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Spectral properties of weakly asymptotically almost periodic semigroups in the sense of Stepanov

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Analisi matematica. — *Spectral properties of weakly asymptotically almost periodic semigroups in the sense of Stepanov.* Nota di VALENTINA CASARINO, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — The spectral structure of the infinitesimal generator of strongly measurable, asymptotically S^p-almost periodic semigroups is investigated.

KEY WORDS: Semigroups of class (A); Asymptotically S^p-almost periodic functions; Spectrum.

RIASSUNTO. — *Proprietà spettrali di semigruppi debolmente asintoticamente quasi periodici nel senso di Stepanov.* Si studia la struttura spettrale del generatore infinitesimale di semigruppi fortemente misurabili, debolmente asintoticamente quasi periodici nel senso di Stepanov.

According to the approximation theorem, proved for scalar-valued functions by N. N. Bogoliubov and extended to vector-valued functions by S. Bochner, the space of almost periodic functions defined by H. Bohr coincides with the closure of the class of all trigonometric polynomials under the uniform convergence. Of course, completions of this space with respect to other metrics yield, in principle, different classes of almost periodic functions. For scalar valued maps, by using the metric

$$d(f, g) = \sup_{t \in \mathbb{R}} \left(\frac{1}{L} \int_t^{t+L} |f(s) - g(s)|^p \, ds \right)^{1/p},$$

where $p \in [1, +\infty)$ and $L > 0$, W. Stepanov defined a class of almost periodic functions, for which continuity fails, and only measurability and integrability in the sense of Lebesgue are required. L. Amerio and G. Prouse extended this definition to functions with values in a Banach space.

For a strongly continuous semigroup T of linear bounded operators on a complex Banach space $\mathcal{E}$, H. Henriquez proved that the definitions of almost periodicity given by H. Bohr and by W. Stepanov are equivalent; he proved also the equivalence between the asymptotic almost periodicity (as defined by M. Fréchet) and the asymptotic S^p-almost periodicity, that will be defined in n. 1.

In [13] E. Vesentini investigates which constraints on the spectral structure of the infinitesimal generator of a strongly continuous semigroup are generated by very weak hypothesis on the almost periodic behaviour of the semigroup, for example by the existence of some $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$ for which the function $t \mapsto \langle T(t)x, \lambda \rangle$ is asymptotically almost periodic. Under this assumption, the equivalence proved by H. Henriquez does not hold, so that also the case of a strongly continuous semigroup can be considered. When T is strongly continuous, one finds that almost every result of [13] can be estab-

(*) Nella seduta del 19 giugno 1997.

lished without substantial modifications, by substituting the hypothesis of asymptotic almost periodicity with that of asymptotic S^p-almost periodicity for some $p \in [1, +\infty)$. Therefore it may now be considered the case of a semigroup $T: \mathbb{R}_+ \to \mathcal{L}(\mathcal{E})$, which is only strongly measurable for $t > 0$; this assumption being closer to the original definition of W. Stepanov.

Some of the results which are only announced in this paper will be established in one of the chapters of the author's doctoral dissertation.

1. Let f be a $L^p_{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$ function, for some $1 \leq p < +\infty$ and let $\varepsilon > 0$. A *Stepanov ε-period* for f is a real number τ_ε for which

$$\left\{ \int_0^1 \|f(t+s+\tau_\varepsilon) - f(t+s)\|^p \, ds \right\}^{1/p} \leq \varepsilon \quad \text{for every } t \in \mathbb{R}.$$

A function f in $L^p_{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$ is said *almost periodic in the sense of Stepanov* (or *S^p-almost periodic*) if for every $\varepsilon > 0$ there is a relatively dense set of Stepanov ε-periods for f.

For a continuous function $f \in L^p_{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$ the almost periodicity in the sense of Bohr implies that of Stepanov. The converse is false, except, as is well known, in the case of uniformly continuous functions.

S. Bochner pointed out that almost periodicity in the sense of Stepanov can be reduced to that of Bohr. More precisely, a function $f: \mathbb{R} \to \mathcal{E}$ is almost periodic in the sense of Stepanov if, and only if, the function $\tilde{f}(t) = \{f(t+s): s \in [0, 1]\}$ from $\mathbb{R}$ to $L^p([0, 1]; \mathcal{E})$ is almost periodic in the sense of Bohr.

A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{E})$ is said to be *asymptotically almost periodic in the sense of Stepanov* (or *asymptotically S^p-almost periodic*) if the function $\tilde{f}: \mathbb{R}_+ \to L^p([0, 1]; \mathcal{E})$ is asymptotically almost periodic.

A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$ is said to be *S^p-bounded* if

$$\|f\|_{S^p} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < +\infty .$$

The space, which will be denoted by S^p_s, of all S^p-bounded functions is a Banach space for the norm $\|\cdot\|_{S^p}$. The following theorem is a version of Bochner's criterion for asymptotically almost periodic functions in the sense of Stepanov.

THEOREM 1.1. *If f belongs to S^p_s, the following assertions are equivalent*:

1) *f is S^p-almost periodic*;

2) *the set of all translates $f(\bullet + s)$ of f by all $s \in \mathbb{R}$ is relatively compact in S^p_s.*

Also for S^p-almost periodic or asymptotically S^p-almost periodic functions the limit

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t e^{-i\theta s} f(s) \, ds$$

exists for all $\theta \in \mathbb{R}$ and vanishes for all values of θ with the possible exception of an at most countable set of values. These values of θ and the corresponding values of the limit are again called the *Fourier exponents* (or the *frequencies*) and the *Fourier coefficients* for f.

