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Characterization of the domain of an elliptic operator of infinitely many variables in $L^2(\mu)$ spaces

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Analisi matematica. — *Characterization of the domain of an elliptic operator of infinitely many variables in $L^2(\mu)$ spaces.* Nota (*) del Corrisp. GIUSEPPE DA PRATO.

ABSTRACT. — We consider an elliptic operator associated to a Dirichlet form corresponding to a differential stochastic equation of potential form. We characterize the domain of the operator as a subspace of $W^{2,2}(\mu)$, where μ is the invariant measure of the differential stochastic equation.

KEY WORDS: Elliptic equations; Kolmogorov equations; Dirichlet forms.

RIASSUNTO. — *Caratterizzazione del dominio di un operatore ellittico con infinite variabili in spazi $L^2(\mu)$.* Si considera un operatore ellittico associato alla forma di Dirichlet corrispondente a un'equazione differenziale stocastica di tipo potenziale. Si caratterizza il dominio dell'operatore come un sottospazio di $W^{2,2}(\mu)$, dove μ è la misura invariante dell'equazione differenziale stocastica.

1. INTRODUCTION

Let H, K be separable Hilbert spaces with K continuously and densely embedded on H. We are given a linear operator $A: D(A) \subset H \to H$ and a nonlinear mapping $U: K \to H$, such that

HYPOTHESIS 1. (*i*) *A is self-adjoint and there exists $\omega > 0$ such that $\langle Ax, x\rangle_H \leqslant - \omega |x|_H^2$, for all $x \in D(A)$.*

(*ii*) $A^{-1} \in \mathcal{L}_1(H)$ (1).

(*iii*) *$U \in C^2(K; H)$ and*

$$\langle D^2 U(x)y, y\rangle_H \geqslant 0, \qquad \forall y \in K.$$

We set $Q = (-1/2)A^{-1}$, and denote by $\mu_0 = \mathcal{N}(0, Q)$, the Gaussian measure with mean 0 and covariance operator Q. We set moreover

$$\mu(x) = e^{-2U(x)} \mu_0(dx).$$

As well known, see *e.g.* [3], μ and ν are the invariant measures of the stochastic systems (2)

$$dZ = AZ\,dt + dW_t, \tag{1.1}$$

and

$$dX = (AX - DU(X))\,dt + dW_t, \tag{1.2}$$

respectively. We denote by P_t, $t \geqslant 0$, the transition semigroup in $L^2(H; \nu)$ correspond-

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(1) $\mathcal{L}(H)$ is the Banach algebra of all linear bounded operators on H, endowed with the sup norm $\|\cdot\|$. By $\mathcal{L}_1(H)$ (norm $\|\cdot\|_{\mathcal{L}_1(H)}$) we mean the Banach space of all trace-class operators on H, and by $\mathcal{L}_2(H)$ (norm $\|\cdot\|_{\mathcal{L}_2(H)}$) the Hilbert space of all Hilbert-Schmidt operators in H. If $T \in \mathcal{L}_1(H)$, the trace of T is denoted by $\mathrm{Tr}\, T$.

(2) By W_t we mean the cylindrical white noise on H.

ing to equation (1.2), and by $\mathcal{A}_U$ its infinitesimal generator, formally defined as

$$\mathcal{A}_U \varphi = (1/2)\,\mathrm{Tr}\,[D^2\varphi] + \langle Ax - DU(x), D\varphi\rangle_H \,. \tag{1.3}$$

The Dirichlet form corresponding to P_t, $t \geq 0$:

$$a(\varphi, \psi) = \int_H \langle D\varphi(x), D\psi(x)\rangle \mu(dx),$$

has been extensively studied, see *e.g.* [5]. Instead, the domain of $\mathcal{A}_U$, as a linear operator on $L^2(H;\mu)$, has not be studied at our knowledge. When $U = 0$ a characterization of $\mathcal{A}_0$, was presented in [1, 2] and when H is finite-dimensional case in [4]. The main goal of this paper is to characterize the domain of $\mathcal{A}_U$ for a general U.

We will use the following notations. We shall denote by $\{e_k\}$ a complete orthonormal system of eigenvectors of A and by $\{-\mu_k\}$ the corresponding sequence of eigenvalues:

$$Ae_k = -\mu_k e_k\,, \qquad k \in \mathbf{N}.$$

We have clearly

$$Qe_k = \lambda_k e_k\,, \qquad k \in \mathbf{N},$$

where $\lambda_k = -\mu_k^{-1}/2$.

For any $k \in \mathbf{N}$ we denote by $D_k\varphi$ the derivative of φ in the direction of e_k, and we set $x_k = \langle x, e_k\rangle_H$, $x \in H$. It is well known that D_k is closable. We shall still denote by D_k its closure.

