{4}{\mu}[\sigma_2^3 + (1+\varepsilon)\sigma_2]t\right\},$$

and for $t \geqslant \tilde{\sigma}_1 x$

$$|\varphi(x, t)| \leqslant \frac{8N_f}{\pi}\zeta(5)\left(\frac{2t}{\tilde{\sigma}_1\pi}\right)^4 \frac{2+t}{\sigma_2\mu} \exp\left\{\frac{4}{\mu}[\sigma_2^3 + (1+\varepsilon)\sigma_2]t\right\},$$

$$|\varphi_x(x, t)| \leqslant \frac{8N_f}{\pi}\zeta(4)\left(\frac{2t}{\tilde{\sigma}_1\pi}\right)^3 \frac{2+t}{\sigma_2\mu} \exp\left\{\frac{4}{\mu}[\sigma_2^3 + (1+\varepsilon)\sigma_2]t\right\}.$$

Hence for $\operatorname{Re} z > \xi^*$

$$(6.22)\qquad |\widehat{\varphi}(z,x)| \leqslant \left| \int_0^{\widetilde{\sigma}_1 x} e^{-z\tau} P_0(\widetilde{\sigma}_1 x) e^{\xi^* \widetilde{\sigma}_1 x} d\tau \right| + \left| \int_{\widetilde{\sigma}_1 x}^{+\infty} e^{-zt} P_0(t) e^{\xi^* t} dt \right|,$$

$$(6.23)\qquad |\widehat{\varphi}_x(z,x)| \leqslant \left| \int_0^{\widetilde{\sigma}_1 x} e^{-z\tau} P_1(\widetilde{\sigma}_1 x) e^{\xi^* \widetilde{\sigma}_1 x} d\tau \right| + \left| \int_{\widetilde{\sigma}_1 x}^{+\infty} e^{-zt} P_1(t) e^{\xi^* t} dt \right|,$$

where

$$P_0(t) = \frac{8N_f}{\pi}\zeta(5)\left(\frac{2t}{\widetilde{\sigma}_1 \pi}\right)^4 \frac{2+t}{\sigma_2 \mu}, \qquad P_1(t) = \frac{8N_f}{\pi}\zeta(4)\left(\frac{2t}{\widetilde{\sigma}_1 \pi}\right)^3 \frac{2+t}{\sigma_2 \mu}.$$

From (6.21) we get

$$(6.24)\qquad \begin{cases} B_1(z,x) + B_2(z,x) = \widehat{\varphi}(z,x) + \psi_0(z,x), \\ \omega_1(z) B_1(z,x) + \omega_2(z) B_2(z,x) = \widehat{\varphi}_x(z,x) + \psi_1(z,x), \end{cases}$$

where

$$B_1(z,x) = \left[c_1(z) + A_1(z) \int_0^x e^{-\omega_1(z)s} \widehat{f}(z,s)\, ds \right] e^{\omega_1(z)x},$$

$$B_2(z,x) = \left[c_2(z) + A_2(z) \int_0^x e^{-\omega_2(z)s} \widehat{f}(z,s)\, ds \right] e^{\omega_2(z)x},$$

$$\psi_0(z,x) = \left[c_1(z) + A_1(z) \int_0^x e^{\omega_1(z)s} \widehat{f}(z,s)\, ds \right] e^{-\omega_1(z)x} + \\ + \left[c_2(z) + A(z) \int_0^x e^{\omega_2(z)s} \widehat{f}(z,s)\, ds \right] e^{-\omega_2(z)x},$$

$$\psi_1(z,x) = -\omega_1(z) \left[c_1(z) + A_1(z) \int_0^x e^{\omega_1(z)s} \widehat{f}(z,s)\, ds \right] e^{-\omega_1(z)x} - \\ - \omega_2(z) \left[c_2(z) + A_2(z) \int_0^x e^{\omega_1(z)s} \widehat{f}(z,s)\, ds \right] e^{-\omega_2(z)x}.$$

From (6.24) we get

$$(6.24)^*\qquad \begin{cases} [\omega_1(z) - \omega_2(z)] B_1(z,x) = \\ \qquad = \widehat{\varphi}_x(z,x) - \omega_2(z)\widehat{\varphi}(z,x) + \psi_1(z,x) - \omega_2(z)\psi_0(z,x), \\ [\omega_1(z) - \omega_2(z)] B_2(z,x) = \\ \qquad = \widehat{\varphi}(z,x)\omega_1(z) + \psi_0(z,x)\omega_1(z) - \widehat{\varphi}_x(z,x) - \psi_1(z,x) \end{cases}$$