From the decomposition theorem for asymptotically almost periodic functions in the sense of Bohr the following, analogous result follows [6]:

THEOREM 1.2. *Let $f\colon (0, +\infty) \to \mathcal{E}$ be asymptotically S^p-almost periodic for some $1 \leq p < +\infty$. There exist $g \in L^p_{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$ and $q \in L^p_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{E})$ such that:*

a) g is S^p-almost periodic;

b) $\tilde{q}(t) = \{q(t+s)\colon s \in [0, 1]\}$ belongs to $C_0(\mathbb{R}_+; L^p([0, 1]; \mathcal{E}))$, i.e. *$\tilde{q}$ is a continuous function from $\mathbb{R}_+$ to $L^p([0, 1]; \mathcal{E})$ vanishing at infinity;*

c) $f(t) = g_{|\mathbb{R}_+}(t) + q(t)$ for every $t > 0$.

In analogy to the case of asymptotically almost periodic functions, g and q are said, respectively, the *principal term* and the *correction term* for f.

A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$ is called *weakly S^p-almost periodic* (or *weakly asymptotically S^p-almost periodic* if $f \in L^p_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{E})$) if for every $\lambda \in \mathcal{E}'$ the function $t \mapsto \langle f(t), \lambda \rangle$ is S^p-almost periodic (or asymptotically S^p-almost periodic) (1).

2. Let the semigroup $T(t)\colon [0, +\infty) \to \mathcal{L}(\mathcal{E})$ be strongly measurable for $t > 0$, *i.e.* for every $x \in \mathcal{E}$ there exists a sequence of finitely valued functions strongly convergent almost everywhere to the function $t \mapsto T(t)x$. According to a classical result of R. S. Philips, T is strongly continuous for $t > 0$. Let $\omega_0 = \inf\limits_{t>0} (1/t) \log \|T(t)\|$ be the type of the semigroup T. Given a linear operator X, its range will be denoted by $\mathcal{R}X$.

A semigroup T is said to be of class (A) if the linear subspace $\mathcal{E}_0 = \bigcup\limits_{t>0} \mathcal{R}T(t)$ is dense in $\mathcal{E}$ and if there exists $\omega_1 > \omega_0$ such that, for every $\zeta \in \mathbb{C}$, $\Re\zeta > \omega_1$, there is a linear bounded operator $R(\zeta)$ on $\mathcal{E}$ satisfying:

a) $R(\zeta)x = \displaystyle\int_0^{+\infty} e^{-\zeta t} T(t)x\,dt$ for every $x \in \mathcal{E}_0$;

b) $\|R(\zeta)\|$ is bounded in the halfplane $\Re\zeta > \omega_1$, and

c) $\lim\limits_{\zeta \to +\infty} \zeta R(\zeta)x = x$ for every $x \in \mathcal{E}$ (*i.e.* $T(\cdot)x$ is Abel-summable to x for every $x \in \mathcal{E}$).

(1) Observe that this definition is different from that given by L. Amerio and G. Prouse [1]; according to their definition, a function f is called weakly S^p-almost periodic if the associated map $\tilde{f}$ from $\mathbb{R}$ to $L^p([0, 1]; \mathcal{E})$ is weakly almost periodic.

Now, let X_0 be the linear operator defined on the linear submanifold of $\mathcal{E}$:

$$\mathcal{D}(X_0) = \left\{ x \in \mathcal{E}: \lim_{t \to 0} \frac{1}{t} (T(t)x - x) \text{ exists in } \mathcal{E} \right\}$$

by $X_0 x = \lim_{t \to 0} t^{-1}(T(t)x - x)$.

X_0 is called the infinitesimal operator of T. In general, the operator X_0 is neither closed nor densely defined; however, $\mathcal{D}(X_0)$ is dense in $\mathcal{E}_0$. The smallest closed extension of X_0, when it exists, is denoted by X and is called the infinitesimal generator of T. E. Hille and R. S. Phillips proved that, if T is of class (A), the infinitesimal generator exists and $R(\zeta) = R(\zeta, X)$ for $\Re\zeta > \omega_1$, where $R(\zeta, X)$ is the resolvent operator of X.

For semigroups of class (A), E. Hille and R. S. Phillips built a duality theory; more precisely, given a semigroup T of class (A), one defines $\mathcal{E}^+$ as the closure of $\mathcal{D}(X')$ in $\mathcal{E}'$, and X^+ as the part of X' in $\mathcal{E}^+$, *i.e.* the restriction of X' to the linear space

$$\mathcal{D}(X^+) = \{\phi \in \mathcal{D}(X'): X'\phi \in \mathcal{E}^+\}.$$

They proved that X^+ is the infinitesimal generator of a semigroup of class (A) $T^+: \mathbb{R}_+ \to \mathcal{L}(\mathcal{E}^+)$. Moreover, given

$$\Gamma = \left\{\phi \in \mathcal{E}': \lim_{t \to 0^+} \|T'(t)\phi - \phi\| = 0\right\},$$

they proved that $\overline{\Gamma} = \mathcal{E}^+$, showing that $\mathcal{E}^+$ is the largest subspace of $\mathcal{E}'$, on which the semigroup of adjoint operators $\{T'(t)\}$, defined by $T'(t) = T(t)'$ for every $t > 0$, has suitable continuity properties.

Finally, for semigroups of class (A) the following spectral mapping theorem [7] holds:

Proposition 2.1. *If T is a semigroup of class* (A) *with infinitesimal generator X and if x is an eigenvector for X with eigenvalue ζ, for some $\zeta \in \mathbb{C}$, then x is an eigenvector for $T(t)$ with eigenvalue $e^{\zeta t}$, for every $t \geq 0$.*

In the following, $T: (0, +\infty) \to \mathcal{L}(\mathcal{E})$ will denote a strongly measurable semigroup and of type $\omega_0 \leq 0$ of continuous linear operators on $\mathcal{E}$, such that:

h_1) for every $x \in \mathcal{E}$ $\int_0^1 \|T(s)x\| ds < +\infty$;

h_2) there are $t_0 > 0$ and $M > 0$ such that $\|T(t)x\| \leq M\|x\|$ for all $t \geq t_0$ and for all $x \in \mathcal{E}$.

