Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

Giuseppe Savaré, Augusto Visintin

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **8** (1997), n.1, p. 49–89.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1997_9_8_1_49_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1997.

Rend. Mat. Acc. Lincei
s. 9, v. 8:49-89 (1997)

Variational convergence of nonlinear diffusion equations: applications to concentrated capacity problems with change of phase

Memoria (*) **di** Giuseppe Savaré - Augusto Visintin

Abstract. — We study a variational formulation for a Stefan problem in two adjoining bodies, when the heat conductivity of one of them becomes infinitely large. We study the «concentrated capacity» model arising in the limit, and we justify it by an asymptotic analysis, which is developed in the general framework of the abstract evolution equations of monotone type.

Key words: Stefan problem; Concentrated capacity; Variational convergence; Subdifferential operators; Abstract evolution equations.

Riassunto. — *Convergenza variazionale di equazioni di diffusione non lineari: applicazioni ai problemi di cambiamento di fase in capacità concentrata*. Si studia la formulazione variazionale del problema di Stefan in due corpi adiacenti, in uno dei quali la conducibilità termica tende all'infinito. Utilizzando e sviluppando alcuni concetti e metodi della teoria della Γ-convergenza e delle equazioni di evoluzione astratte negli spazi di Hilbert, si riesce a giustificare il modello limite, che rientra nella classe dei problemi in «capacità concentrata».

0. Introduction

Let us consider the heat conduction in two adjoining bodies Ω_1, Ω_2 in the presence or not of a change of phase. If the thermal conductivity of Ω_2 along the normal direction to the common boundary $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ becomes infinitely large, a possible way to study the limit situation is to assume that the temperature in Ω_2 depends only on the coordinates on the surface Γ and to model the phenomenon by a system of two coupled parabolic (or elliptic, in the stationary case) equations in Ω_1 and on Γ.

This is a particular case of the wide class of the «concentrated capacity problems», according to the name introduced by Tichonov (1950) for the elliptic/parabolic boundary value problems, which involve second order tangential derivatives on the boundary. Among the many physical phenomena which can be modeled in this way, we recall the diffusion in fractured media [8], the plates and junctions in elastic multi-structures [9], the electric transmission through high conducting materials [33].

The interest of studying the Stefan model in a concentrated capacity was pointed out by Rubinstein [34] and a mathematical formulation (allowing a change of phase in Ω_2) and many related results in some particular important geometrical situations have been given by Fasano, Primicerio, and Rubinstein in [20] (see also [37, 2]).

In a recent series of papers [25-29] (for other references, comments and various related questions, see also [30]), Magenes established very general uniqueness and existence results under various assumptions on the heat diffusion in Ω_1 (in the presence or

(*) Presentata nella seduta del 13 giugno 1996 dal Socio E. Magenes.

not of a change of phase), on the topology of Γ (which may or may not coincide with a connected component of $\partial\Omega_1$) and on the boundary conditions imposed on $\partial\Omega_1\backslash\overline{\Gamma}$. The basic idea of these papers is to reduce the coupled system in Ω_1, Γ to a unique evolution equation on Γ, where the heat conduction properties are described by a suitable choice of a Riemannian metric and the source term takes account of the heat exchange between Ω_1 and Γ; this term is related to the solution itself via a non–local operator of Steklov-Poincaré type, which is also non-linear when a change of phase occurs in Ω_1. It is clear that the study of this operator can be very complicated and requires fine parabolic estimates; in particular, when Γ has a boundary and Neumann boundary conditions are imposed on the remaining part $\partial\Omega_1\backslash\overline{\Gamma}$, subtle technical difficulties arise (cf. [29]).

Our approach goes back to the original coupled problem in Ω_1, Ω_2; we shall see that the natural variational formulation (which can be re-interpreted in the framework of abstract evolution equations as developed by Brezis in [5, 6]) is well adapted to pass to the limit and the resulting problem preserves the same abstract structure. A quite general existence, uniqueness and convergence result is then given by applying the same abstract theory.

In this way we can determine how the resulting conductivity properties of Γ are influenced by its geometry and the corresponding properties of Ω_2: in the simplest case of a constant conductivity along the tangent directions, we shall see that the Riemannian metric induced on Γ does not coincide with the standard one induced by the surrounding space (as it happens in the simplified planar case studied in [20, 2]) but it also depends on the principal curvatures of Γ and on the thickness of Ω_2.

Our abstract theory is also applied to study another asymptotic limit which leads to equations in a concentrated capacity. Following the approach of [22, 33], who considered the linear case of the heat equation in a simple geometric situation, the global conductivity and the heat capacity blow up together with the shrinking of Ω_2 to Γ. When these two processes are suitable balanced, we obtain in the limit a concentrated capacity on Γ, without an explicit dependence on its geometry as above (for the modelization of different asymptotic behaviors of the conductivity, the capacity, and the thickness of the layer, see *e.g.* [35, 7, 1, 11], and the book [36]; another geometric situation, allowing self-contact domains, is studied in [38]).

We decided to develop the results we need in an abstract form, since it does not require more effort and can be employed in many different applications, such as porous media equations, homogenization of nonlinear diffusion equations (see [14]), problems where the concentrated capacity lies on manifolds of codimension higher than 1, etc.

From the abstract point of view, this possibility is equivalent to give an answer (as we try to do) to the following general question: what are the most general notions of convergence for *all* the data, which are compatible with the type of nonlinear diffusion equation we want to study. It is easy to conjecture that the general theory of the variational convergences (see [3, 12 and the references therein]) plays a fundamental role here; in particular, the convergence in the sense of Mosco (cf. [31, 32]) seems very natural because of the convex structure of the problems.

The plan of the paper is the following: in the next section we introduce more precisely the asymptotic problems we shall deal with, starting from that of transmission in Ω_1, Ω_2; our main results on the concentrated capacity models are then given in the second section. The abstract theory is presented in the third part of this paper; the fourth one contains the related proofs and the last one is devoted to detail the link between abstract and concrete situations; in the appendix we collect some useful properties of differential calculus on Γ, referring to [15, 16] for a very complete and detailed development of this argument.

The variational formulation of the problems and the links with the theory of evolution equations of monotone type are the common contribution of both authors; the variational convergence tools and the asymptotic analysis have been developed by the first author.