For $z = \xi + i0$, $\xi > 0$, we have

$$\omega_1(\xi) > 0\,, \qquad \lim_{\xi \to +\infty} \frac{\omega_1(\xi)}{\xi} = 2^{-1/2}[\nu + (\nu^2 - 4t_0)^{1/2}]^{1/2}\,,$$

$$\omega_2(\xi) > 0\,, \qquad \lim_{\xi \to +\infty} \frac{\omega_2(\xi)}{\xi} = 2^{-1/2}[\nu - (\nu^2 - 4t_0)^{1/2}]^{1/2}\,.$$

Assume $z = \xi + i0$ and ξ_1 such that $\xi_1 > \xi^* \widetilde{\sigma}_1$ and such that for $\xi > \xi_1$: $\omega_1(\xi) > \xi_1$, $\omega_2(\xi) > \xi_1$.

From (6.22), (6.23) and hypothesis $\mathcal{H}_1$) we get

$$\lim_{x \to +\infty} e^{-\xi_1 x}[\widehat{\varphi}_x(\xi, x) - \omega_2(\xi)\,\widehat{\varphi}(\xi, x) + \psi_1(\xi, x) - \omega_2(\xi)\,\psi_0(\xi, x)] = 0\,,$$

$$\lim_{x \to +\infty} e^{-\xi_1 x}[\widehat{\varphi}(\xi, x)\,\omega_1(\xi) + \psi_0(\xi, x)\,\omega_1(\xi) - \widehat{\varphi}_x(\xi, x) - \psi_1(\xi, x)] = 0\,.$$

It follows from (6.24)* that

$$\lim_{x \to +\infty} e^{-\xi_1 x} B_1(\xi, x) = \lim_{x \to +\infty} e^{-\xi_1 x} B_2(\xi, x) = 0$$

for $\xi > \xi_1$, implies

$$C_1(\xi) = -A_1(\xi) \int_0^{+\infty} e^{-\omega_1(\xi)s} \widehat{f}(\xi, s)\,ds\,, \qquad C_2(\xi) = -A_2(\xi) \int_0^{+\infty} e^{-\omega_2(\xi)s} \widehat{f}(\xi, s)\,ds\,.$$

We need now the following lemma.

6.II. *The two integrals*

$$\int_0^{+\infty} e^{-\omega_1(z)s} \widehat{f}(z, s)\,ds\,, \qquad \int_0^{+\infty} e^{-\omega_2(z)s} \widehat{f}(z, s)\,ds\,,$$

represent two holomorphic functions of z in the half-plane $\operatorname{Re} z > 0$.

For proving this lemma it is sufficient to demonstrate that the two harmonic functions $\operatorname{Re}\omega_1(z)$, $\operatorname{Re}\omega_2(z)$ are positive for $\operatorname{Re} z > 0$.

From now on we shall indicate by $O(w^k)$ *any* complex valued function $f(w)$ such that: *i)* $f(w)$ is defined in a disc D_ϱ: $|w| < \varrho$ of the w complex plane; *ii)* for any $w \in D_\varrho$ we have $f(w) = w^k f_0(w)$, where $f_0(w)$ is a bounded function in D_ϱ.

Setting

$$a = 2\nu(1 + \varepsilon) - 4\,, \qquad b = \nu^2 - 4t_0\,,$$

we have for $|z| > \varrho^{-1}$ with ϱ sufficiently small

$$\omega_1(z) = 2^{-1/2} z \left\{ \nu + \frac{1 + \varepsilon}{z} + b^{1/2} \sum_{k=0}^{\infty} \binom{1/2}{k} \left[\frac{a}{b}\frac{1}{z} + \frac{(1 + \varepsilon)^2}{b}\frac{1}{z^2} \right]^k \right\}^{1/2} =$$

$$= 2^{-1/2} z \left\{ \nu + b^{1/2} + \frac{1 + \varepsilon}{z} + \frac{1}{2}\frac{a}{b^{1/2}}\frac{1}{z} + O\left(\frac{1}{z^2}\right) \right\}^{1/2} =$$

$$= 2^{-1/2} z(\nu + b^{1/2})^{1/2} \left\{ 1 + \left[\frac{1+\varepsilon}{\nu + b^{1/2}} + \frac{1}{2} \frac{a}{b^{1/2}(\nu + b^{1/2})} \right] \frac{1}{z} + O\left(\frac{1}{z^2}\right) \right\}^{1/2} =$$

$$= 2^{-1/2} z(\nu + b^{1/2})^{1/2} \left\{ 1 + \sum_{k=1}^{\infty} \binom{1/2}{k} \left[\frac{1+\varepsilon}{\nu + b^{1/2}} + \frac{1}{2} \frac{a}{b^{1/2}(\nu + b^{1/2})} + O\left(\frac{1}{z^2}\right) \right]^k \frac{1}{z^k} \right\}.$$