It will now be shown as the mean ergodic theorem, proved for strongly continuous semigroups in [5], can be generalized to strongly measurable semigroup.

Set

$$\mathcal{F} = \left\{ x \in \mathcal{E}: \lim_{t \to +\infty} \frac{1}{t} \int_0^t T(s)x\, ds \text{ exists in } \mathcal{E} \right\}.$$

For every $x \in \mathcal{F}$ define

$$Px = \lim_{t \to +\infty} \frac{1}{t} \int_0^t T(s)x\,ds\,.$$

THEOREM 2.2. *Let T be a strongly measurable semigroup of type $\omega_0 \leqslant 0$ on a complex Banach space $\mathcal{E}$, satisfying h_1) and h_2). Let P and $\mathcal{F}$ be defined as above. Then:*

1) $PT(r)x = T(r)Px = Px$ *for every* $x \in \mathcal{F}$ *and all* $r > 0$;

2) P *is a linear bounded projection operator on* $\mathcal{F}$, *with* $\|P\| \leqslant M$;

3) $\overline{\mathcal{R}X_0} \subset \ker P$;

4) $\mathcal{R}P = \ker X_0$ *and* $P_{|\ker X_0} = I$;

5) $\mathcal{R}P \cap \ker P = \{0\}$ *and* $\mathcal{F} = \ker P \oplus \mathcal{R}P$.

If, in addition, T is of class (A), *then*:

1*b*) $\mathcal{F} = \ker X \oplus \overline{\mathcal{R}X}$;

2*b*) $\ker P = \overline{\mathcal{R}X}$.

PROOF. 1) If $x \in \mathcal{F}$ and $r > 0$, then

$$T(r)Px = \lim_{t \to +\infty} \frac{1}{t} \int_r^{r+t} T(s)x\,ds = \lim_{t \to +\infty} \frac{1}{t}\left(\int_0^{r+t} - \int_0^r\right) T(s)x\,ds =$$

$$= \lim_{t \to +\infty} \frac{r+t}{t}\,\frac{1}{r+t} \int_0^{r+t} T(s)x\,ds = Px\,,$$

since, by h_1) and the local boundedness of $\|T(t)\|$, it follows that $\int_0^r T(s)x\,ds$ is bounded. Analogously, $PT(r)x = Px$ for all $x \in \mathcal{F}$ and $r > 0$, *i.e.* 1) holds.

2) If $x \in \mathcal{F}$, then

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t T(s)x\,ds = \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^t T(s)x\,ds\,,$$

so that, by h_2), $\|P\| \leqslant M$. Moreover, from 1) it follows

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t T(s)Px\,ds = Px\,,$$

and therefore P is a projector.

3) Let $v \in \overline{\mathcal{R}X_0}$. For every $\varepsilon > 0$ there is some $w \in \mathcal{R}(X_0)$ for which $\|v - w\| < \varepsilon$.

Let $z \in \mathcal{D}(X_0)$ be such that $X_0 z = w$. Now:

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t T(s)(v-w)\,ds = \lim_{t\to+\infty} \frac{1}{t}\int_{t_0}^t T(s)(v-w)\,ds = 0\,,$$

by h_2) and the arbitrariness of ε.

Since, moreover,

$$\left\| \int_0^t T(s)X_0 z\,ds \right\| = \|T(t)z - z\| \leqslant (M+1)\|z\| \qquad \text{for all } t \geqslant t_0\,,$$

the conclusion follows.

4) If $x \in \mathcal{F}$, then $((T(t)-I)/t)Px = (Px - Px)/t = 0$, so that $X_0 Px = 0$.

Moreover, since, for every $x \in \mathcal{D}(X_0)$ such that $\int_0^1 \|T(t)X_0 x\|\,dt < \infty$ the equality

$$T(t)x - x = \int_0^t T(\tau)X_0 x\,d\tau\,,$$

holds, then, if $x \in \ker X_0$, one has $\lim_{t\to+\infty}(1/t)\int_0^t T(s)x\,ds = x$ and therefore $\ker X_0 \subset \mathcal{R}P$.

5) It is obvious, since P is a projector.

1*b*) If T is of class (A), it is easy to see that $\overline{\mathcal{R}X_0} = \overline{\mathcal{R}X}$ and $\ker X_0 = \ker X$, and, therefore, from 3) and 4) it follows $\ker X \oplus \overline{\mathcal{R}X} \subset \mathcal{F}$. The proof of the converse inclusion paraphrases that of E. Vesentini in the continuous case and it is reported for the sake of completeness. If there exists $x_0 \in \mathcal{F}\setminus(\ker X \oplus \overline{\mathcal{R}X})$, then there is some $\lambda_0 \in \mathcal{E}'\setminus\{0\}$, vanishing on $\ker X \oplus \overline{\mathcal{R}X}$ and such that $\langle x_0, \lambda_0\rangle \neq 0$. Thus for all $x \in \mathcal{D}(X)\langle Xx, \lambda_0\rangle = 0$, so that $\lambda_0 \in \mathcal{D}(X')$ and $X'\lambda_0 = 0$. That implies $\lambda_0 \in \mathcal{D}(X^+)$ and $X^+\lambda_0 = 0$, whence $T^+(t)\lambda_0 = \lambda_0$ for every $t \geqslant 0$. Now

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \langle T(s)x_0, \lambda_0\rangle\,ds = \langle x_0, \lambda_0\rangle \neq 0\,,$$

which is absurd since $Px_0 \in \ker X$.

2*b*) It follows from 1*b*) and 5). ■

It is worth noticing that 3) and 4) imply that

$$\ker X_0 \cap \overline{\mathcal{R}X_0} = \{0\} \tag{2.1}$$

and, when T is of class (A), also that

$$\ker X \cap \overline{\mathcal{R}X} = \{0\}\,. \tag{2.2}$$

Let now the Banach space $\mathcal{E}$ be weakly sequentially complete.

