Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

Maria Letizia Bertotti, Sergey V. Bolotin

Doubly asymptotic trajectories of Lagrangian systems and a problem by Kirchhoff

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **8** (1997), n.2, p. 93–100.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1997_9_8_2_93_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1997.

Rend. Mat. Acc. Lincei
s. 9, v. 8:93-100 (1997)

Analisi matematica. — *Doubly asymptotic trajectories of Lagrangian systems and a problem by Kirchhoff.* Nota di MARIA LETIZIA BERTOTTI e SERGEY V. BOLOTIN, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — We consider Lagrangian systems with Lagrange functions which exhibit a quadratic time dependence. We prove the existence of infinitely many solutions tending, as $t \to \pm\infty$, to an «equilibrium at infinity». This result is applied to the Kirchhoff problem of a heavy rigid body moving through a boundless incompressible ideal fluid, which is at rest at infinity and has zero vorticity.

KEY WORDS: Lagrangian systems; Routh method; Doubly asymptotic trajectories; Calculus of variations.

RIASSUNTO. — *Soluzioni doppiamente asintotiche di sistemi lagrangiani ed un problema di Kirchhoff.* Consideriamo sistemi lagrangiani con funzione lagrangiana dipendente in modo quadratico dal tempo. Proviamo l'esistenza di infinite soluzioni che tendono, quando $t \to \pm\infty$, ad un «equilibrio all'infinito». Il risultato è applicato al problema di Kirchhoff di un corpo rigido mobile in un fluido ideale incomprimibile illimitato, che è in quiete all'infinito ed ha vorticità nulla.

1. INTRODUCTION AND FORMULATION OF THE MAIN RESULT

Motivated by a classical problem, which is described in § 2, we consider in this *Note* a class of Lagrangian systems, whose Lagrange function is quadratic in time. Specifically, we assume the n-dimensional configuration manifold M to be compact and the Lagrange function $L \in C^2(TM \times \mathbb{R})$ to be of the form

$$L(x, \dot{x}, t) = K(x, \dot{x}) + t\langle w(x), \dot{x}\rangle + t^2 U(x) + V(x)\,. \tag{1.1}$$

Here $K(x, \dot{x}) = \langle B(x)\dot{x}, \dot{x}\rangle/2$, $B(x)$ is a symmetric positive definite operator for all $x \in M$, $\langle\cdot,\cdot\rangle$ denotes the pairing between covectors and vectors, w is a covector field on M, and U and V are functions on M. The equations of motion are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0\,. \tag{1.2}$$

Although this formula is written in local coordinates, the left hand side is a well defined covector independent of the choice of local coordinates.

A system with the Lagrangian (1.1) has no equilibria unless the equations

$$\nabla U(x) = 0 \qquad \text{and} \qquad \nabla V(x) = w(x)$$

have a common solution. Even if this is not the case, there are solutions similar to equilibria. Indeed (as one can easily see by performing a transformation of time $\tau = t^2/2$ and by taking the limit for $\tau \to \infty$) as $t \to \pm\infty$, the system of equations (1.2) tends to an autonomous system with Lagrange function $L_\infty(x, x') = K(x, x') + \langle w(x), x'\rangle + U(x)$. Thus, any critical point $\bar{x}$ of U may be thought of as an

(*) Nella seduta del 10 gennaio 1997.

«equilibrium at infinity» and it makes sense to look for solutions of (1.2), which asymptotically tend to $\bar{x}$ in the past and in the future.

We establish the existence of infinitely many doubly asymptotic solutions, under a condition, which we describe next.

Suppose that there exists a unique minimum point x_0 of the function U on M and suppose that it is nondegenerate. Without loss of generality, we calibrate the Lagrange function by adding a time derivative so that $w(x_0) = 0$ and $\nabla w(x_0)$ is an antisymmetric operator.

Define then a function W on M by the formula

$$W(x) = U(x) - \langle w(x), B(x)^{-1} w(x)\rangle / 2 . \tag{1.3}$$

Under the assumptions above, the point x_0 is a critical point of W, but not necessarily a minimum.

THEOREM 1.1. *Let $x_0 \in M$ be a nondegenerate unique minimum point of W. Then there exist infinitely many trajectories of the system* (1.2), *such that $x(t) \to x_0$ and $\dot{x}(t) \to 0$ as $t \to \pm\infty$.*

The proof of Theorem 1.1 relies on the use of variational methods. In § 3 below, we give an outline of it and we refer to [2] for the complete proof, for more details and for other results.

