Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Luciano Carbone, Riccardo De Arcangelis

On integral representation, relaxation and homogenization for unbounded functionals

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **8** (1997), n.2, p. 129–135.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1997_9_8_2_129_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1997.

Rend. Mat. Acc. Lincei
s. 9, v. 8:129-135 (1997)

Calcolo delle variazioni. — *On integral representation, relaxation and homogenization for unbounded functionals.* Nota di LUCIANO CARBONE e RICCARDO DE ARCANGELIS, presentata (*) dal Socio E. Magenes.

ABSTRACT. — A theory of integral representation, relaxation and homogenization for some types of variational functionals taking extended real values and possibly being not finite also on large classes of regular functions is presented. Some applications to gradient constrained relaxation and homogenization problems are given.

KEY WORDS: Integral representation; Relaxation; Homogenization.

RIASSUNTO. — *Sulla rappresentazione integrale, il rilassamento e l'omogeneizzazione di funzionali non limitati.* Si presenta una teoria della rappresentazione integrale, rilassamento ed omogeneizzazione per alcune famiglie di funzionali variazionali a valori reali estesi, anche non finiti su ampie classi di funzioni regolari. Si forniscono alcune applicazioni a problemi di rilassamento ed omogeneizzazione con vincoli sul gradiente.

1. INTRODUCTION

In this *Note* we present some topics of a theory of integral representation for some types of variational functionals taking extended real values, and possibly being not finite also on large classes of regular functions, that we tried to develop in order to frame in a general context some classes of relaxation and homogenization problems in Calculus of Variations.

The model functional we consider comes from some gradient constrained variational problems in elastic-plastic torsion theory and electrostatics (cf. for example [12, 14, 16]) and is of the type $u \mapsto \int_\Omega f(\nabla u)\,dx$, where Ω is a regular bounded open subset of $\boldsymbol{R}^n$, u varies in a space of functions having gradients and f is a Borel function on $\boldsymbol{R}^n$ taking its values in $[0, +\infty]$. We observe explicitly that, from a variational point of view, the admissibility of the value $+\infty$ in the range of f is equivalent to the consideration of the pointwise constraint $\nabla u(x) \in \{z \in \boldsymbol{R}^n : f(z) < +\infty\}$ for a.e. $x \in \Omega$ on the gradients of the admissible functions.

When f is just real valued, problems of semicontinuity, relaxation and homogenization for such functionals are well studied in literature also in more general settings (cf. for example [1-3, 5, 9, 13, 15]) and are approached also by using abstract characterizations of the real valued functionals, depending on an open set Ω and a function u, that can be described as integrals of the type considered.

On the contrary when the functionals are admitted to take the value $+\infty$ also on bounded sets of regular functions, *i.e.* they are what we call unbounded functionals, such characterizations are no more available in literature, unless in some particular cases (cf. [8] where functionals taking the value 0 and $+\infty$ only are studied).

(*) Nella seduta del 7 febbraio 1997.

For such functional we first prove some abstract integral representation results on various sets of functions under different sets of assumptions.

We prove the adequacy of our abstract theory by obtaining several new results extending some classical ones concerning gradient constrained relaxation (cf. [13] where the constraint is a ball) and the more advanced ones concerning gradient constrained homogenization (cf. [7] where the constraint is described by a bounded nonnegative function).

Moreover the constructed general frame lets us to enlighten some apparently new phenomena (see Theorem 4 and Example 5).

Eventually we emphasize that the homogenization results are directly related to the previously recalled elastic-plastic torsion theory (cf. [2, 12]).

2. Results and comments

One of the integral representation results we prove concerns functionals defined in Sobolev spaces.

For every bounded open set Ω, $p \in [1, +\infty]$ we consider a functional $F(\Omega, \cdot)$: $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n) \to [0, +\infty]$ and introduce the following conditions (for every $z \in \boldsymbol{R}^n$, $x_0 \in \boldsymbol{R}^n$, $r > 0$ we set $u_z : x \in \boldsymbol{R}^n \mapsto z \cdot x$ and denote by $Q_r(x_0)$ the open cube of $\boldsymbol{R}^n$ with faces parallel to the coordinate planes centred in x_0 and with sidelength r)

(1) $F(\Omega, u_z + c) = F(\Omega, u_z)$ for every bounded open set Ω, $z \in \boldsymbol{R}^n$, $c \in \boldsymbol{R}$,