If, given $x \in \mathcal{E}$, the limit

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)x, \lambda \rangle ds \tag{2.3}$$

exists for all $\lambda \in \mathcal{E}'$, then there is some $Qx \in \mathcal{E}$ for which the limit (2.3) is equal to $\langle Qx, \lambda \rangle$. Define

$$\mathcal{G} = \left\{ x \in \mathcal{E} \colon \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)x, \lambda \rangle ds \text{ exists for every } \lambda \in \mathcal{E}' \right\}.$$

THEOREM 2.3. *Let T be a strongly measurable semigroup, of type $\omega_0 \leq 0$, satisfying h_1) and h_2), on a complex, weakly sequentially complete Banach space $\mathcal{E}$.*
Let Q and $\mathcal{G}$ be defined as above. Then:

1) $QT(r)x = T(r)Qx = Qx$ *for every* $x \in \mathcal{G}$ *and all* $r > 0$;
2) Q *is a linear bounded projection operator on* $\mathcal{G}$, *with* $\|Q\| \leq M$;
3) $\overline{\mathcal{R}X_0} \subset \ker Q$, $\mathcal{R}Q = \ker X_0$ *and* $Q_{|\ker X_0} = I$;
4) $\mathcal{R}Q \cap \ker Q = \{0\}$ *and* $\mathcal{G} = \ker Q \oplus \mathcal{R}Q$.

If, in addition, T is of class (A), *then*:

1*b*) $\mathcal{G} = \ker X \oplus \overline{\mathcal{R}X}$;

2*b*) $\ker Q = \overline{\mathcal{R}X}$.

PROOF. 1) If $x \in \mathcal{G}$ and $r > 0$, then:
$\langle T(r)Qx, \lambda \rangle = \langle Qx, T'(r)\lambda \rangle =$

$$= \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)x, T'(r)\lambda \rangle ds = \lim_{t \to +\infty} \frac{1}{t} \int_r^{r+t} \langle T(s)x, \lambda \rangle ds = \langle Qx, \lambda \rangle$$

for every $\lambda \in \mathcal{E}'$, and therefore $T(r)Qx = Qx$ for every $r > 0$. Analogously

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)T(r)x, \lambda \rangle ds = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)x, \lambda \rangle ds$$

if $x \in \mathcal{G}$, and therefore $QT(r) = Q$ for any $r > 0$ on $\mathcal{G}$.

2) If $x \in \mathcal{G}$, then

$$\langle Qx, \lambda \rangle = \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^t \langle T(s)x, \lambda \rangle ds,$$

so that, by h_2), $|\langle Qx, \lambda \rangle| \leq M\|x\| \cdot \|\lambda\|$ for all $\lambda \in \mathcal{E}'$. Moreover, since

$$\langle Q^2x, \lambda \rangle = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)Qx, \lambda \rangle ds = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle Qx, \lambda \rangle ds = \langle Qx, \lambda \rangle$$

for every $\lambda \in \mathcal{E}'$, then $Q^2x = Qx$ on $\mathcal{G}$.

3) On $\overline{\mathcal{R}X_0}$ and on $\ker X_0 P$ is defined, and therefore also Q is defined and it coincides with P, whence the thesis follows.

4) It is obvious by 2).

1*b*) and 2*b*) They follow from 1*b*) and 2*b*) in Theorem 2.2. ■

In virtue of the fact that the dual of a Banach space $\mathcal{E}'$ is always sequentially weak-star complete (as a consequence of the Banach-Steinhaus theorem), E. Vesentini proves in [13] a version of the mean ergodic theorem on the dual space of $\mathcal{E}$, whose proof requires only few changes with respect to the original one, when the continuity hypothesis is dropped. Set

$$\mathcal{K}' = \left\{ \lambda \in \mathcal{E}' : \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)x, \lambda \rangle ds \text{ exists for every } x \in \mathcal{E} \right\}.$$

Given $\lambda \in \mathcal{K}'$, there is some $R\lambda \in \mathcal{E}'$ such that

$$\langle x, R\lambda \rangle = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)x, \lambda \rangle ds \text{ for all } x \in \mathcal{E}.$$

Theorem 2.4. *Let T be a strongly measurable semigroup, of type $\omega_0 \leq 0$, satisfying h_1) and h_2), on a complex Banach space $\mathcal{E}$.*
Let R and $\mathcal{K}'$ be defined as above. Then:

1) $RT'(r)\lambda = T'(r)R\lambda = R\lambda$ *for every* $\lambda \in \mathcal{K}'$ *and all* $r > 0$;

2) R *is a linear bounded projector on* $\mathcal{K}'$, *with* $\|R\lambda\| \leq M\|\lambda\|$ *for every* $\lambda \in \mathcal{K}'$;

3) *if* $\lambda \in \mathcal{K}'$, *then* $\langle X_0 x, R\lambda \rangle = 0$ *for every* $x \in \mathcal{D}(X_0)$;

4) $\mathcal{R}R \cap \ker R = \{0\}$ *and* $\mathcal{K}' = \ker R \oplus \mathcal{R}R$.

If, in addition, T is of class (A), *then*:

1*b*) $\mathcal{R}R = \ker X^+$;

2*b*) $\ker X^+ \oplus \overline{\mathcal{R}X^+} \subset \mathcal{K}'$ *and* $R\lambda = \lim_{t \to +\infty} (1/t) \int_0^t T^+(s)\lambda\, ds$ *for each* $\lambda \in \ker X^+ \oplus \overline{\mathcal{R}X^+}$.

Proof. 1) The proof is very similar to that of 1) in Theorem 2.3.

2) If $\lambda \in \mathcal{K}'$, 1) implies that for every $x \in \mathcal{E}$

$$\frac{1}{t} \int_0^t \langle T(s)x, R\lambda \rangle ds = \frac{1}{t} \int_0^t \langle x, T'(s)R\lambda \rangle ds = \frac{1}{t} \int_0^t \langle x, R\lambda \rangle ds = \langle x, R\lambda \rangle,$$

i.e. $R\lambda \in \mathcal{K}'$ and $R^2\lambda = R\lambda$. A standard application of h_1) and h_2) yields $\|R\lambda\| \leq M\|\lambda\|$ for every $\lambda \in \mathcal{K}'$.