REMARKS.

1. The condition on W in Theorem 1.1 cannot be removed. In [2] an example is given showing that, in general, if x_0 is a nondegenerate minimum of U, but not of W, then Theorem 1.1 doesn't hold.

2. The function W is an analog of the Hagedorn function [8], which plays a role in the stability theory for Lagrangian systems with gyroscopic forces. An analogous function for time-dependent Lagrangian systems was introduced in [4], where the existence of asymptotic solutions was established by variational methods. For time-periodic Lagrangian systems, under an assumption similar to that of Theorem 1.1, the existence of one homoclinic solution has been proved in [3].

3. The compactness assumption on the manifold M can be removed if suitable additional completeness conditions are postulated.

We illustrate next a general situation, in which systems of the type we study arise.

Imagine we start with a natural mechanical system [1]: we have a configuration manifold N, a kinetic energy $T = T(q, \dot{q})$ of class C^2, which is a positive definite homogeneous quadratic form in the velocity $\dot{q}$ and a generalized force Q of class C^1, which is a covector field on N.

Suppose that the kinetic energy is invariant under a one-parameter transformation group $\mathbb{R}$ and the force field $Q = \nabla\phi$ is potential and invariant under the action of the same group. Let v be the vector field on N corresponding to the symmetry group. Suppose that the projection $F = \langle Q, v \rangle$ of the force field on v is a nonzero constant. So, the potential ϕ is not invariant. The fibration of N to the orbits of the group action turns out to be trivial: N is diffeomorphic to $M\{x\} \times \mathbb{R}\{y\}$, where M is a smooth manifold, and the group action corresponds to the translation $(x, y) \in M \times \mathbb{R} \to (x, y + s) \in \in M \times \mathbb{R}$, $s \in \mathbb{R}$. We identify N with $M \times \mathbb{R}$. Then $\phi = V(x) + Fy$ and the force field Q takes the form $Q(x, y) = (\nabla V(x), F) \in T_x^* M \times \mathbb{R}$, where V is a C^2 function on M. Thus we have a constant generalized force F in the direction of the coordinate y. Since the kinetic energy doesn't depend on y, the coordinate y can be actually ignored in determining x along the solutions. For that reason we refer to y as to a cyclic coordinate.

The kinetic energy T is a C^2 function on $TM \times \mathbb{R}$:

$$T(x, \dot{x}, \dot{y}) = \langle A(x)\dot{x}, \dot{x} \rangle / 2 + \langle b(x), \dot{x} \rangle \dot{y} + c(x)\dot{y}^2/2\,,$$

where $A(x)$ is a symmetric positive definite operator for all $x \in M$, b is a covector field on M and c is a positive function on M.

Let $p_y = T_{\dot{y}} = \langle b(x), \dot{x} \rangle + c(x)\dot{y}$ be the generalized momentum corresponding to the coordinate y. Then the system of Lagrange's equations takes the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = \frac{\partial V}{\partial x}\,, \qquad \dot{p}_y = F\,. \tag{1.4}$$

Solving the second equation in (1.4) yields $p_y(t) = Ft + p_y(0)$. Performing a time shift, without loss of generality, we put $p_y(t) = Ft$. Now, we are in the position to apply the classical Routh method of ignoring a cyclic coordinate y. Since T is positive definite in the velocity, the equation $p_y = Ft$ can be solved for $\dot{y}$ in terms of x, $\dot{x}$ and t: $\dot{y} = g(x, \dot{x}, t) = (Ft - \langle b(x), \dot{x} \rangle)/c(x)$. The Routh function L is then defined on $TM \times \mathbb{R}$ as

$$L(x, \dot{x}, t) = (T(x, \dot{x}, \dot{y}) - Ft\dot{y})\big|_{\dot{y} = g(x, \dot{x}, t)} + V(x) =$$
$$= (\langle A(x)\dot{x}, \dot{x} \rangle - \langle b(x), \dot{x} \rangle^2 / c(x))/2 + \frac{\langle b(x), \dot{x} \rangle}{c(x)} Ft - \frac{F^2}{2c(x)} t^2 + V(x)\,.$$

By the Routh theorem [1], the components $x(t)$ of the trajectories of the natural mechanical system with $p_y(t) = Ft$ satisfy the nonautonomous Lagrangian system (1.2) with Lagrange function L and configuration space M. Notice that L is of the form (1.1).