(2) $F(\Omega - x_0, u_z(\cdot + x_0)) = F(\Omega, u_z)$

for every bounded open set Ω, $z \in \boldsymbol{R}^n$, $x_0 \in \boldsymbol{R}^n$,

(3) $F(\Omega_1, u) \leqslant F(\Omega_2, u)$

whenever Ω_1, Ω_2 are bounded open sets with $\Omega_1 \subseteq \Omega_2$, $u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$,

(4) $F(\Omega_1, u) + F(\Omega_2, u) \leqslant F(\Omega_1 \cup \Omega_2, u)$

whenever Ω_1, Ω_2 are disjoint bounded open sets, $u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$,

(5) $F(\Omega_1 \cup \Omega_2, u) \leqslant F(\Omega_1, u) + F(\Omega_2, u)$

whenever Ω_1, Ω_2 are bounded open sets, $u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$,

(6) $F(\Omega, u) = \sup \{F(A, u) : A \text{ open set}, \overline{A} \subseteq \Omega\}$

for every bounded open set Ω, $u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$,

(7) $\limsup_{r \to 0+} r^{-n} F(Q_r(x_0), u) \geqslant F(Q_1(x_0), u(x_0) + \nabla u(x_0) \cdot (\cdot - x_0))$

for every $u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$, x_0 a.e. in $\boldsymbol{R}^n$,

(8) for every bounded open set Ω $F(\Omega, \cdot)$ is sequentially weakly-$W^{1,p}(\Omega)$

(weakly* -$W^{1,\infty}(\Omega)$ if $p = +\infty$) -lower semicontinuous,

(9) $F(\Omega_2, u) \leqslant F(\Omega_1, u)$ whenever Ω_1, Ω_2 are bounded open sets with

$\Omega_1 \subseteq \Omega_2$, $|\Omega_2 \backslash \Omega_1| = 0$, $u \in W^{1,\infty}_{\mathrm{loc}}(\boldsymbol{R}^n)$.

THEOREM 1. *Let* $p \in [1, +\infty]$. *For every bounded open set* Ω *let* $F(\Omega, \cdot)$: $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n) \to [0, +\infty]$ *be a functional verifying* (1)-(9), *then there exists* $f: \boldsymbol{R}^n \to$ $\to [0, +\infty]$ *convex and lower semicontinuous such that*

$$F(\Omega, u) = \int_\Omega f(\nabla u)\,dx \quad \text{for every bounded open set } \Omega,\ u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n). \tag{10}$$

Conversely, given $f: \boldsymbol{R}^n \to [0, +\infty]$ *convex, lower semicontinuous and defined, for every bounded open set* Ω, *the functional* $F(\Omega, \cdot)$ *by* (10), *it turns out that conditions* (1)-(9) *are fulfilled by F.*

In the case of functionals defined in BV-spaces we consider for every bounded open set Ω a functional $F(\Omega, \cdot)$: $BV_{\mathrm{loc}}(\boldsymbol{R}^n) \to [0, +\infty]$, then, having in mind the Goffman-Serrin formula that naturally extends the right-hand side of (10) to BV-functions, we look for necessary and sufficient conditions on the family $\{F(\Omega, \cdot): \Omega$ bounded open set$\}$ to deduce integral representation results of the type

$$F(\Omega, u) = \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| \tag{11}$$

$$\text{for every bounded open set } \Omega,\ u \in BV_{\mathrm{loc}}(\boldsymbol{R}^n),$$

where f^∞ is the recession function of f, ∇u denotes the density of the absolutely continuous part of the vector measure Du with respect to Lebesgue measure, $D^s u$ its singular part and $dD^s u/d|D^s u|$ is the Radon-Nikodym derivative of $D^s u$ with respect to the total variation $|D^s u|$ of $D^s u$.

We first observe by means of an example that, in general, conditions analogous to the ones proposed in the case of Sobolev spaces are not sufficient in order to achieve an integral representation result as in (11), then we introduce the following assumptions, the first two replacing (2) and (8),

$$F(\Omega - x_0, u(\cdot + x_0)) = F(\Omega, u) \tag{12}$$

$$\text{for every bounded open set } \Omega,\ u \in BV_{\mathrm{loc}}(\boldsymbol{R}^n),\ x_0 \in \boldsymbol{R}^n,$$

(13) for every bounded open set Ω

$F(\Omega, \cdot)$ is sequentially weakly* -$BV(\Omega)$-lower semicontinuous,

(14) for every bounded open set Ω $F(\Omega, \cdot)$ is convex

and prove the following result.