3) If $\lambda \in \mathcal{H}'$, then

$$t^{-1}\langle T(t)x - x, R\lambda\rangle = t^{-1}\langle x, T'(t)R\lambda - R\lambda\rangle = 0,$$

and therefore, if $x \in \mathcal{D}(X_0)$, then $\langle X_0 x, R\lambda\rangle = 0$.

4) Obvious.

1*b*) It follows from 3) that for every $x \in \mathcal{D}(X)$ $\langle Xx, R\lambda\rangle = 0$, and therefore $R\lambda \in \mathcal{D}(X')$ with $X'R\lambda = 0$. Thus $R\lambda \in \mathcal{D}(X^+)$ and $X^+R\lambda = 0$. *Viceversa*, if $\lambda \in \ker X^+ \setminus \{0\}$, then $T^+(t)\lambda = \lambda$ for all $t > 0$ by Proposition 2.1, and

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle T(s)x, \lambda\rangle ds = \langle x, \lambda\rangle \qquad \text{for all } x \in \mathcal{E},$$

so that $\lambda \in \mathcal{H}'$ and $R\lambda = \lambda$.

2*b*) Obvious, since T^+ is a semigroup of class (A) and of type $\omega_0 \leqslant 0$ on the Banach space $\mathcal{E}^+$. ■

Corollary 2.5. *If the limit* (2.3) *exists for all* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}^+$, *then* $\mathcal{H}' = \mathcal{E}^+$ *and* $R \in \mathcal{L}(\mathcal{E}^+)$.

If T is a strongly measurable semigroup, possibly of type $\omega_0 \leqslant 0$ or of class (A), then the semigroup $\{e^{-i\theta t}T(t)\}$ satisfies the same properties, and $X_0 - i\theta I$ and $X - i\theta I$ represent, respectively, the infinitesimal operator and, if it exists, the infinitesimal generator of the new semigroup. Thus Theorems 2.2, 2.3 and 2.4 can be reformulated in terms of the semigroup $\{e^{-i\theta t}T(t)\}$, by introducing the spaces $\mathcal{F}_{i\theta}$, $\mathcal{G}_{i\theta}$ and $\mathcal{H}_{i\theta}$, and the operators $P_{i\theta}$, $G_{i\theta}$ and $R_{i\theta}$.

Remark 2.6. E. Hille and R. S. Phillips investigated the behaviour at infinity of semigroups of class (A) and of type $\omega_0 \leqslant 0$, with the aid of the Abel means, *i.e.* in terms of the operator $Qx = \lim_{\zeta \to 0} \zeta R(\zeta, X)x$, for all x for which the limit exists. It turns out that Q is a projector defined on $\ker X \oplus \overline{\mathcal{R}X}$ and that under these assumptions:

h_1') for every $x \in \mathcal{E}$ $\int_0^1 \|T(s)x\|^p ds < +\infty$ for some $p \in (1, +\infty)$;

h_2') there are $t_0 > 0$ and $M > 0$ such that $\|T(t)x\| \leqslant M\|x\|$ for all $t \geqslant t_0$ and for all $x \in \mathcal{E}$;

it coincides with the Cesaro limit P, whose properties have been investigated in Theorem 2.2. Moreover, the construction of Hille and Phillips can be extended to the case of weakly sequentially complete Banach spaces or of dual Banach spaces, getting, also in this framework, equivalence with the Cesaro approach. Another possible approach is given by the mean ergodic theorem of P. Masani [8], establishing that, for strongly measurable semigroups satisfying h_1), the set of all $x \in \mathcal{E}$ for which Px is defined coincides with $\mathfrak{Z} \oplus \overline{R}$, where $\mathfrak{Z} = \bigcap_{t>0} \ker[T(t) - I]$ and $R = \bigcup_{t>0} \mathcal{R}[T(t) - I]$. Note that Masani's theorem does not help in settling the question, whether there is a connection between the frequencies of an asymptotically S^p-almost periodic function arising from a semigroup and the spectrum of the infinitesimal generator, unless additional hypotheses on the existence of the generator are assumed.

3. Recall that, if X is an effective extension of X_0, then $p\sigma(X_0) \subset p\sigma(X)$, whereas $r\sigma(X) \subset r\sigma(X_0)$.

THEOREM 3.1. *Let* $T\colon (0, +\infty) \to \mathcal{L}(\mathcal{E})$ *be a strongly measurable semigroup on* $\mathcal{E}$, *of type* $\omega_0 \leq 0$, *satisfying* h_1) *and* h_2).

1) *If there are* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}'$ *for which the function* $t \mapsto \langle T(t)x, \lambda\rangle$ *is non constant and asymptotically* S^p*-almost periodic, for some* $1 \leq p < +\infty$, *then the set of all frequencies of this function is contained in the set* $[(p\sigma(X_0) \cup r\sigma(X_0)) \cap i\mathbb{R}]/i$.

2) *If* T *is of class* (A) *and if there are* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}'$ *for which the function* $t \mapsto \langle T(t)x, \lambda\rangle$ *is non constant and asymptotically* S^p*-almost periodic, for some* $1 \leq p < +\infty$, *then the set of all frequencies of this function is contained in the set* $[(p\sigma(X_0) \cup r\sigma(X)) \cap \cap i\mathbb{R}]/i$.

Conversely, for every $i\theta \in (p\sigma(X) \cup r\sigma(X)) \cap i\mathbb{R}$, *there are* $x \in \mathcal{D}(X)$ *and* $\lambda \in \mathcal{E}'$ *for which* $\langle x, \lambda\rangle \neq 0$ *and* θ *is a frequency of the periodic function* $t \mapsto \langle T(t)x, \lambda\rangle$.