At the end we get the asymptotic estimate

$$\omega_1(z) = 2^{-1/2} z(\nu + b^{1/2})^{1/2} \{ 1 + \gamma_1/z + O(z^{-2}) \}, \tag{6.25}$$

where

$$\gamma_1 = [(1+\varepsilon)/(\nu + b^{1/2}) + a/(2b^{1/2}(\nu + b^{1/2}))]/2.$$

Hence for $|z| > \varrho^{-1}$

$$\operatorname{Re} \omega_1(z) = 2^{-1/2}(\nu + b^{1/2})^{1/2} \gamma_1 + 2^{-1/2}(\nu + b^{1/2})^{1/2} \xi + O(|z|^{-1})$$

which implies for $\operatorname{Re} z \geqslant 0$, $|z| > r_1$, with r_1 sufficiently large,

$$\operatorname{Re} \omega_1(z) > 0. \tag{6.26}$$

Let us now assume $z = 0 + i\eta$. We have

$$\omega_1(i\eta) = (|\eta|^{1/2}/2)(1 + i \operatorname{sign} \eta) \{ 1 + \varepsilon + i\nu\eta + [(1+\varepsilon)^2 - b\eta^2 + ia\eta]^{1/2} \}^{1/2} =$$

$$= (|\eta|^{1/2}/2)(1 + i \operatorname{sign} \eta) \{ 1 + \varepsilon + \varrho^{1/2} \cos(\theta/2) + i[\nu\eta + \varrho^{1/2} \sin(\theta/2] \}^{1/2},$$

where

$$\varrho = \varrho(\eta) = \{ [(1+\varepsilon)^2 - b\eta^2]^2 + a^2\eta^2 \}^{1/2},$$

$$\cos(\theta/2) = 2^{-1/2}[1 + \cos\theta]^{1/2}, \qquad \sin(\theta/2) = 2^{-1/2}[1 - \cos\theta]^{1/2},$$

$$\cos\theta = [(1+\varepsilon)^2 - b\eta^2]/\varrho.$$

Then

$$w_1(i\eta) = (|\eta|^{1/2}/2)(1 + i \operatorname{sign} \eta) R^{1/2} [\cos(\psi/2) + i \sin(\psi/2)], \tag{6.27}$$

where

$$R = R(\eta) = \{ [1 + \varepsilon + 2^{-1/2}(\varrho(\eta) + (1+\varepsilon)^2 - b\eta^2)^{1/2}]^2 +$$
$$+ [\nu\eta + 2^{-1/2}(\varrho(\eta) - (1+\varepsilon)^2 + b\eta^2)^{1/2}]^2 \}^{1/2},$$

$$\cos(\psi/2) = 2^{-1/2}[1 + \cos\psi]^{1/2}, \qquad \sin(\psi/2) = 2^{-1/2}[1 - \cos\psi]^{1/2},$$

$$\cos\psi = \{ (1+\varepsilon) + 2^{-1/2}[\varrho + (1+\varepsilon)^2 - b\eta^2]^{1/2} \}/R(\eta) \tag{6.28}$$

$$\sin\psi = \{ \nu\eta + 2^{-1/2}[\varrho - (1+\varepsilon)^2 + b\eta^2]^{1/2} \}/R(\eta). \tag{6.29}$$

By using a procedure analogous to the one employed for deriving (6.25), we get for $|\eta|$ large enough

$$(1+\varepsilon) + 2^{-1/2}[\varrho + (1+\varepsilon)^2 - b\eta^2]^{1/2} = 1 + \varepsilon + a/2b + O(\eta^{-2}) \tag{6.30}$$

$$\nu\eta + 2^{-1/2}[\varrho - (1+\varepsilon)^2 + b\eta^2]^{1/2} = (\nu \operatorname{sign} \eta + b^{1/2})|\eta| + O(|\eta|^{-1}). \tag{6.31}$$

We have from (6.28), (6.29), (6.30), (6.31) that

$$[\cos\psi]_{\eta=0}=1\,,\qquad [\sin\psi]_{\eta=0}=0\,,$$
$$\lim_{\eta\to-\infty}\cos\psi=0\,,\qquad \lim_{\eta\to-\infty}\sin\psi=-1\,,$$
$$\lim_{\eta\to+\infty}\cos\psi=0\,,\qquad \lim_{\eta\to+\infty}\sin\psi=1\,.$$

On the other hand the function $\cos\psi$ is continuous and positive for $\eta\in(-\infty,+\infty)$ and the function $\sin\psi$ is continuous in $(-\infty,+\infty)$, negative for $\eta\in(-\infty,0)$ and positive for $\eta\in(0,+\infty)$.