1. The transmission problem

We are given two disjoint (strongly) Lipschitz open sets Ω_1, Ω_2 of $\mathbb{R}^N$ such that their common boundary

$$\overline{\Gamma} := \partial\Omega_1 \cap \partial\Omega_2 \tag{1.1}$$

is the closure of a regular $(N-1)$-submanifold Γ with boundary $\Gamma' := \overline{\Gamma}\backslash\Gamma$ (1); the remaining parts of the two boundaries are denoted by

$$\Gamma_i = \partial\Omega_i\backslash\overline{\Gamma}, \qquad i = 1, 2, \tag{1.2}$$

and the exterior unit normal to $\partial\Omega_i$ is denoted by $\boldsymbol{n}_i$ (of course, $\boldsymbol{n}_1 = -\boldsymbol{n}_2$ on Γ); here and in the following we assume that the index i takes the integer values 1, 2.

We choose a pair of continuous functions $\varrho_i : \overline{\Omega}_i \mapsto \mathbb{R}$ and a pair of $N \times N$ symmetric matrices A_i, continuously depending on $x \in \overline{\Omega}_i$. We assume that ϱ_i, A_i satisfy in their domains

$$A_i = A_i^T, \qquad \text{and } \alpha \leqslant \varrho_i \leqslant M, \qquad \alpha|\boldsymbol{v}|^2 \leqslant A_i\boldsymbol{v}\cdot\boldsymbol{v} \leqslant M|\boldsymbol{v}|^2, \quad \forall \boldsymbol{v} \in \mathbb{R}^N, \tag{1.3}$$

where α, $M > 0$ are two fixed positive constants. $\overline{\boldsymbol{n}}_i := A_i\boldsymbol{n}_i$ are the related conormal vectors.

As usual for the weak formulation of the Stefan problem, we introduce two monotone functions $\beta_i : \mathbb{R} \mapsto \mathbb{R}$ satisfying

$$\beta_i(0) = 0, \qquad \liminf_{|s|\to\infty} \beta_i(s)/s > 0, \tag{1.4}$$

and, for a constant $c_\beta > 0$,

$$(\beta_i(s) - \beta_i(t))(s-t) \geqslant c_\beta|\beta_i(s) - \beta_i(t)|^2, \qquad \forall s, t \in \mathbb{R}. \tag{1.5}$$

Finally we fix a time interval $]0, T[$, $T > 0$, and we set

$$Q_i := \Omega_i \times]0,T[, \qquad \Sigma_i := \Gamma_i \times]0,T[, \qquad \Sigma := \Gamma \times]0,T[, \qquad \Sigma' := \Gamma' \times]0,T[.$$

We shall consider the following transmission problem

(1) The regularity of Γ (say C^2) is not necessary to state and solve the next transmission problem, but it will be needed by the subsequent developments. In order to fix our ideas, we shall assume that Γ' is not empty; in the other case, some technical details become simpler.

PROBLEM *TP*. *Given*

$$f_i: Q_i \mapsto \mathbb{R}, \qquad u_{0,i}: \Omega_i \mapsto \mathbb{R}$$

we look for

$$\theta_i, u_i: Q_i \mapsto \mathbb{R} \text{ with } \theta_i = \beta_i(u_i)$$

which satisfy the parabolic differential equations

$$\varrho_i(\partial u_i / \partial t) - \operatorname{div}(A_i \nabla \theta_i) = f_i, \quad \text{in } Q_i,$$

the transmission conditions

$$\theta_1 = \theta_2, \qquad \partial\theta_1/\partial\bar{\boldsymbol{n}}_1 = -(\partial\theta_2/\partial\bar{\boldsymbol{n}}_2) \quad \text{on } \Sigma,$$

the initial Cauchy conditions

$$u_i(x, 0) = u_{0,i}(x) \quad \text{in } \Omega_i,$$

and the lateral boundary conditions of variational type (i.e. *Dirichlet, Neumann or mixed*) *on the remaining parts* Σ_i; *in order to fix our ideas, we consider the Neumann case* (2)

$$\partial\theta_i/\partial\bar{\boldsymbol{n}}_i = g_i \quad \text{on } \Sigma_i.$$

The following weak formulation is naturally associated to *TP* (see [29,13]).

PROBLEM *wTP*. *If*

(1.6) $$f_i \in L^2(Q_i), \quad g_i \in L^2(\Sigma_i), \quad u_{0,i} \in L^2(\Omega_i),$$

we say that $\{(\theta_i, u_i)\}_{i=1,2}$ *is a weak solution of TP if*

(1.7) $$u_i \in L^2(Q_i), \quad \theta_i \in L^2(0, T; H^1(\Omega_i)), \quad \text{with } \theta_i = \beta_i(u_i) \text{ a.e. in } Q_i,$$

(1.8) $$\theta_1|_\Sigma = \theta_2|_\Sigma \quad \text{in the sense of traces},$$

and

(1.9) $$\sum_i \int_{Q_i} \left\{ -\varrho_i u_i \frac{\partial v_i}{\partial t} + A_i \nabla\theta_i \cdot \nabla v_i \right\} dx\, dt =$$

$$= \sum_i \left\{ \int_{\Omega_i} \varrho_i u_{0,i} v_i(x, 0)\, dx + \int_{Q_i} f_i v_i\, dx\, dt + \int_{\Sigma_i} g_i v_i\, d\mathcal{H}^{N-1}\, dt \right\}$$

for every couple of test functions $v_i \in H^1(0, T; H^1(\Omega_i))$ *with*

(1.10) $$v_1|_\Sigma = v_2|_\Sigma \quad \text{and } v_i(\cdot, T) = 0 \text{ on } \Omega_i, \quad \text{in the sense of traces}.$$

We have

PROPOSITION 1.1. *For every choice of the data* $f_i, g_i, u_{0,i}$ *satisfying* (1.6), *there exists a unique solution of the previous problem. Moreover*

(1.11) $$u_i: [0, T] \mapsto L^2(\Omega_i) \quad \text{is uniformly bounded and weakly continuous}.$$

(2) Which is more complicated from the technical point of view (cf. [29]); the other (homogeneous) boundary conditions require only small changes.