We finally recall the definition of Sobolev spaces. We denote by $W^{1,2}(H;\mu)$ the linear space of all functions $\varphi \in L^2(H;\mu)$ such that $D_k\varphi \in L^2(H;\mu)$ for all $k \in \mathbf{N}$ and

$$\int_H |D\varphi(x)|^2 \mu(dx) = \sum_{k=1}^{\infty} \int_H |D_k\varphi(x)|^2 \mu(dx) < +\infty\,.$$

$W^{1,2}(H;\mu)$, endowed with the inner product

$$\langle \varphi, \psi\rangle_1 = \int_H \varphi(x)\psi(x)\mu(dx) + \int_H \langle D\varphi(x), D\psi(x)\rangle_H \mu(dx),$$

is a Hilbert space.

In a similar way we can define the Sobolev space $W^{2,2}(H;\mu)$ consisting of all functions $\varphi \in W^{1,2}(H;\mu)$ such that $D_h D_k \varphi \in L^2(H;\mu)$ for all $h, k \in \mathbf{N}$ and

$$\int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)} \mu(dx) = \sum_{h,k=1}^{\infty} \int_H |D_h D_k \varphi(x)|^2 \mu(dx) < +\infty\,.$$

$W^{2,2}(H;\mu)$, endowed with the inner product

$$\langle \varphi, \psi\rangle_2 = \langle \varphi, \psi\rangle_1 + \sum_{h,k=1}^{\infty} \int_H D_h D_k \varphi(x)\, D_h D_k \psi(x)\, \mu(dx) =$$

$$= \langle \varphi, \psi\rangle_1 + \int_H \langle D^2\varphi(x), D^2\psi(x)\rangle_{\mathcal{L}_2(H)} \mu(dx)$$

is a Hilbert space.

We shall also need some weighted Sobolev spaces. Let $B\colon D(B)\subset H\to H$ be a self-adjoint operator such that

$$\langle Bx,x\rangle_H\geq \beta\,|x|^2\,,$$

for some $\beta>0$. Then we consider the linear operator D_B in $L^2(H;\mu)$:

$$D_B\,\varphi(x)=\sqrt{B}\,D\varphi(x)\,,\qquad x\in H\,,$$

defined on all $\varphi\in W^{1,2}(H;\mu)$ such that $D\varphi(x)\in D(\sqrt{B})$ μ-a.e. and $\sqrt{B}\,D\varphi\in L^2(H;\mu)$. It is easy to see that D_B is closable; we still denote by D_B its closure. We define $W_B^{1,2}(H;\mu)$ as the domain of the closure of $D(B)$. $W_B^{1,2}(H;\mu)$, endowed with the norm

$$\|\varphi\|^2_{W_B^{1,2}(H;\mu)}=\int_H(|\varphi(x)|^2+|\sqrt{B}\,D\varphi(x)|^2)\,\mu(dx)\,,$$

is a Banach space.

2. The main result

We start with an identity relating operator $\mathcal{A}_U$ with the Dirichlet form

$$a(\varphi,\psi)=\int_H\langle D\varphi(x),D\psi(x)\rangle_H\,\nu(dx)\,,$$

see also [5].

Proposition 2.1. *Assume that Hypothesis* 1 *holds. Then for any* $\varphi\in D(\mathcal{A}_U)$ *one has*

$$\int_H(\mathcal{A}\varphi)(x)\,\varphi(x)\,\nu(dx)=-\frac{1}{2}\int_H|D\varphi(x)|^2\,\nu(dx)\,. \tag{2.1}$$

Proof. It is enough to prove the identity

$$I:=\int_H\langle Ax-DU(x),D\varphi(x)\rangle_H\,\varphi(x)\,\nu(dx)= \tag{2.2}$$

$$=-\frac{1}{2}\int_H\mathrm{Tr}\,[D^2\varphi(x)]\,\varphi(x)\,\nu(dx)-\frac{1}{2}\int_H|D\varphi(x)|^2\,\nu(dx)\,,$$

for cylindrical functions. Let $\varphi\in L^2(H;\mu)$ be a cylindrical function:

$$\varphi(x)=\psi(x_1,\dots,x_n)\,,$$

where $n\in\mathbf{N}$ and $\psi\colon\boldsymbol{R}^n\to\boldsymbol{R}$ is a Borel function. Denoting by Q_n the orthogonal projection of Q on the linear subspace of H spanned by $e_1,\dots,e_n$, we have

$$I=(2\pi)^{-n/2}[\det Q_n]^{-1/2}\int_{H_n}\langle Ax-DU(x),D\psi(x)\rangle_H\,\psi(x)\,e^{\langle Ax,x\rangle_H-2U(x)}\,dx\,\cdot$$