THEOREM 2. *For every bounded open set* Ω *let* $F(\Omega, \cdot)$: $BV_{\mathrm{loc}}(\boldsymbol{R}^n) \to [0, +\infty]$ *be a functional verifying* (1), (3)-(6) *written with* $BV_{\mathrm{loc}}(\boldsymbol{R}^n)$ *in place of* $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$, (7) *written with* $C^1(\boldsymbol{R}^n)$ *in place of* $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$, (9) *and* (12)-(14), *then there exists* $f: \boldsymbol{R}^n \to [0, +\infty]$ *convex and lower semicontinuous such that* (11) *holds.*

Conversely, given $f: \boldsymbol{R}^n \to [0, +\infty]$ *convex, lower semicontinuous and defined, for every bounded open set* Ω, *the functional* $F(\Omega, \cdot)$ *by* (11), *it turns out that conditions* (1), (3)-(6) *written with* $BV_{\mathrm{loc}}(\boldsymbol{R}^n)$ *in place of* $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$, (7) *written with* $C^1(\boldsymbol{R}^n)$ *in place of* $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$, (9) *and* (12)-(14) *are fulfilled by F.*

Such characterizations are obtained also by making use of fine measure theoretic results for increasing set functions, in particular of inner regularity nature, of a blow-up condition and of an identification property for unbounded functionals from their values on regular functions that we prove in abstract settings.

By making use of the above characterizations we obtain some gradient constrained relaxation results, a particular case of which is the following one relative to Dirichlet problems.

For every subset C of $\boldsymbol{R}^n$ and every $f\colon \boldsymbol{R}^n \to [0, +\infty]$ we denote by I_C the indicator function of C defined by $I_C(z) = 0$ if $z \in C$ and $I_C(z) = +\infty$ if $z \in \boldsymbol{R}^n \setminus C$, by f^{**} the bipolar of f, by $\operatorname{co} f$ the convex envelope of f and by $\operatorname{sc}^- f$ the lower semicontinuous envelope of f.

THEOREM 3. *Let $g\colon \boldsymbol{R}^n \to [0, +\infty[$ be continuous and verifying $|z| \leq g(z)$ for every $z \in \boldsymbol{R}^n$ and let C be a convex subset of $\boldsymbol{R}^n$ with $C^\circ \neq \emptyset$, then for every convex bounded open set Ω, $\beta \in L^\infty(\Omega)$*

$$\inf\left\{\int_\Omega g(\nabla u)\,dx + \int_\Omega \beta u\,dx \colon u \in W_0^{1,\infty}(\Omega),\ \nabla u(x) \in C \text{ for a.e. } x \in \Omega\right\} =$$

$$= \min\left\{\int_\Omega (g + I_C)^{**}(\nabla u)\,dx + \int_\Omega ((g + I_C)^{**})^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| + \right.$$

$$\left. + \int_{\partial\Omega} ((g + I_C)^{**})^\infty(-u\,\boldsymbol{n})\,dH^{n-1} + \int_\Omega \beta u\,dx \colon u \in BV(\Omega)\right\},$$

H^{n-1} being the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$ and $\boldsymbol{n}$ the unit outward vector normal to $\partial\Omega$.

Theorem 3 relies on some auxiliary results among which the following one is particularly interesting.

THEOREM 4. *Let $g\colon \boldsymbol{R}^n \to [0, +\infty[$ be continuous and C be a convex subset of $\boldsymbol{R}^n$ with $C^\circ \neq \emptyset$, then*

$$\inf\left\{\liminf_h \int_\Omega g(\nabla u_h)\,dx \colon \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\boldsymbol{R}^n),\right. \tag{15}$$

$$\left. \text{for every } h \in \boldsymbol{N}\ \nabla u_h(x) \in C \text{ for a.e. } x \in \Omega,\ u_h \to u \text{ in weak}^*\text{-}W^{1,\infty}(\Omega)\right\} =$$

$$= \int_\Omega \operatorname{co}(\operatorname{sc}^-(g + I_C))(\nabla u)\,dx \qquad \text{for every bounded open set } \Omega,\ u \in W^{1,\infty}(\Omega).$$

Theorem 4 is specified by the following example in which a phenomenon is pointed out that does not occur when no constraints are taken into account, *i.e.* $C = \boldsymbol{R}^n$; it shows that in some cases in Theorem 4 it may occur $(g + I_C)^{**}(z) < \operatorname{co}(\operatorname{sc}^-(g + I_C))(z)$ for some $z \in \boldsymbol{R}^n$, contrary to what happens in the unconstrained case where $g^{**} = \operatorname{co}(\operatorname{sc}^- g)$.