PROOF. The proof is very similar to that of E. Vesentini. ■

Also when the continuity assumption on T is dropped, the above results can be improved if the space $\mathcal{E}$ is reflexive. Indeed, the following two results hold:

LEMMA 3.2. *Let* X *be a linear operator, defined on* $\mathcal{D}(X) \subset \mathcal{E}$. *If* $\ker(X - i\theta I) \oplus \oplus \overline{\mathcal{R}(X - i\theta I)} = \mathcal{E}$, *then* $i\theta \notin r\sigma(X)$.

LEMMA 3.3. *If* $\mathcal{E}$ *is reflexive, then* $i\mathbb{R} \cap r\sigma(X) = \emptyset$.

In the following, T will denote a semigroup of class (A) and of type $\omega_0 \leq 0$, satisfying h_1) and h_2), with infinitesimal operator X_0 and infinitesimal generator X.
Lemma 3.3 implies

COROLLARY 3.4. *Let* $\mathcal{E}$ *be a reflexive Banach space. If* $t \mapsto \langle T(t)x, \lambda\rangle$ *is a non-constant asymptotically* S^p*-almost periodic function for some* $1 \leq p < +\infty$, *for some* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}'$, *then the set of all frequencies of this function is contained in* $[p\sigma(X_0) \cap \cap i\mathbb{R}]/i$.

Theorem 2.3 yields also

THEOREM 3.5. *Let* $\mathcal{E}$ *be a weakly sequentially complete Banach space* $\mathcal{E}$. *If* T *is weakly asymptotically* S^p*-almost periodic for some* $1 \leq p < +\infty$, *then* $\mathcal{E} = \ker(X - i\theta I) \oplus \oplus \overline{\mathcal{R}(X - i\theta I)}$ *and, therefore,* $r\sigma(X) \cap i\mathbb{R} = \emptyset$.

Theorem 2.4 and Corollary 2.5 yield

THEOREM 3.6. *If the functions* $t \mapsto \langle T(t)x, \lambda\rangle$ *are asymptotically* S^p*-almost periodic for* *some* $1 \leqslant p < +\infty$, *for all* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}^+$, *then the frequencies of these functions are contained in* $[p\sigma(X^+) \cap i\mathbb{R}]/i$.

E. Vesentini has recently proved that, if X is a linear, densely defined operator, then $p\sigma(X') = k\sigma(X)$, where the compression spectrum $k\sigma(X)$ is defined by

$$k\sigma(X) = \{\zeta \in \mathbb{C} : \overline{R(X - \zeta I)} \neq \mathcal{E}\}.$$

Moreover, if X generates a strongly continuous semigroup, then $p\sigma(X^+) = k\sigma(X)$. His proof also holds when X is the infinitesimal generator of a semigroup of class (A), and therefore in the following it will be assumed

$$p\sigma(X') = p\sigma(X^+) = k\sigma(X).$$

A complex number ζ is said to be of index ν (where ν is a positive integer) with respect to a linear operator X in case $(X - \zeta I)^{\nu+1} \equiv 0$ implies $(X - \zeta I)^{\nu} \equiv 0$ and there is x_0 such that $(X - \zeta I)^{\nu} x_0 = 0$ and $(X - \zeta I)^{\nu-1} x_0 \neq 0$.

ζ has index zero if, by definition, $X - \zeta I$ has an inverse. If no such integer ν exists, ζ is said to be of infinite index.

THEOREM 3.7. *Let T be a semigroup of class* (A) *and of type* $\omega_0 \leqslant 0$, *satisfying* h_1) *and* h_2), *and let* $\mathcal{E} = \ker(X - i\theta I) \oplus \overline{\mathcal{R}(X - i\theta I)}$ *for some* $\theta \in \mathbb{R}$. *Then the complex number* $i\theta$ *is of index* 0 *or* 1 *with respect to* $X - i\theta I$.

If $i\theta$ *has index* 0, *then* θ *cannot be a frequency for any asymptotically* S^p*-almost periodic function* $t \mapsto \langle T(t)x, \lambda\rangle$, *with* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}'$.

If $i\theta$ *has index* 1, *then* $i\theta$ *belongs to* $p\sigma(X_0)$, *and* θ *is a frequency of* $t \mapsto T(t)x$, *for some* $x \in \mathcal{E}$.

PROOF. If $(X - i\theta I)^2 x = 0$ for some $x \in \mathcal{D}(X^2)$, then $(X - i\theta I)x \in \ker(X - i\theta I) \cap \cap\, \mathcal{R}(X - i\theta I)$, so that from (2.2) $(X - i\theta I)x = 0$ follows.

Suppose now that $i\theta$ is of index 0. Then $X - i\theta I$ has an inverse, and therefore $\ker(X - i\theta I) = \{0\}$. Since $\mathcal{E} = \ker(X - i\theta I) \oplus \overline{\mathcal{R}(X - i\theta I)}$, then $i\theta$ doesn't belong to $r\sigma(X)$, and therefore $i\theta \in c\sigma(X) \cup r(X)$. Theorem 1.3 yields now the thesis.

If $i\theta$ has index 1, then $i\theta \in p\sigma(X)$ and θ is a frequency of a periodic function $t \mapsto T(t)x$. ■

COROLLARY 3.8. *Let T be a semigroup of class* (A) *and of type* $\omega_0 \leqslant 0$, *satisfying* h_1) *and* h_2), *weakly asymptotically* S^p*-almost periodic for some* $1 \leqslant p < +\infty$ *on a weakly sequentially complete Banach space. Then for every* $\theta \in \mathbb{R}$ $i\theta$ *is of index* 1.