We may assume that the function $\psi=\psi(\eta)$, where

$$\psi(\eta)=\arcsin\left(\nu\eta+2^{-1/2}[\varrho-(1+\varepsilon)^2+b\eta^2]^{1/2}\right)/R(\eta)\,,$$

is continuous in $(-\infty,+\infty)$ and is such that

$$-\pi/2<\psi(\eta)<0\quad \text{in } (-\infty,0)\,,$$
$$0<\psi(\eta)<\pi/2\quad \text{in } (0,+\infty)\,,$$
$$\psi(0)=0\,.$$

Since from (6.27) we get

$$\operatorname{Re}\omega_1(i\eta)=(|\eta|^{1/2}/2)[R(\eta)]^{1/2}[\cos 2^{-1}\psi(\eta)-\operatorname{sign}\eta)\sin 2^{-1}\psi(\eta)]\,,$$

we deduce that $\operatorname{Re}\omega_1(i\eta)$ is positive for any η, except for $\eta=0$ where it vanishes.

Then the harmonic function $\operatorname{Re}\omega_1(z)$, because of (6.26), is positive for each z of the domain: $\operatorname{Re}z>0$, $|z|<2r_1$. Hence $\operatorname{Re}\omega_1(z)$ is positive in the half-plane $\operatorname{Re}z>0$.

For proving the same results for $\operatorname{Re}\omega_2$ we have to use an analogous procedure which we briefly summarize. First, we get the asymptotic estimate

$$\omega_2(z)=2^{-1/2}z(\nu-b^{1/2})^{1/2}\{1+\gamma_2/z+O(z^{-2})\}\,, \tag{6.32}$$

where

$$\gamma_2=[(1+\varepsilon)/(\nu-b^{1/2})-a/(2b^{1/2}(\nu-b^{1/2}))]/2\,.$$

Hence, for $|z|$ large enough,

$$\operatorname{Re}\omega_2(z)=2^{-1/2}(\nu-b^{1/2})\gamma_2+2^{-1/2}(\nu-b^{1/2})^{1/2}\xi+O(z^{-2})\,,$$

which implies, for $\operatorname{Re}z\geqslant 0$, $|z|>r_2$, with r_2 sufficiently large, that

$$\operatorname{Re}\omega_2(z)>0\,. \tag{6.33}$$

Let us now consider the following new hypotheses on $f(x,t)$:

$\mathcal{H}_2$) *$f(x,t)$ is integrable in Q.*

$\mathcal{H}_{3,q}$) *The derivative $f_{t^k}(x,t)$ $(k=2,\ldots,q)$ exists and is continuous and bounded in $\overline{Q}$ and $f(x,0)\equiv f_t(x,0)\equiv\ldots\equiv f_{t^{k-2}}(x,0)\equiv 0$ for $x>0$.*

We are now in a position to state and to prove the main theorem which gives the asymptotics of $\varphi(x,t)$ and of its derivatives when $t\to+\infty$.

6.III. *Under hypotheses* $\mathcal{H}_0$), $\mathcal{H}_1$), $\mathcal{H}_2$), *we have for any* $x > 0$

$$(6.34) \qquad \lim_{t \to +\infty} \frac{\varphi(x,t)}{t} = 0, \qquad \lim_{t \to +\infty} \frac{\varphi_x(x,t)}{t^\sigma} = 0, \qquad \textit{for any } \sigma > 0,$$

$$(6.35) \qquad \lim_{t \to +\infty} \frac{\varphi_t(x,t)}{\log t} = 0, \qquad \lim_{t \to +\infty} \varphi_{x^p t^q}(x,t) = 0, \qquad \textit{for } p + q = 2.$$

Under hypotheses $\mathcal{H}_0$), $\mathcal{H}_1$), $\mathcal{H}_2$), $\mathcal{H}_{3,2}$) *we have for any* $x > 0$ *and for* $p + q = 3$

$$(6.36) \qquad \lim_{t \to +\infty} \varphi_{x^p t^q}(x,t) = 0.$$

Under hypotheses $\mathcal{H}_0$), $\mathcal{H}_1$), $\mathcal{H}_2$), $\mathcal{H}_{3,3}$) *we have for any* $x > 0$ *and for* $p + q = 4$

$$(6.37) \qquad \lim_{t \to +\infty} \varphi_{x^p t^q}(x,t) = 0.$$

Assume $a > \xi^*$ and consider for any fixed $x > 0$ the function $\widehat{\varphi}_{pq}(z,x)$ in the strip $S(0,a)$. This function is holomorphic in $S(0,a)$....

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