REMARK 1.2. By applying various results on the weak maximum principles, on the abstract evolution equations or on the L^1-contraction semigroups, we could give several other existence and regularity results under different assumptions on the data (3). We limit ourselves to this setting, since we are more interested to show how the fundamental structure of these equations is preserved by the limiting process we shall introduce in a moment. ■

The remaining part of this section is devoted to present two particular geometric situations, where the shape of Ω_2 and (some of) the data on Ω_2 are related to a perturbation parameter ε going to 0; the asymptotic behavior of the corresponding solutions is the object of our investigation.

In order to describe the geometric model, we introduce the following definitions and assumptions.

DEFINITION 1.3. *For every $x \in \mathbb{R}^N$ let $d_\Gamma(x)$ be the distance of x from Γ; we shall assume that*

$$(1.12) \qquad d_\Gamma(x) := \inf_{y \in \Gamma} |x - y| \quad \textit{is a function of class } C^2(\overline{\Omega}_2).$$

In particular, this regularity implies that for every point $x \in \Omega_2$ there exists a unique projection x_Γ on Γ satisfying

$$(1.13) \qquad |x - x_\Gamma| = d_\Gamma(x)$$

so that we can define a C^1 unit vector field $\boldsymbol{n}(x)$, normal to Γ (see [10, 2.5.4])

$$(1.14) \qquad \boldsymbol{n}(x) := \nabla d_\Gamma(x) = (x - x_\Gamma)/d_\Gamma(x) \ (^4).$$

For every $x \in \overline{\Omega}_2$ we call s_x the intersection of Ω_2 with the straight line passing through x and parallel to $\boldsymbol{n}(x)$:

$$s_x := \{y \in \overline{\Omega}_2 : \exists \lambda \in \mathbb{R}, y = x + \lambda \boldsymbol{n}(x)\}.$$

REMARK 1.4. Even if the requirement (1.12) about the regularity of d_Γ is not necessary to prove the following Theorem 1.7 (the differentiability of d_Γ would be enough), we stated it to unify our assumptions. (1.12) is equivalent to assume that Γ is of class C^2 and Ω_2 is contained in a suitable neighborhood of Γ, depending on its curvatures (see [15, 5.5, 5.6]). ■

First of all, we consider the case which originally motivated the introduction of the concentrated capacity models.

(3) *E.g.* if the time derivative of g_i is a square integrable function on Σ_i, we have

$$\theta_i \in H^1_{\mathrm{loc}}(]0, T]; L^2(\Omega_i)) \cap L^\infty_{\mathrm{loc}}(]0, T]; H^1(\Omega_i)),$$

the analogous global result (*i.e.* near the origin) holding if $\beta_i(u_{0,i}) \in H^1(\Omega_i)$, with $\beta_1(u_{0,1}) = \beta_2(u_{0,2})$ on Γ.

(4) Observe that $\boldsymbol{n}(x)$ is defined also on Γ, where it coincides with $\boldsymbol{n}_1(x) = -\boldsymbol{n}_2(x)$.

Case I: blow up of the normal conductivity.

For every $\varepsilon > 0$ we perturb the problem TP by adding to the conductivity matrix A_2 the normal term $\varepsilon^{-1} \boldsymbol{n}\boldsymbol{n}^T$, $\varepsilon > 0$; we obtain a family of transmission problems TP_I^ε where A_2 is replaced by

$$A_2^\varepsilon(x) := A_2(x) + \varepsilon^{-1} \boldsymbol{n}(x)\, \boldsymbol{n}^T(x)\,. \tag{1.15}$$

If $\{(\theta_i^\varepsilon, u_i^\varepsilon)\}_{\varepsilon > 0}$ is the family of the corresponding solutions of TP_I^ε, we want to prove the existence of their limit (θ_i, u_i) as ε goes to 0 and to characterize it.

To this aim it is natural to introduce the closed subspace $H_{\boldsymbol{n}}^1(\Omega_2)$ of $H^1(\Omega_2)$, consisting of functions which are constant along s_x for $\mathcal{H}^{N-1}$-a.e. $x \in \Gamma$:

$$H_{\boldsymbol{n}}^1(\Omega_2) := \{v \in H^1(\Omega_2) \colon \boldsymbol{n}^T \cdot \nabla v = 0\}\,. \tag{1.16}$$

We also set

$$L_{\boldsymbol{n}}^2(\Omega_2) := \overline{H_{\boldsymbol{n}}^1(\Omega_2)}^{L^2(\Omega_2)}\,, \qquad \Pi_{\boldsymbol{n}} := \text{orthogonal projection of } L^2(\Omega_2) \text{ on } L_{\boldsymbol{n}}^2(\Omega_2)\,,$$

and we have

THEOREM 1.5. *Let $(\theta_i^\varepsilon, u_i^\varepsilon)$, $\varepsilon > 0$, be the solution of the problems TP_I^ε previously defined; as ε goes to 0*

$$\theta_i^\varepsilon \to \theta_i \quad \textit{strongly in } L^2(Q_i) \textit{ and weakly in } L^2(0, T; H^1(\Omega_i)) \tag{1.17}$$

and, for every fixed $t \in]0, T]$

$$u_1^\varepsilon(\cdot, t) \rightharpoonup u_1(\cdot, t)\,, \qquad \Pi_{\boldsymbol{n}} u_2^\varepsilon(\cdot, t) \rightharpoonup u_2(\cdot, t)\,, \qquad \textit{weakly in } L^2(\Omega_i)\,, \tag{1.18}$$

the last convergence of (1.17) *being also strong if $u_{0,2}$ belongs to $L_{\boldsymbol{n}}^2(\Omega_2)$. Moreover, (θ_i, u_i) is the unique solution of the following (weak) limit formulation.*

PROBLEM wLP_I. *Find (θ_i, u_i) satisfying* (1.7), (1.8) *and*

$$u_2 \in L_{\boldsymbol{n}}^2(\Omega_2)\,, \qquad \theta_2 \in H_{\boldsymbol{n}}^1(\Omega_2) \qquad \textit{for a.e. } t \in]0, T[\,, \tag{1.19}$$

such that (1.9) *holds for every couple of test functions $v_i \in H^1(0, T; H^1(\Omega_i))$ with* (1.10) *and*

$$v_2 \in H_{\boldsymbol{n}}^1(\Omega_2) \qquad \textit{for a.e.} \quad t \in]0, T[\,. \tag{1.20}$$

We postpone to the next section the interpretation of this problem as the weak formulation of a system of two coupled evolution equations, one of which is set on the manifold Γ.