E. Vesentini proved that a strongly continuous and uniformly bounded group T, such that $T_{|\mathbb{R}_+}$ is asymptotically almost periodic, is strongly almost periodic. This result, combined with that of H. Henriquez, leads to the following

PROPOSITION 3.9. *If T is a uniformly bounded and strongly continuous group such that* $T_{|\mathbb{R}_+}$ *is asymptotically* S^p*-almost periodic for some* $1 \leqslant p < +\infty$, *then T is a strongly almost periodic group.*

4. It will now be shown that the closure of the intersection between $i\mathbb{R}$ and $p\sigma(X) \cup r\sigma(X)$ is discrete, in particular in case T is weakly asymptotically S^p-almost periodic.

Recall that, if $f\colon \mathbb{R}_+ \to \mathbb{C}$ is an asymptotically S^p-almost periodic function, then for every $\varphi \in \mathbb{R}$ the map $g\colon t \mapsto e^{-it\varphi} f(t)$ from $\mathbb{R}_+$ to $\mathbb{C}$ is asymptotically S^p-almost periodic, *i.e.* the application $\tilde{g}$ from $\mathbb{R}_+$ in $L^p([0, 1]; \mathbb{C})$ is asymptotically almost periodic; for every $\varepsilon > 0$ there exists, therefore, an inclusion length $l > 0$, which depends on φ, containing an ε-period for $\tilde{g}$; l will be denoted by $l(\varepsilon, \varphi)$.

THEOREM 4.1. *Let the semigroup T be of class* (A). *If the function $t \mapsto \langle T(t)x, \lambda\rangle$ is asymptotically S^p-almost periodic for some $1 \leqslant p < +\infty$, for every $x \in \mathcal{D}(X)$ and $\lambda \in \mathcal{D}(X^+)$, and if the following conditions hold*:

i) *there are $x_0 \in \mathcal{D}(X)$, $\lambda_0 \in \mathcal{D}(X^+)$ and $\zeta \in \mathbb{C}$ such that $\langle x_0, \lambda_0\rangle \neq 0$ and either x_0 is an eigenvector of X with eigenvalues ζ, or λ_0 is an eigenvector of X^+ with eigenvalue ζ*;

ii) *for some $0 < \varepsilon < \sqrt{2}\,|\langle x_0, \lambda_0\rangle|$ it holds* $\sup\{l(\varepsilon, \varphi)\colon \varphi \in \mathbb{R}\} < +\infty$;

then the set $(p\sigma(X) \cup p\sigma(X'))\cap i\mathbb{R}$ has no accumulation point.

PROOF. Since the function $t \mapsto \langle T(t)x_0, \lambda_0\rangle$ is asymptotically S^p-almost periodic, for every $\varepsilon > 0$ there are $l > 0$ and $K \geqslant 0$ such that, for all $s \geqslant 0$, the interval $[s, s + l]$ contains some τ for which:

$$\left(\int_0^1 |\langle T(t+\tau+u)x_0, \lambda_0\rangle - \langle T(t+u)x_0, \lambda_0\rangle|^p\, du\right)^{1/p} \leqslant \varepsilon \tag{4.1}$$

whenever $t, t + \tau \geqslant K$, *i.e.*

$$\left(\int_0^1 |e^{\zeta(t+\tau+u)} - e^{\zeta(t+u)}|^p\, du\right)^{1/p} |\langle x_0, \lambda_0\rangle| \leqslant \varepsilon\,,$$

that is

$$|e^{\zeta\tau} - 1|\; |\langle x_0, \lambda_0\rangle| \left(\int_0^1 |e^{\zeta(t+u)}|^p\, du\right)^{1/p} \leqslant \varepsilon\,,$$

which holds whenever (4.1) holds. If $\zeta = \xi + i\theta$, for some $\xi, \theta \in \mathbb{R}$, then the last inequality is equivalent to:

$$|e^{\zeta\tau} - 1|\; |\langle x_0, \lambda_0\rangle| \left(\int_0^1 |e^{\xi(t+u)}|^p\, du\right)^{1/p} \leqslant \varepsilon\,. \tag{4.2}$$

Fix $t > K$. Let s increase to $+\infty$ (and, therefore, also τ). If $\Re\zeta > 0$, then $|e^{\zeta\tau} - 1|$ tends to infinity and this yields a contradiction, and therefore $\xi \leqslant 0$. Suppose now

$\xi = 0$. Then (4.2) becomes

$$|e^{i\theta\tau} - 1|\,|\langle x_0, \lambda_0\rangle| < \varepsilon\,,$$

which, exactly as in [13], if $0 < \varepsilon < \sqrt{2}\,|\langle x_0, \lambda_0\rangle|$ leads to the inequality $|\theta| > \pi/l$.

For every $\phi \in \mathbb{R}$, the semigroup $\{e^{-it\phi}T(t)\}$, generated by $X - i\phi$, is also of class (A) and satisfies the hypotheses of theorem. In order to prove the thesis, it suffices to apply the result above to $X - i\phi$ and $X^+ - i\phi$, whose point spectra are the images of $p\sigma(X)$ and of $p\sigma(X^+)$ under the translation by $-i\phi$, and then to apply (3.1). ■

Observe that condition *ii)* in Theorem 4.1, combined with asymptotic S^p-almost periodicity of the family of maps $t \mapsto \langle T(t)x, \lambda\rangle$ for some $1 \leqslant p < +\infty$, for every $x \in \mathcal{D}(X)$ and $\lambda \in \mathcal{D}(X^+)$, does not imply uniform asymptotic S^p-almost periodicity for that family.

COROLLARY 4.2. *If the semigroup T of class* (A) *is weakly asymptotically S^p-almost periodic for some $1 \leqslant p < +\infty$, then the set $(p\sigma(X) \cup p\sigma(X')) \cap i\mathbb{R}$ has no accumulation point.*

Recall that a sequence of positive numbers $\{\zeta_n : n \in \mathbb{N}\}$ is said to be *lacunary* if there exists $\varrho > 1$ for which $\zeta_{n+1} > \varrho\zeta_n$ for every n.