$$\cdot(2\pi)^{-n/2}\,[\det Q_n]^{-1/2}\int_{H_n}\langle De^{\langle Ax,x\rangle_H-2U(x)},D\psi(x)\rangle_H\,\psi(x)\,dx\,\cdot$$

$$\cdot (2\pi)^{-n/2} [\det Q_n]^{-1/2} \sum_{k=1}^{n} \int_{H_n} \langle D_k e^{\langle Ax, x\rangle_H - 2U(x)}, D_k \psi(x)\rangle_H \psi(x)\, dx \cdot$$

$$\cdot (2\pi)^{-n/2} [\det Q_n]^{-1/2} \sum_{k=1}^{n} \int_{H_n} [D_k^2 \psi(x)\psi(x) + |D_k \psi(x)|^2] e^{\langle Ax, x\rangle_H - 2U(x)} dx =$$

$$= -\frac{1}{2}\int_H [\mathrm{Tr}\,[D^2\psi(x)]\,\psi(x) + |D\psi(x)|^2]\,\nu(dx).$$

We are now ready to give a characterization of $D(\mathcal{A}_U)$. We write $\mathcal{A}_U$ as

$$(2.3)\quad (\mathcal{A}_U \varphi)(x) = \frac{1}{2}\sum_{h=1}^{\infty} D_h^2 \varphi(x) - \sum_{h=1}^{\infty} \mu_h x_h D_h \varphi(x) - \sum_{h=1}^{\infty} D_h U(x) D_h \varphi(x).$$

THEOREM 2.2. *Assume that Hypothesis* 1 *holds. Then we have*

$$(2.4)\quad D(\mathcal{A}_U) = \Big\{\varphi \in W^{2,2}(H;\mu) \cap W_A^{1,2}(H;\mu):$$

$$\int_H \langle D^2 U(x) D\varphi(x), D\varphi(x)\rangle_H \mu(dx) < +\infty \Big\}.$$

PROOF. It is enough to prove the following identity

$$(2.5)\quad \frac{1}{2}\int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\mu(dx) - \int_H \langle D\varphi(x), AD\varphi(x)\rangle \mu(dx) =$$

$$= 2\int_H |f(x)|^2 \mu(dx) - 2\int_H f(x)\Big\langle Ax + \frac{1}{2} Q^{-1}x, D\varphi(x)\Big\rangle \mu(dx).$$

By differentiating (2.3) with respect to x_j, we obtain

$$\mathcal{A}(D_j\varphi)(x) - \mu_j D_j \varphi(x) + \sum_{h=1}^{\infty} D_j D_h U(x) D_h \varphi(x) = D_j f(x).$$

Multiplying both sides by $D_j\varphi(x)$, integrating with respect to μ and recalling (2.1) we find

$$\frac{1}{2}\int_H |DD_j\varphi(x)|^2\mu(dx) + \mu_j \int_H |D_j\varphi(x)|^2 \mu(dx) =$$

$$= \sum_{h=1}^{\infty}\int_H D_j D_h U(x) D_h\varphi(x)\,\nu\;(dx) - \int_H D_j\varphi(x) D_j f(x)\,\nu(dx).$$

Summing up on j we find

$$\frac{1}{2}\int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\mu(dx) - \int_H \langle D\varphi(x), AD\varphi(x)\rangle\mu(dx) +$$

$$+ \int_H \langle D\varphi(x), D^2U(x)D\varphi(x)\rangle \mu(dx) = \int_H f(x)\{\mathrm{Tr}\,[D^2\varphi(x)] - \langle Q^{-1}x, D\varphi(x)\rangle\}\,\mu(dx),$$

and the conclusion follows. ∎

References

[1] G. Da Prato, *Perturbations of Ornstein-Uhlenbeck semigroups.* Preprint Scuola Normale Superiore n. 39, Pisa 1996; Rendiconti del Seminario Matematico di Trieste, to appear.

[2] G. Da Prato, *Regularity results for Kolmogorov equations in $L^2(H, \mu)$ spaces and applications.* Preprint Scuola Normale Superiore n. 40, Pisa 1996; Ukrainian Mathematical Journal, to appear.

[3] G. Da Prato - J. Zabczyk, *Ergodicity for infinite dimensions.* Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1996.

[4] A. Lunardi, *On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures.* Preprint Scuola Normale Superiore, Pisa 1995; Trans. Amer. Math. Soc., to appear.

[5] Z. M. Ma - M. Rockner, *Introduction to the theory of (non symmetric) Dirichlet forms.* Springer-Verlag, 1992.

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