Moreover observe that it is always true that $(g + I_C)^{**}(z) = \mathrm{sc}^-(\mathrm{co}(g + I_C))(z)$ for every $z \in \boldsymbol{R}^n$.

EXAMPLE 5. Let $n = 2$, $C = \{(z_1, z_2) \in \boldsymbol{R}^2 : z_1 > 0\}$ and let g be defined by

$$(16)\qquad g : (z_1, z_2) \in \boldsymbol{R}^2 \mapsto \begin{cases} 1 & \text{if } z_1 \leqslant 0, \\ 1 - z_1 \exp(z_2^2) & \text{if } 0 < z_1 \leqslant \exp(-z_2^2), \\ 0 & \text{if } z_1 > \exp(-z_2^2), \end{cases}$$

then

$$(g + I_C)^{**} : (z_1, z_2) \in \boldsymbol{R}^2 \mapsto \begin{cases} +\infty & \text{if } z_1 < 0, \\ 0 & \text{if } z_1 \geqslant 0, \end{cases}$$

$$(17)\qquad \mathrm{co}(\mathrm{sc}^-(g + I_C)) : (z_1, z_2) \in \boldsymbol{R}^2 \mapsto \begin{cases} +\infty & \text{if } z_1 < 0, \\ 1 & \text{if } z_1 = 0, \\ 0 & \text{if } z_1 > 0. \end{cases}$$

REMARK 6. We observe that if C is as in Example 5 and g is given by (16) then $\mathrm{co}(\mathrm{sc}^-(g + I_C))$ in (17) is convex but not lower semicontinuous.

By virtue of this and of Theorem 4 it results that for every bounded open set Ω the functional in the left-hand side of (15) is not even strongly $W^{1,\infty}(\Omega)$-lower semicontinuous. This feature is due to the presence of an effective constraint, in fact, when $C = \boldsymbol{R}^n$, the left-hand side of (15) turns out to be sequentially weakly*-$W^{1,\infty}(\Omega)$-lower semicontinuous for every bounded open set Ω.

Relaxation results for nonhomogeneous Dirichlet and Neumann gradient constrained problems in Sobolev spaces and in BV-spaces are also proved.

Finally the following homogenization result for Dirichlet problems with constraints on the gradient is obtained.

Let $p \in]n, +\infty]$, let f, φ be functions verifying

$$(18)\qquad \begin{aligned} & f : (x, z) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \mapsto f(x, z) \in [0, +\infty[, \\ & f(\cdot, z) \text{ measurable and }]0, 1[^n\text{-periodic for every } z \in \boldsymbol{R}^n, f(x, \cdot) \text{ convex for a.e.} \\ & x \in \boldsymbol{R}^n, \\ & \varphi : x \in \boldsymbol{R}^n \mapsto \varphi(x) \in [0, +\infty],\ \varphi\ ]0, 1[^n\text{-periodic and in } L^p(]0, 1[^n), \\ & \text{for every } z \in \boldsymbol{R}^n,\ f(\cdot, \varphi(\cdot) z) \in L^1(]0, 1[^n), \end{aligned}$$

$f_{\mathrm{hom}} : \boldsymbol{R}^n \to [0, +\infty]$ be given by

$$(19)\qquad f_{\mathrm{hom}}(z) = \inf\left\{ \int_{]0,1[^n} f(x, z + \nabla u)\, dx : u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n),\ u\ ]0, 1[^n\text{-periodic}, \right.$$
$$\left. |z + \nabla u(x)| \leqslant \varphi(x) \text{ for a.e. } x \in]0, 1[^n \right\} \text{ for every } z \in \boldsymbol{R}^n$$

and let us introduce the following conditions

$$\text{for every } z \in \boldsymbol{R}^n \; f(\cdot, z) \in L^1(]0, 1[^n), \tag{20}$$

$$\text{for a.e. } x \in \boldsymbol{R}^n \; f(x, 0) = \min_{z \in \boldsymbol{R}^n} f(x, z). \tag{21}$$

THEOREM 7. *Let f, φ be as in* (18), *f_{hom} be given by* (19) *and, if $\{z \in \boldsymbol{R}^n : f_{\text{hom}}(z) < +\infty\}^\circ = \emptyset$, let us assume that* (20) *and* (21) *hold. Then for every bounded open set Ω, $\beta \in L^{p'}(\Omega)$ $(p' = p/(p-1)$, $p' = 1$ if $p = +\infty)$ the values*

$$\min\left\{\int_\Omega f(hx, \nabla u)\,dx + \int_\Omega \beta u\,dx : u \in W_0^{1,p}(\Omega),\ |\nabla u(x)| \leq \varphi(hx) \text{ for a.e. } x \in \Omega\right\} \tag{22}$$

converge to

$$\min\left\{\int_\Omega f_{\text{hom}}(\nabla u)\,dx + \int_\Omega \beta u\,dx : u \in W_0^{1,\infty}(\Omega)\right\} \tag{23}$$

and the minimizers of the problems in (22) *have subsequences converging uniformly in Ω to minimizers of the problem in* (23).