Now we focus our attention to the evident common structure of wTP_I^ε and wLP_I, in order to conjecture an abstract result.

Let us consider the Hilbert space

$$H := L_{\varrho_1}^2(\Omega_1) \times L_{\varrho_2}^2(\Omega_2) \text{ (5)} \tag{1.21}$$

(5) *I.e.* the L^2-spaces with respect to the measures $\varrho_i \cdot \mathcal{L}$, $\mathcal{L}$ being the usual Lebesgue measure on $\mathbb{R}^N$. By (1.3), they coincides (up to equivalent norms) with the usual $L^2(\Omega_i)$.

whose elements we denote by $U := (u_1, u_2)$. On H is defined the (cyclically) monotone operator

$$(1.22) \qquad \Lambda : H \mapsto H, \qquad \Lambda(U) := [\beta_1(u_1), \beta_2(u_2)].$$

On the linear subspace of H

$$(1.23) \qquad V := \{\Theta = (\theta_1, \theta_2) \in H^1(\Omega_1) \times H^1(\Omega_2) : \theta_1|_\Gamma = \theta_2|_\Gamma\}$$

we define the (weakly) coercive bilinear forms

$$(1.24) \qquad a_\varepsilon(\Theta, V) := \int_{\Omega_1} A_1 \nabla\theta_1 \cdot \nabla v_1 \, dx + \int_{\Omega_2} \left(A_2 \nabla\theta_2 \cdot \nabla v_2 + \frac{1}{\varepsilon} \frac{\partial \theta_2}{\partial n} \frac{\partial v_2}{\partial n} \right) dx,$$

and the time-dependent linear functionals $L(t) \in V'$, $t \in]0, T[$

$$(1.25) \qquad \langle L(t), V \rangle := \sum_i \left\{ \int_{\Omega_i} f_i(x,t) v_i dx + \int_{\Gamma_i} g_i(x,t) v_i \, d\mathcal{H}^{N-1}(x) \right\}.$$

The weak formulation wTP_I^ε consists in the search of

$$(1.26) \qquad U^\varepsilon \in L^2(0, T; H), \qquad \Theta^\varepsilon \in L^2(0, T; V)$$

such that

$$(1.27) \qquad \Theta^\varepsilon(t) = \Lambda U^\varepsilon(t) \quad \text{for a.e. } t \in]0, T[$$

and (cf. (1.9))

$$(1.28) \qquad \int_0^T \{-(U^\varepsilon, V_t) + a_\varepsilon(\Theta^\varepsilon, V)\} \, dt = (U_0, V(0))_H + \int_0^T \langle L, V \rangle \, dt$$

for any choice of $V \in H^1(0, T; V_\varepsilon)$ with $V(T) = 0$; here $U_0 := (u_{0,1}, u_{0,2})$. When $\varepsilon = 0$ we simply have to define

$$(1.29) \qquad V_0 := \{\Theta = (\theta_1, \theta_2) \in H^1(\Omega_1) \times H^1_n(\Omega_2) : \theta_1|_\Gamma = \theta_2|_\Gamma\}$$

and

$$(1.30) \qquad a_0(\Theta, V) := \int_{\Omega_1} A_1 \nabla\theta_1 \cdot \nabla v_1 \, dx + \int_{\Omega_2} A_2 \nabla\theta_2 \cdot \nabla v_2 \, dx$$

and to repeat the same requirements (1.26), ..., (1.28) (6).

The possibility of this substitution in the limit is justified by the following basic fact:

PROPOSITION 1.6. *For $\varepsilon > 0$ define $V_\varepsilon := V$ and, for $\varepsilon \geq 0$,*

$$(1.31) \qquad \mathfrak{a}_\varepsilon(\Theta) := \begin{cases} a_\varepsilon(\Theta, \Theta) & \text{if } \Theta \in V_\varepsilon, \\ +\infty & \text{if } \Theta \in H \setminus V_\varepsilon. \end{cases}$$

(6) Unlike V, V_0 is not dense in H; this fact gives rise to non-uniqueness of the component U^0 of the limit solution. Adding the further condition $U^0 \in H_0 := \overline{V_0}^H$, we overcome this difficulty, thanks to a compatibility property between Λ and (the orthogonal projection Π_0 of H onto) H_0.

Then as ε *goes to* 0, α_ε *converges to* α_0 *in the sense of Mosco* (cf. [3, Thm. 3.20 and sect. 3]). ■

We shall see in the abstract setting of the third section that the combination of this convergence (which also allows to vary A, L, and U_0 with respect to ε) with a kind of uniform coercivity on the couple a_ε, A_ε, are the good assumptions to study in a general context the asymptotic behavior of a family of nonlinear diffusion equations.

Case II: blow up of the global conductivity when Ω_2 *shrinks to* Γ.

We consider a family of contractions in the direction of the vector field $-\boldsymbol{n}(x)$ (see definition 1.3):

$$G^\varepsilon(x) := \varepsilon x + (1-\varepsilon)x_\Gamma = x_\Gamma + \varepsilon d_\Gamma(x)\,\boldsymbol{n}(x), \qquad 0 < \varepsilon \leqslant 1, \tag{1.32}$$

and we call Ω_2^ε, s_x^ε the shrinked sets

$$\Omega_2^\varepsilon := G^\varepsilon(\Omega_2), \qquad s_x^\varepsilon := G^\varepsilon(s_x).$$

As before, we have a family of problems TP_{II}^ε, where we also have to assign a varying set of data in Ω_2^ε

$$\varrho_2^\varepsilon, \qquad A_2^\varepsilon, \qquad f_2^\varepsilon, \qquad u_{0,2}^\varepsilon\,(^7), \tag{1.33}$$

while keeping fixed the remaining ones on Ω_1.