2. Kirchhoff problem

The concrete physical problem which prompted our interest concerns a rigid body moving in an infinite volume of incompressible ideal fluid. This problem was studied in the nineteenth century by Kirchhoff, who developed the model, which is our starting point [9, 12].

Assume that the fluid is at rest at infinity and has zero vorticity, so that the flow is potential and assume the impenetrability and attachment of the flow on the surface of the rigid body. Under these conditions, the motion on the fluid is completely determined by the motion of the body. In turn, according to Kirchhoff model, the effect of the fluid pressures on the surface of the rigid body can be represented by an addition to the «inertia» of the body as explained below. The body and the fluid can be treated together as a six degrees of freedom Lagrangian system, whose configuration space N is diffeomorphic to $SO(3) \times \mathbb{R}^3$.

Let $-P\gamma$, where γ is the unit vertical vector, be the sum of the weight of the body and the Archimedus force. Depending on the density of the body, P can be positive or negative. Denote by O the point of the body where the force $P\gamma$ is applied. Let ω and v be the angular velocity of the body and the velocity of the point O with respect to an inertial system (represented in a coordinate frame connected to the rigid body). The kinetic energy of the system body-fluid is given by Kirchhoff's formula [9, 12]:

$$T = \langle A\omega, \omega\rangle/2 + \langle B\omega, v\rangle + \langle Cv, v\rangle/2, \tag{2.1}$$

where the matrices A, B, C are constant and the symmetric matrices A, C and

$$D = A - B^T C^{-1} B \tag{2.2}$$

are positive definite. The matrix A is a sum of the inertia tensor of the body and an additional inertia tensor corresponding to the liquid. The eigenvalues of the matrix C are sums of the mass of the body and the so called additional masses, certain constant coefficients describing inertial properties of the fluid, which are determined by the configuration of the solid surface.

The kinetic energy T is invariant under the symmetry group $SO(3) \times \mathbb{R}^3$ acting on the configuration space by rotations and translations. The force field $-P\gamma$ is invariant under the action of the group $S^1 \times \mathbb{R}^3$, where $S^1 \subset SO(3)$ corresponds to rotations about the vertical. Thus we are in a situation similar to those described in § 1. The force $-P\gamma$ plays the role of the generalized force Q in § 1 and the coordinate y is now the height of the point O. The only difference is that the symmetry group is $S^1 \times \mathbb{R}^3$ and so there are additional integrals of motion. Let $p_\alpha = \langle p, \alpha\rangle$, $p_\beta = \langle p, \beta\rangle$ and $p_\gamma = \langle p, \gamma\rangle$, where α and β denote two horizontal fixed vectors, be the components of momentum $p = T_v$. Then p_α, p_β and $p_\gamma + Pt$ are integrals of motion. Without loss of generality we can put $p_\gamma = -Pt$. For simplicity we assume that the horizontal momentum is zero: $p_\alpha = p_\beta = 0$. We also assume that the integral of vertical angular momentum $J_\gamma = \langle J, \gamma\rangle$, where $J = T_\omega$, is zero. The last assumption is necessary for the variational methods to work. In contrast, the subsequent results can be generalized to the case of nonzero p_α and p_β: only, the Routh function contains in that case an additional term.

The quotient manifold which gives the reduced configuration space is

$$M = (SO(3) \times \mathbb{R}^3)/(S^1 \times \mathbb{R}^3) \cong S^2 = \{\gamma \in \mathbb{R}^3 : |\gamma| = 1\}.$$

The Routh function on $TS^2 \times \mathbb{R}$ turns out to be:

$$L(\gamma, \dot\gamma, t) = \frac{\det D}{2} \frac{\langle D^{-1}\dot\gamma, \dot\gamma\rangle}{\langle D\gamma, \gamma\rangle} - Pt \frac{\langle D\gamma \times B^T C^{-1}\gamma, \dot\gamma\rangle}{\langle D\gamma, \gamma\rangle} + t^2 U(\gamma), \tag{2.3}$$

where D is the matrix (2.2) and

$$U(\gamma) = -\frac{P^2}{2}\left(\langle C^{-1}\gamma, \gamma\rangle + \frac{\langle C^{-1}B\gamma, \gamma\rangle^2}{\langle D\gamma, \gamma\rangle}\right). \tag{2.4}$$

Here, the unit vertical vector $\gamma \in S^2$ has to be thought of as represented in a coordinate frame connected to the body.