Again the use of the above mentioned fine inner regularity results allows also an analogous study of the homogenization of Neumann gradient constrained problems.

For sake of semplicity we exposed in this *Note* only particular cases of results holding in more general contexts. The results in their general form and the proofs will appear in a work of the authors in course of publication.

ACKNOWLEDGEMENTS

Work performed as part of the research project «Relaxation and Homogenization Methods in the Study of Composite Materials» of the Progetto Strategico C.N.R.-1995 «Applicazioni della Matematica per la Tecnologia e la Società».

REFERENCES

[1] L. AMBROSIO - G. DAL MASO, *On the relaxation in $BV(\Omega; \boldsymbol{R}^n)$ of quasi-convex integrals.* J. Funct. Anal., 109, 1992, 76-97.

[2] A. BENSOUSSAN - J. L. LIONS - G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures.* Studies in Mathematics and its Applications, 5, North-Holland, Amsterdam 1978.

[3] G. BOUCHITTÉ - G. DAL MASO, *Integral representation and relaxation of convex local functionals on $BV(\Omega)$.* Ann. Sc. Norm. Sup. Pisa, 20, 1993, 483-533.

[4] H. BREZIS - M. SIBONY, *Equivalence de deux inéquations variationnelles et applications.* Arch. Rational Mech. Anal., 41, 1971, 254-265.

[5] G. BUTTAZZO, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman Research Notes in Mathematics Series, 207, Longman Scientific & Technical, Harlow 1989.

[6] L. CARBONE - S. SALERNO, *Further results on a problem of homogenization with constraints on the gradient.* J. Analyse Math., 44, 1984-85, 1-20.

[7] A. CORBO ESPOSITO - R. DE ARCANGELIS, *Homogenization of Dirichlet problems with nonnegative bounded constraints on the gradient.* J. Analyse Math., 64, 1994, 53-96.

[8] A. Corbo Esposito - R. De Arcangelis, *A characterization of families of function sets described by constraints on the gradient.* Ann. Inst. Henri Poincaré - Analyse non linéaire, 11, 1994, 553-609.
[9] G. Dal Maso, *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and Their Applications, 8, Birkhäuser, Boston-Basel-Berlin 1993.
[10] E. De Giorgi - T. Franzoni, *Su un tipo di convergenza variazionale.* Atti Acc. Lincei Rend. fis., s. 8, 58, 1975, 842-850.
[11] E. De Giorgi - G. Letta, *Une notion générale de convergence faible pour des fonctions croissantes d'ensemble.* Ann. Sc. Norm. Sup. Pisa, 4, 1977, 61-99.
[12] G. Duvaut - J. L. Lions, *Inequalities in Mechanics and Physics.* Grundlehren der mathematischen Wissenschaften, 219, Springer-Verlag, Berlin-Heidelberg-New York 1976.
[13] I. Ekeland - R. Temam, *Convex Analysis and Variational Problems.* Studies in Mathematics and its Applications, 1, North-Holland, Amsterdam 1976.
[14] R. Glowinski - H. Lanchon, *Torsion élastoplastique d'une barre cylindrique de section multiconnexe.* J. Mécanique, 12, 1973, 151-171.
[15] C. Goffman - J. Serrin, *Sublinear functions of measures and variational integrals.* Duke Math. J., 31, 1964, 159- 178.
[16] J. Rauch - M. Taylor, *Electrostatic screening.* J. Math. Phys., 16, 1975, 284-288.

L. Carbone:
Dipartimento di Matematica e Applicazioni «R. Cacciappoli»
Università degli Studi di Napoli «Federico II»
Complesso Monte S. Angelo
Via Cintia - 80126 Napoli

R. De Arcangelis:
Dipartimento di Ingegneria dell'Informazione e Matematica Applicata
Università degli Studi di Salerno
Via S. Allende - 84081 Baronissi SA