The main assumption on the data of f_2^ε, $u_{0,2}^\varepsilon$ is that they give rise, in the limit, to a suitable distribution on the lower dimensional manifold Γ. More precisely, we assume that there exist

$$\varrho_\Gamma \in C^0(\Gamma), \qquad A_\Gamma \in C^0(\Gamma; \mathbb{R}^{N\times N}); \qquad f_\Gamma \in L^2(\Sigma), \qquad u_{0,\Gamma} \in L^2(\Gamma), \tag{1.34}$$

such that

$$\begin{cases} \lim\limits_{\varepsilon\to 0} \varepsilon^{-1} \displaystyle\int_{Q_2^\varepsilon} |f_2^\varepsilon(x,t) - f_\Gamma(x_\Gamma, t)|^2\, dx\, dt = 0, \\ \lim\limits_{\varepsilon\to 0} \varepsilon^{-1} \displaystyle\int_{\Omega_2^\varepsilon} |u_{0,2}^\varepsilon(x) - u_{0,\Gamma}(x_\Gamma)|^2\, dx = 0, \end{cases} \tag{1.35}$$

and

$$\begin{cases} \lim\limits_{\varepsilon\to 0} \sup\limits_{x\in\Omega_2^\varepsilon} [\,|\varepsilon A_2^\varepsilon(x) - A_\Gamma(x_\Gamma)| + |\varepsilon\varrho_2^\varepsilon(x) - \varrho_\Gamma(x_\Gamma)|\,] = 0, \\ \text{with } A_\Gamma,\quad \varrho_\Gamma \text{ satisfying (1.3) on } \Gamma. \end{cases} \tag{1.36}$$

If $\{(\theta_i^\varepsilon, u_i^\varepsilon)\}_{\varepsilon>0}$ is the family of the solutions of the problems TP_{II}^ε, we want to characterize their limit, as $\varepsilon \to 0$, or, more precisely, the limit of

$$(\theta_1^\varepsilon, u_1^\varepsilon) \quad \text{in } Q_1, \qquad \text{and } (\bar\theta_2^\varepsilon, \bar u_2^\varepsilon) \quad \text{on } \Sigma,$$

(7) For the sake of simplicity, here we assume $g_2^\varepsilon \equiv 0$.

where, for $\mathcal{H}^{N-1}$-a.e. $x \in \Gamma$, $\overline{\theta}_2^\varepsilon(x)$, $\overline{u}_2^\varepsilon(x)$ are the mean values on s_x^ε of θ_2^ε, u_2^ε respectively:

$$\overline{\theta}_2^\varepsilon(x) := \fint_{s_x^\varepsilon} \theta_2^\varepsilon(y)\, d\mathcal{H}^1(y), \qquad \overline{u}_2^\varepsilon(x) := \fint_{s_x^\varepsilon} u_2^\varepsilon(y)\, d\mathcal{H}^1(y). \tag{1.37}$$

2. The limit problems in the concentrated capacity

Let us briefly recall some basic definitions on differential calculus and Sobolev spaces on Γ. The tangent and the normal spaces to Γ at the point x are defined by

$$\mathcal{T}_x := \{\boldsymbol{v} \in \mathbb{R}^N : \boldsymbol{n}(x) \cdot \boldsymbol{v} = 0\}, \qquad \mathcal{N}_x := \{\boldsymbol{v} \in \mathbb{R}^N : \boldsymbol{v} = \lambda \boldsymbol{n}(x), \text{ for some } \lambda \in \mathbb{R}\},$$

the orthogonal projection on $\mathcal{T}_x$ being given by

$$P_x : \mathbb{R}^N \mapsto \mathcal{T}_x, \qquad P_x \boldsymbol{v} := [I - \boldsymbol{n}(x)\boldsymbol{n}^T(x)]\boldsymbol{v}. \tag{2.1}$$

The principal curvatures $\kappa_1(x), \ldots, \kappa_{N-1}(x)$ of Γ at x are the eigenvalues, besides 0, of the differential matrix of $\boldsymbol{n}$ (see [21, p. 355])

$$S(x) := -D\boldsymbol{n}(x) = -D^2 d_\Gamma(x), \qquad \in C^0(\overline{\Omega}_2). \tag{2.2}$$

If v^* is a regular extension to Ω_2 of a function $v : \Gamma \mapsto \mathbb{R}$, it is easy to check that the tangential gradient

$$\nabla_\Gamma v : x \in \Gamma \mapsto \mathcal{T}_x, \qquad \nabla_\Gamma v(x) := P_x[\nabla v^*(x)],$$

is well defined and it is independent of the extension v^*.

A (regular) tangential vector field is a mapping

$$\boldsymbol{v} : \Gamma \mapsto \mathbb{R}^N \text{ such that } \boldsymbol{v}(x) \in \mathcal{T}_x, \qquad \forall x \in \Gamma.$$

For this kind of vector fields, we define the divergence on Γ as

$$\operatorname{div}_\Gamma \boldsymbol{v} := \operatorname{div} \boldsymbol{v}^* - (\partial(\boldsymbol{v}^* \cdot \boldsymbol{n})/\partial \boldsymbol{n}), \tag{2.3}$$

$\boldsymbol{v}^*$ being an extension of $\boldsymbol{v}$ as before; also in this case, it is possible to check that $\operatorname{div}_\Gamma \boldsymbol{v}$ does not depend on the extension $\boldsymbol{v}^*$ (cf. [15, sect. 6]). When $\boldsymbol{v}$ is not tangential, it will be useful to define

$$\operatorname{div}_\Gamma \boldsymbol{v} := \operatorname{div}_\Gamma(P_x \boldsymbol{v}). \tag{2.4}$$

If $\boldsymbol{v}$ is a regular tangential vector field and w is a regular function, the following Green's formula holds on Γ

$$-\int_\Gamma \operatorname{div}_\Gamma \boldsymbol{v} w \, d\mathcal{H}^{N-1} = \int_\Gamma \boldsymbol{v} \cdot \nabla_\Gamma w \, d\mathcal{H}^{N-1} - \int_{\Gamma'} w(\boldsymbol{v} \cdot \boldsymbol{n}')\, d\mathcal{H}^{N-2} \tag{2.5}$$

where $\boldsymbol{n}'(x)$ is the outward unit normal to Γ' in the tangent space $\mathcal{T}_x$.