We are therefore dealing with a Lagrangian of the form (1.1).

Since the function W in (1.3) turns out to be rather complicated and it is also not easy to describe the critical points of the function U in (2.4) explicitly, for simplicity we assume that $B = 0$. Then the Routh function (2.3) takes the simple form

$$L(\gamma, \dot\gamma, t) = \frac{\det A}{2}\frac{\langle A^{-1}\dot\gamma, \dot\gamma\rangle}{\langle A\gamma, \gamma\rangle} + t^2 U(\gamma), \qquad U(\gamma) = -\frac{P^2}{2}\langle C^{-1}\gamma, \gamma\rangle. \tag{2.5}$$

In this case, since $L(-\gamma, -\dot\gamma, t) = L(\gamma, \dot\gamma, t)$, we can regard the system as a Lagrangian system on the projective plane $\mathbb{R}P^2$.

The quadratic form U has three pairs of critical points on S^2. Let c_i be the eigenvalues of the matrix C and e_i the corresponding unit eigenvectors. If c_1 is the smallest eigenvalue and $c_1 < c_{2,3}$, then $\gamma_\pm = \pm e_1$ are the equilibrium points given by the minimum of U on S^2. Correspondingly, there is a unique minimum point of U on $\mathbb{R}P^2$. We distinguish two classes of doubly asymptotic trajectories: doubly asymptotic to the same point γ_+ or γ_- as $t \to \pm\infty$, or connecting different points γ_+ and γ_-. The former solutions may be called homoclinics, the latter ones heteroclinics. Theorem 1.1 implies the following proposition.

Proposition 2.1. *If $B = 0$ and $c_1 < c_{2,3}$, there exists an infinite number of homoclinic motions of the body such that $\gamma(t) \to e_1$ and $\dot\gamma(t) \to 0$ as $t \to \pm\infty$ and also an infinite number of heteroclinic motions such that $\gamma(t) \to \pm e_1$ and $\dot\gamma(t) \to 0$ as $t \to \pm\infty$.*

The doubly asymptotic solutions correspond to the following motions of the rigid body [10]. As $t \to +\infty$, the body is falling down in the fluid (or floating up, depending on whether P is positive or negative), its mass center moves asymptotically along a straight line and the solid doesn't rotate. The same holds for $t \to -\infty$, with reversed time direction. The homoclinic and heteroclinic trajectories differ in the following way: for homoclinic solutions the orientation of the body is the same, up to a rotation about the vertical, as $t \to \pm\infty$ and for heteroclinic ones the body turns upside down as $-\infty < t < \infty$. Physically, the direction of e_1 and $-e_1$ is that, in which the resistance of the fluid is minimal.

Example. Suppose that the body has three orthogonal symmetry planes and the distribution of masses is also symmetric. Then $B = 0$ and the Routh function takes the form (2.5), where the matrices A and C can be diagonalized simultaneously: $A = \operatorname{diag}(a_1, a_2, a_3)$ and $C = \operatorname{diag}(c_1, c_2, c_3)$. Hence the Lagrange's equations with the Lagrangian (2.5) have three invariant submanifolds in the phase space TS^2, namely $N_i = \{(\gamma, \dot\gamma) \in TS^2 : \gamma_i = \dot\gamma_i = 0\}$ with $i = 1, 2, 3$. For example, on N_3 we can use the generalized coordinate $\varphi \bmod 2\pi$ defined by $\gamma_1 = \cos\varphi$ and $\gamma_2 = \sin\varphi$. Passing to the projective plane means performing the transformation $\theta = 2\varphi$. Then, up to a function of t and a constant multiplier, the Lagrangian $L_3 = L|_{N_3}$ takes the form $L_3 = \dot\theta^2/2 + kt^2(1-\cos\theta)$ with $k = P^2(c_2 - c_1)/(c_1 c_2 a_3)$. The equation of motion is:

$$\ddot\theta = kt^2 \sin\theta\,. \tag{2.6}$$

In this case, Theorem 1.1 yields the following

PROPOSITION 2.2. *For any $m \in \mathbb{Z}$ there exists a solution $\theta(t)$ of equation* (2.6) *such that* $\lim_{t\to-\infty} \theta(t) = 0$ *and* $\lim_{t\to\infty} \theta(t) = 2\pi m$. *Thus the body performs $m/2$ full rotations around a horizontal axis. For even m these solutions are homoclinic trajectories and for odd m heteroclinic ones.*