Let f: $\mathbb{R} \to \mathcal{E}$ be an almost periodic function. According to a result given by C. Corduneanu for scalar valued, almost periodic functions in the sense of Bohr, whose proof holds also for vector-valued functions, if the set of all frequencies of f has the infinity as unique limit point and if the absolute values of the frequencies constitute a lacunary sequence, then the Fourier series of f

$$\sum_{n=1}^{+\infty} a_n e^{i\theta_n \bullet}$$

converges to f uniformly on $\mathbb{R}$.

Let now f: $(0, +\infty) \to \mathcal{E}$ be an asymptotically S^p-almost periodic function. Then, by Theorem 1.2, there are an S^p-almost periodic function g and a map $q \in L^p_{\mathrm{loc}}(\mathbb{R}_+ ; \mathcal{E})$ such that $f(t) = g(t) + q(t)$ for all $t > 0$. In [6] it is proved that, if $q \in L^p_{\mathrm{loc}}(\mathbb{R}_+ ; \mathcal{E})$ is such that $\tilde{q} \in C_0(\mathbb{R}_+ ; L^p([0, 1]; \mathcal{E}))$ and if w: $\mathbb{R}_+ \to \mathbb{C}$ is a continuous function such that $|w(t)| \leqslant 1$, then

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t w(s)q(s)\,ds = 0\,,$$

and therefore the Fourier coefficients of f coincide with those of its principal term g.

Suppose now T is a semigroup of class (A) and of type $\omega_0 \leqslant 0$ satisfying h_1) and h_2), and such that $t \mapsto \langle T(t)x, \lambda\rangle$ is asymptotically S^p-almost periodic for some $1 \leqslant p < +\infty$, for every $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$. Let $\{\theta_1, \theta_2, \dots\}$ be an ordering of the countable set $[(p\sigma(X) \cup r\sigma(X)) \cap i\mathbb{R}]/i$. For $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}^+$, the Fourier series of the principal term of the function $t \mapsto \langle T(t)x, \lambda \rangle$ is

$$\sum_{n=1}^{+\infty} e^{i\theta_n t} \langle x, R_{i\theta_n} \lambda \rangle, \tag{4.3}$$

as a consequence of Corollary 2.5. Thus the convergence theorem of Corduneanu yields:

THEOREM 4.3. *If the set of the numbers belonging to* $[(p\sigma(X) \cup r\sigma(X)) \cap i\mathbb{R}]/i$, *taken in absolute value, is a lacunary sequence, then the Fourier series* (4.3) *converges uniformly on* $\mathbb{R}$ *to the principal term of the function* $t \mapsto \langle T(t)x, \lambda \rangle$ *for every* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}^+$.

COROLLARY 4.4. *If the group* T *is weakly* S^p*-almost periodic, under the same hypothesis of Theorem* 4.3 *the Fourier series* (4.3) *converges uniformly on* $\mathbb{R}$ *to* $t \mapsto \langle T(t)x, \lambda \rangle$ *for every* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}^+$.

COROLLARY 4.5. *If the group* T *is strongly* S^p*-almost periodic, under the same hypothesis of Theorem* 4.2 *the Fourier series*

$$\sum_{n=1}^{+\infty} e^{i\theta_n t} P_{i\theta_n} x$$

converges uniformly on $\mathbb{R}$ *to the function* $t \mapsto T(t)x$ *for every* $x \in \mathcal{E}$.

5. Also the results that E. Vesentini obtained for uniformly asymptotically almost periodic semigroups can be proved for a semigroup T of class (A) and of type $\omega_0 \leqslant 0$ satisfying h_1) and h_2), such that the family of maps $\{t \mapsto \langle T(t)x, \lambda \rangle : x \in \mathcal{E}, \lambda \in \mathcal{E}^+\}$ be uniformly asymptotically S^p-almost periodic for some $1 \leqslant p < +\infty$.

This means that for every $\varepsilon > 0$ there exists $l > 0$ such that for every $s \geqslant 0$, the interval $[s, s+l]$ contains some τ for which:

$$\left(\int_0^t |\langle T(t+\tau+u)x, \lambda \rangle - \langle T(t+u)x, \lambda \rangle|^p \, du \right)^{1/p} \leqslant \varepsilon$$

for all $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}^+$, for every $t, t+\tau \geqslant K$, for some $K = K(\varepsilon, x, \lambda)$.

If θ is a frequency of $t \mapsto \langle T(t)x, \lambda \rangle$, then $i\theta \in [p\sigma(X) \cup p\sigma(X^+)] \cap i\mathbb{R}$. Let τ be a Stepanov ε-period for this function.

If $i\theta \in p\sigma(X)$, then there is some $x \in \mathcal{D}(X)$ for which $Xx = i\theta x$. As in n. 4, this implies

$$|\langle x, \lambda \rangle| \, |e^{\xi\tau} - 1| < \varepsilon, \tag{5.1}$$

for all $\lambda \in \mathcal{E}'$. Analogously, if $i\theta \in p\sigma(X^+)$, then, for some $\lambda \in \mathcal{D}(X^+)$ and for every $x \in \mathcal{E}$, (5.1) holds. According to the definition given by Y. Meyer, a subset of real numbers Λ is called *harmonious* if for each $\varepsilon > 0$ the set

$$\bigcap_{\lambda \in \Lambda} \{\tau : |e^{i\tau\lambda} - 1| \leqslant \varepsilon\}$$

is relatively dense in $\mathbb{R}$. Thus

THEOREM 5.1. *Let T be a semigroup of class* (A) *and of type $\omega_0 \leqslant 0$, satisfying h_1) and h_2). If the family $\{t \mapsto \langle T(t)x, \lambda \rangle : x \in \mathcal{E}, \lambda \in \mathcal{E}^+\}$ is uniformly asymptotically S^p-almost periodic for some $1 \leqslant p < +\infty$, then $\{[p\sigma(X) \cup p\sigma(X')] \cap i\mathbb{R}\}/i$ is harmonious.*

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