Remark 2.1. In this framework, the usual Laplace-Beltrami operator, induced by the Euclidean metric on Γ, has the simple form

$$\Delta_\Gamma v := \operatorname{div}_\Gamma(\nabla_\Gamma v). \qquad \blacksquare$$

Remark 2.2. The notions of ∇_Γ, $\operatorname{div}_\Gamma$ are usually given in an intrinsic way via local coordinates, which do not require any embedding of Γ in an Euclidean space. We use this simpler (but, maybe, less elegant) approach, since it is more direct and it shows the strict relation with the differential-geometric properties of the ambient space. ■

Now we introduce the usual (Hilbertian) Sobolev spaces on Γ (for the intrinsic definitions, see [4]). $H^1(\Gamma)$ is the usual completion of $C^1(\bar{\Gamma})$ (8) with respect to the norm induced by the scalar product

$$(u,v)_{H^1(\Gamma)} := \int_\Gamma [u(x)v(x) + \nabla_\Gamma u(x)\cdot\nabla_\Gamma v(x)]\, d\mathcal{H}^{N-1}(x),$$

so that ∇_Γ becomes a linear and continuous operator between $H^1(\Gamma)$ and

$$L^2(\Gamma;\mathcal{T}(\Gamma)) := \{v \in L^2(\Gamma;\mathbb{R}^N) : v(x) \in \mathcal{T}_x, \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma\}. \tag{2.6}$$

$H_0^1(\Gamma)$ is the closure in $H^1(\Gamma)$ of the C^1-functions with compact support in Γ and $H^{-1}(\Gamma)$ is its dual space. Via (2.5) we extend div_Γ to a linear and continuous operator from $L^2(\Gamma;\mathcal{T}(\Gamma))$ to $H^{-1}(\Gamma)$. If a vector field $v \in L^2(\Gamma;\mathcal{T}(\Gamma))$ with $\mathrm{div}_\Gamma v \in L^2(\Gamma)$ satisfies

$$-\int_\Gamma \mathrm{div}_\Gamma v w\, d\mathcal{H}^{N-1} = \int_\Gamma v\cdot\nabla_\Gamma w\, d\mathcal{H}^{N-1}, \qquad \forall w \in H^1(\Gamma), \tag{2.7}$$

we say (in a formal way, but this argument could be made more precise, see [24]) that it has a vanishing normal component on Γ', that is $v\cdot n' = 0$.

We can finally state our main results on the two problems of the previous section. For the sake of simplicity, we assume that

(G1) For every $x \in \Gamma$, s_x is a segment of (regular) length $\ell(x)$, with

$$0 < \ell_0 \leq \ell(x) \leq \ell_1 < +\infty .$$

(G2) $S(x)$ is bounded on Γ and there exists a constant $\eta > 0$ such that

$$\det(I - \lambda S(x)) \geq \eta, \qquad \forall x \in \Gamma, \qquad 0 \leq \lambda \leq \ell(x).$$

(G3) $g_2(x) = 0$ on Σ_2.

THEOREM I. *Let us assume that* (G1, 2, 3) *hold together to* (1.6), *and let us denote by* (θ_i, u_i) *the limits of the solutions of* TP_I^ε *given by Theorem* 1.5. *Let* $(\tilde{\theta}_2, \tilde{u}_2)$ *denote the traces on* Σ *of* (θ_2, u_2) (9). *Then* $\{(\theta_1, u_1), (\tilde{\theta}_2, \tilde{u}_2)\}$ *is the unique weak solution of the follo-*

(8) Of course, when $\mathcal{H}^{N-1}(\Gamma) = +\infty$, we have to choose C^1 functions with compact support in $\bar{\Gamma}$.

(9) The trace operator maps continuously $H_n^1(\Omega_2)$ onto $H^1(\Gamma)$ and (it can be continuously extended by density from) $L_n^2(\Omega_2)$ onto $L^2(\Gamma)$.

wing system of coupled equations

$$\begin{cases} \theta_1 = \beta_1(u_1) & \text{in } Q_1, \\ \varrho_1 \dfrac{\partial u_1}{\partial t} - \operatorname{div}(A_1 \nabla \theta_1) = f_1 & \text{in } Q_1, \\ \dfrac{\partial \theta_1}{\partial \overline{\boldsymbol{n}}_1} = g_1 & \text{on } \Sigma_1, \\ u_1(x, 0) = u_{0,1}(x) & \text{in } \Omega_1; \end{cases}$$

$$\theta_1 = \widetilde{\theta}_2 \quad \text{on } \Sigma;$$

$$\begin{cases} \widetilde{\theta}_2 = \beta_2(\widetilde{u}_2) & \text{in } \Sigma, \\ \widehat{\varrho}_2 \dfrac{\partial \widetilde{u}_2}{\partial t} - \operatorname{div}_\Gamma (\widehat{A}_2 \nabla_\Gamma \widetilde{\theta}_2) = \widehat{f}_2 - \dfrac{\partial \theta_1}{\partial \overline{\boldsymbol{n}}_1} & \text{in } \Sigma, \\ \dfrac{\partial \widetilde{\theta}_2}{\partial \overline{\boldsymbol{n}}'} = 0 & \text{on } \Sigma', \\ \widetilde{u}_2(x, 0) = \widehat{u}_{0,2}(x) & \text{in } \Gamma, \end{cases}$$

where $\widehat{f}_2$, $\widehat{u}_{0,2}$, $\widehat{\varrho}_2$, and $\widehat{A}_2$ can be explicitly computed from the corresponding values f_2, $u_{0,2}$, ϱ_2, A_2, and from the matrix $S(x)$; $\overline{\boldsymbol{n}}'$ is the conormal vector $\overline{\boldsymbol{n}}' := \widehat{A}_2 \boldsymbol{n}'$.

We shall give the general formulae in the last section; let us consider here the special case of $N = 3$, $A_2 = I$. Using more familiar symbols, we call H and K the mean and the Gaussian curvatures of Γ (oriented by $\boldsymbol{n}$) respectively

(2.8) $$H(x) := (1/2) \operatorname{tr} S(x) = (\kappa_1(x) + \kappa_2(x))/2, \qquad K(x) := \kappa_1(x) \kappa_2(x).$$

Now for a generic point $x \in \Gamma$ we introduce the standard parametrization of the segment s_x

(2.9) $$x_\lambda := x + \lambda \boldsymbol{n}(x), \qquad 0 \leqslant \lambda \leqslant \ell(x),$$

and a deformation measure μ_x on it

(2.10) $$d\mu_x(\lambda) := [1 - 2H(x)\lambda + K(x)\lambda^2]\, d\lambda = \det(1 - \lambda S(x))\, d\lambda, \qquad \lambda \in [0, \ell(x)].$$

We have

(2.11) $$\widehat{f}_2(x, t) := \int_0^{\ell(x)} f_2(x_\lambda, t)\, d\mu_x(\lambda), \qquad \widehat{\varrho}_2(x) := \int_0^{\ell(x)} \varrho(x_\lambda)\, d\mu_x(\lambda),$$

(2.12) $$\widehat{u}_{0,2}(x) := (\widehat{\varrho}_2(x))^{-1} \int_0^{\ell(x)} u_{0,2}(x_\lambda)\, \varrho_2(x_\lambda)\, d\mu_x(\lambda),$$

and finally

$$\widehat{A}_2(x) := \int_0^{\ell(x)} (I - \lambda S(x))^{-2} \, d\mu_x(\lambda). \tag{2.13}$$

Thanks to (G1,2) and to the symmetry of $S(x)$, $\widehat{\varrho}_2$ and $\widehat{A}_2$ still verify on Γ a condition analogous to (1.3).