Equation (2.6) was first obtained and studied by Chaplygin [5]. The qualitative properties of its solutions were analyzed by Kozlov [10, 11]. In particular, in [10] Proposition 2.2 was proved for $m = 1$ via a minimization of the Hamilton action as in [4]. In the same paper, Kozlov posed the question, whether the number of solutions $\theta(t)$ which are doubly asymptotic to 0 is infinite.

3. SKETCH OF THE PROOF OF THEOREM 1.1

According to Hamilton's variational principle, we represent doubly asymptotic solutions of (1.2) as critical points of the action functional on a suitable function space. The manifold M can be smoothly embedded in $\mathbb{R}^N$ for $N = 2n+1$ and, up to a translation, x_0 can be assumed to coincide with the origin of $\mathbb{R}^N$. Denote by $\langle\cdot,\cdot\rangle = |\cdot|^2$ the Euclidean metric in $\mathbb{R}^N$, and also its restriction to M. Denote

$$\mathcal{H} = \{q \in AC(\mathbb{R}, \mathbb{R}^N) \colon \|q\| < \infty\}\,,$$

where $AC(\mathbb{R}, \mathbb{R}^N)$ is the set of absolutely continuous curves in $\mathbb{R}^N$ and

$$\|q\|^2 = \int_{-\infty}^{+\infty} (|\dot q(t)|^2 + t^2 |q(t)|^2)\, dt\,.$$

Let

$$\mathcal{M} = \{q \in \mathcal{H} \colon q(t) \in M \text{ for all } t \in \mathbb{R}\}\,.$$

The set $\mathcal{H}$ is a Hilbert space and $\mathcal{M}$ is a complete Hilbert submanifold in $\mathcal{H}$.

The Hamilton action I is defined on $\mathcal{M}$ as

$$I(q) = \int_{-\infty}^{+\infty} L(q(t), \dot q(t), t)\, dt\,, \qquad q \in \mathcal{M}\,.$$

This integral is convergent for any $q \in \mathcal{M}$.

With the help of a partition of unity, the embedding of M in $\mathbb{R}^N$ can be constructed in such a way that some neighborhood $\mathcal{N}$ of x_0 in M is contained in the linear subspace $\mathbb{R}^n \subset \mathbb{R}^N$. The fact that the function W in (1.3) has a strict nondegenerate minimum at x_0 allows to estimate $L(x, \dot x, t) \geqslant \delta(|\dot x|^2 + t^2|x|^2) + V(x)$ for some positive δ sufficiently small. In turn, this estimate can be used to prove that for any $c > 0$ there exists a $c^* > 0$ such that $\|q\|^2 \leqslant c^*$ and, consequently, $\sup_{t \in \mathbb{R}} t|q(t)|^2 \leqslant 4c^*$ for all $q \in \mathcal{M}^c = \{q \in \mathcal{M}: I(q) < c\}$. This implies that for any $c > 0$ there exists $T = T(c) > 0$ such that $q(t) \in \mathcal{N}$ for all $q \in \mathcal{M}^c$ and for all $|t| > T$. The restriction $I|_{\mathcal{M}^c}$ of the action functional may be splitted as

$$I(q) = J_-(q_-) + J_T(q_T) + J_+(q_+), \qquad q \in \mathcal{M}^c .$$

Here $q_- = q|_{(-\infty, -T]}$, $q_T = q|_{[-T, T]}$ and $q_+ = q|_{[T, +\infty)}$ while J_-, J_T, J_+ respectively denote the action functional on curves defined on $(-\infty, -T], [-T, T], [T, +\infty)$.

This uniform splitting is convenient for investigating the regularity and other properties of I. It allows to handle the problem of unboundedness of the time interval by working not on a manifold, but in $\mathbb{R}^n$. Indeed, since q_- and q_+ take values in $\mathbb{R}^n$, the functionals J_- and J_+ are defined on open sets of a Hilbert space. On the other hand, since the curves q_T are defined on a compact interval, J_T can be studied as for example in [13, 14].