Remark 2.3. This problem can be studied independently from the asymptotic approach (see [25-30]): in this case $\widehat{f}_2$, $\widehat{u}_{0,2}$, $\widehat{\varrho}_2$ and $\widehat{A}_2$ are *a priori* given data and the weak formulation [29] has the same structure as in the previous section, formulae (1.21)-(1.28). Here

$$H := L^2_{\varrho_1}(\Omega_2) \times L^2_{\widehat{\varrho}_2}(\Gamma), \tag{2.14}$$

$$V_0 := \{\Theta = (\theta_1, \widetilde{\theta}_2) \in H^1(\Omega_1) \times H^1(\Gamma) \colon \theta_1|_\Gamma = \widetilde{\theta}_2\}, \tag{2.15}$$

$$a_0(\Theta, V) := \int_{\Omega_1} A_1 \nabla\theta_1 \cdot \nabla v_1 \, dx + \int_\Gamma \widehat{A}_2 \nabla_\Gamma \widetilde{\theta}_2 \cdot \nabla_\Gamma \widetilde{v}_2 \, d\mathcal{H}^{N-1} \tag{2.16}$$

and

$$\langle L(t), V\rangle := \int_{\Omega_1} f_1(x,t) v_1 \, dx + \int_{\Gamma_1} g_1(x,t) v_1 \, d\mathcal{H}^{N-1}(x) + \int_\Gamma \widehat{f}_2(x,t) \, \widetilde{v}_2 \, d\mathcal{H}^{N-1}(x). \tag{2.17}$$

Observe that a careful choice of the (couples of) test functions as in [29] allows us to give a precise meaning to each formula of the system of Theorem I in suitable Sobolev spaces (of negative order, if it is necessary). If Γ is C^∞, we can use the standard distribution setting. ■

Finally, we consider the case II (and the relative notation) of the previous section.

Theorem II. *Let us assume that* (G1,2,3) *hold together to* (1.34) *and* (1.36); *then we have*

$$\theta_1^\varepsilon \to \theta_1 \textit{ strongly in } L^2(0,T; H^1(\Omega_1)), \qquad \overline{\theta}_2^\varepsilon \to \overline{\theta}_2 \textit{ strongly in } L^2(0,T; H^1(\Gamma)),$$

and, for every fixed $t \in [0,T]$,

$$u_1^\varepsilon(\cdot,t) \rightharpoonup u_1(\cdot,t) \textit{ weakly in } L^2(\Omega_1), \qquad \overline{u}_2^\varepsilon(\cdot,t) \rightharpoonup \overline{u}_2(\cdot,t) \textit{ weakly in } L^2(\Gamma).$$

Moreover, $\{(\theta_1, u_1), (\overline{\theta}_2, \overline{u}_2)\}$ *is the unique weak solution of the following coupled system*

$$\begin{cases} \theta_1 = \beta_1(u_1) & \text{in } Q_1\,, \\ \varrho_1 \dfrac{\partial u_1}{\partial t} - \operatorname{div}(A_1 \nabla \theta_1) = f_1 & \text{in } Q_1\,, \\ \dfrac{\partial \theta_1}{\partial \overline{\boldsymbol{n}}_1} = g_1 & \text{on } \Sigma_1\,, \\ u_1(x, 0) = u_{0,1}(x) & \text{in } \Omega_1\,; \end{cases}$$

$$\theta_1 = \overline{\theta}_2 \quad \text{on } \Sigma\,;$$

$$\begin{cases} \overline{\theta}_2 = \beta_2(\overline{u}_2) & \text{in } \Sigma\,, \\ \varrho_\Gamma \dfrac{\partial \overline{u}_2}{\partial t} - \ell^{-1} \operatorname{div}_\Gamma(\ell A_\Gamma \nabla_\Gamma \overline{\theta}_2) = f_\Gamma - \ell^{-1} \dfrac{\partial \theta_1}{\partial \overline{\boldsymbol{n}}_1} & \text{in } \Sigma\,, \\ \dfrac{\partial \overline{\theta}_2}{\partial \overline{\boldsymbol{n}}'} = 0 & \text{on } \Sigma'\,, \\ \overline{u}_2(x, 0) = u_{0,\Gamma}(x) & \text{in } \Gamma, \end{cases}$$

where ℓ *is the thickness of* Ω_2 *defined by* (G1).

REMARK 2.4. Let us note that in the simplest case of constant coefficients of order of magnitude $1/\varepsilon$, *i.e.*

$$\varrho_2^\varepsilon(x) := \varrho_2/\varepsilon\,, \qquad A_2^\varepsilon(x) := A_2/\varepsilon \tag{2.18}$$

we obtain in the limit

$$\varrho_\Gamma(x) = \varrho_2\,, \qquad A_\Gamma(x) = A_2\,, \qquad \forall x \in \Gamma\,, \tag{2.19}$$

without any deformation due to the curvature of Γ. In particular, when $A_2 = I$ and Ω_2 has a uniform thickness, the usual Laplace operator in Ω_2^ε induces the Laplace-Beltrami operator on Γ (see Remark 2.1). ■

3. THE ABSTRACT THEORY

Let H be a (separable) Hilbert space with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$; on H we are given a l.s.c. convex and positive function

$$\phi: H \mapsto [0, +\infty]\,, \qquad \phi(0) = 0\,, \tag{3.1}$$

with domain $D(\phi) := \{u \in H: \phi(u) < +\infty\}$. We shall denote by Λ its subdifferential, defined as

$$\Lambda: H \mapsto 2^H\,, \qquad w \in \Lambda u \Leftrightarrow (w, v-u) \leqslant \phi(v) - \phi(u)\,, \qquad \forall v \in D(\phi)\,. \tag{3.2}$$