The functional I is in $C^1(\mathcal{M})$ and the derivative I' is locally Lipschitz. Moreover, I satisfies the Palais-Smale condition: any sequence $\{q_n\} \subset \mathcal{M}$, for which $I(q_n)$ is bounded and $I'(q_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence. Of course, the quadratic time dependence of the Lagrangian and the condition on the function (1.3) are essential for this compactness property. We point out in this connection that the problem we study is much simpler than the homoclinic problem for time periodic systems. Indeed, in the periodic case the Palais-Smale sequences may have no converging subsequences and a subtle analysis of their behaviour is required for establishing the existence of infinitely many homoclinics (see *e.g.* [7] and the references therein).

In view of the mentioned properties, the Ljusternik-Schnirelmann category theory (see *e.g.* [6]) can be applied. The manifold $\mathcal{M}$ is homotopically equivalent to the loop space $\Omega(M) = \{q \in C^0([0, 1], M): q(0) = x_0 = q(1)\}$ and, since M is a compact manifold, $\operatorname{cat} \Omega(M) = \infty$. Hence, the action functional I has infinitely many critical points on $\mathcal{M}$. The critical points of I correspond to solutions of (1.2) such that $q(t) \to x_0$ as $t \to \pm\infty$. Moreover, one can show that for any solution q such that $q(t) \to x_0$ as $t \to \pm\infty$, also $\dot{q}(t) \to 0$ as $t \to \pm\infty$. Again the condition on W is necessary to draw this conclusion. The sketch of the proof is complete.

References

[1] V. I. Arnold - V. V. Kozlov - A. I. Neishtadt, *Dynamical systems III.* VINITI, Moscow 1985; English transl. in Springer-Verlag, New York-Heidelberg-Berlin 1988.

[2] M. L. Bertotti - S. V. Bolotin, *Doubly asymptotic trajectories of Lagrangian systems in homogeneous force fields.* Annali di Matematica Pura e Applicata, in press.

[3] S. V. Bolotin, *The existence of homoclinic motions.* Vestnik Moskov Univ., ser. I, Matem., Mekhan., 6, 1983, 98-103 (in Russian); English transl. in Moscow Univ. Math. Boll., 38, 1983, 117-123.

[4] S. V. Bolotin - V. V. Kozlov, *On the asymptotic solutions of the equations of dynamics.* Vestnik Moskov Univ., ser. I, Matem., Mekhan., 4, 1980, 84-89 (in Russian); English transl. in Moscow Univ. Math. Boll., 35, 1980, 82-88.

[5] S. A. Chaplygin, *On the motion of heavy bodies in an incompressible fluid, Collected papers 1.* Izd-vo Akad. Nauk SSSR, Leningrad 1933, 133-150 (in Russian).
[6] J. Eells, *A setting for global analysis.* Bull. Amer. Math. Soc., 72, 1966, 751-807.
[7] F. Giannoni - P. H. Rabinowitz, *On the multiplicity of homoclinic orbits on Riemannian manifolds for a class of second order Hamiltonian systems.* Nonlinear Differential Equations and Applications, 1, 1993, 1-46.
[8] P. Hagedorn, *Über die Instabilität konservativer Systeme mit gyroskopischen Kräften.* Arch. Rat. Mech. Anal., 58, 1975, 1-9.
[9] G. Kirchhoff, *Über die Bewegung eines Rotationskörpers in einer Flüssigkeit.* J. für die reine und angewandte Mathematik, 71, 1870, 237-262.
[10] V. V. Kozlov, *On the fall of an heavy rigid body in a ideal fluid.* Meck. Tverd. Tela, 5, 1989, 10-17 (in Russian).
[11] V. V. Kozlov, *On the stability of equilibrium positions in nonstationary force fields.* J. Appl. Math. Mech., 55, 1991, 8-13.
[12] H. Lamb, *Hydrodynamics.* Dover Publications, New York 1945.
[13] R. S. Palais, *Morse theory on Hilbert manifolds.* Topology, 2, 1963, 299-340.
[14] R. S. Palais - S. Smale, *A generalized Morse theory.* Bull. Amer. Math. Soc., 70, 1964, 165-171.

M. L. Bertotti:
Dipartimento di Ingegneria Meccanica e Strutturale
Università degli Studi di Trento
Via Mesiano, 77 - 38050 Trento

S. V. Bolotin:
Department of Mathematics and Mechanics
Moscow State University
Mosca 119899 (Russia)