We consider a symmetric and positive bilinear form $a: V_a \times V_a \mapsto \mathbb{R}$, defined on a subspace (*not necessarily dense*) V_a of H. To the couple (V_a, a) is uniquely associated the generalized quadratic form $\mathfrak{a}: H \mapsto [0, +\infty]$ (see [12, def. 11.7])

$$\mathfrak{a}(u) := \begin{cases} a(u,u) \geqslant 0 & \text{if } u \in V_a\,, \\ +\infty & \text{otherwise}\,; \end{cases}$$

we shall assume that V_a is complete with respect to the Hilbertian norm

$$\|v\|^2_{V_a} := \mathfrak{a}(v) + |v|^2\,, \tag{3.3}$$

or, equivalently, that $\mathfrak{a}$ is l.s.c. with respect to the H-topology [12, 12.16]. We call H_a the closure of V_a in H and $H'_a \subset V'_a$ the dual spaces of H_a and V_a respectively ([10]), $\langle\cdot,\cdot\rangle$ being the duality pairing.

We want to study the problem

Problem $\mathrm{P}(\mathfrak{a}, \phi; L, u_0)$. *Given*

$$L \in L^2(0,T; V'_a) \quad \textit{and } u_0 \in H\,,$$

find $\theta \in L^2(0,T;V_a)$ *and* $u \in L^2(0,T;H)$ *such that*

$$\theta(t) \in \Lambda u(t)\,, \quad \textit{for a.e. } t \in]0,T[\,, \tag{3.4}$$

and

$$\int_0^T \{-(u, v_t) + a(\theta, v)\}\,dt = (u_0, v(0)) + \int_0^T \langle L, v\rangle\,dt\,, \tag{3.5}$$

for any choice of $v \in H^1(0,T;V_a)$ *with* $v(T) = 0$.

Before stating our main results, let us make some remarks about this problem. First of all, if the more usual density hypothesis of V_a into H held, we could identify $H_a = H$ with $H' \subset V'_a$ and (3.5) would be the weak formulation of a Cauchy problem for an abstract differential equation of the type

$$\frac{d}{dt}(\Lambda^{-1}\theta) + A\theta \ni L\,, \qquad (\Lambda^{-1}\theta)(0) \ni u_0\,, \tag{3.6}$$

where $A: V_a \mapsto V'_a$ is the linear operator associated to a and Λ^{-1} is the inverse graph of Λ. Evolution problems of this type have been intensively studied; in particular DiBenedetto and Showalter [17] (whose bibliography we refer to) gives a very general existence result, assuming V_a compactly embedded in H but allowing A to be a nonlinear (maximal monotone and bounded) operator.

In a particular but enlightening case, Brezis [5] exploited the linearity and the *coercivity* of A to rewrite (3.6) as an evolution equation in V'_a governed by a subdifferential operator; thanks to the general theory of such equations (cf. [6]), this approach gives a more detailed insight of the solution of the problem.

Taking account of both these contributions, we decided to formulate the problem in a form which will be well adapted to study the dependence of the solution θ on $\mathfrak{a}$ and

([10]) Since we shall deal with different couples of spaces $V_a \subset H_a$ and in general $H_a \neq H$, we *do not identify* any space with its dual; on the contrary, the dense embedding $H'_a \subset V'_a$ is admissible, since it corresponds to the transpose of the continuous and dense inclusion of V_a in H_a.

ϕ, avoiding compactness assumptions and allowing non-coercive bilinear forms a.

We stress that the general choice of a (possibly) non dense domain V_a is motivated by the limit procedure we shall perform. It is well known that the most natural notion of convergence for evolution problems related to convex functionals is that of Mosco (see the definition later on and the exposition of [3], based on [31, 32]). Since the density of the proper domain of these functionals is not preserved by the Mosco-convergence, we cannot assume this property without restricting the range of possible applications. A significant example is showed by Proposition 1.6.

In order to describe our assumption, we fix a continuous and positive bilinear form

$$b: H \times H \mapsto \mathbb{R}, \qquad \mathfrak{b}(u) := b(u, u) \geq 0, \qquad \forall u \in H,$$

and we assume that

$$(A_\Lambda) \qquad (\Lambda u - \Lambda v, u - v) \geq \mathfrak{b}(\Lambda u - \Lambda v), \qquad \forall u, v \in D(\Lambda),$$

$$(A_\mathfrak{a}) \qquad \exists \alpha > 0: \quad \mathfrak{a}(u) + \mathfrak{b}(u) \geq \alpha |u|^2, \qquad \forall u \in V_a,$$

and, on the data,

$$(A_{L, u_0}) \qquad L \in L^2(0, T; V_a'), \qquad u_0 \in D(\phi).$$

Denoting by $J_a: H \mapsto H_a'$ the linear surjection

$$(3.7) \qquad J_a: H \mapsto H_a', \quad \langle J_a u, v \rangle := (u, v), \qquad \forall u \in H, v \in H_a,$$

we have

THEOREM 1 (uniqueness). *Let us assume that $(A_\Lambda, A_\mathfrak{a})$ hold and the data satisfy (A_{L, u_0}); if (θ^1, u^1), (θ^2, u^2) are two solutions of the problem* $\mathrm{P}(\mathfrak{a}, \phi; L, u_0)$, *then*

$$(3.8) \qquad \theta^1 = \theta^2, \quad \text{and } J_a u^1 = J_a u^2.$$

Moreover $\tilde{u} := J_a u^1 = J_a u^2$ belongs to $H^1(0, T; V_a')$ and satisfies the initial condition $\tilde{u}(0) = J_a u_0$.

We can give some further information about the structure of the set $U = U(\mathfrak{a}, \phi; L, u_0)$ of the (not uniquely determined) components u of the solutions.

We associate to ϕ the convex function

$$(3.9) \quad \tilde{\phi}_a: H_a' \mapsto [0, +\infty], \quad \tilde{\phi}_a(\tilde{v}) := \inf\{\phi(v): J_a v = \tilde{v}\}, \quad D(\tilde{\phi}_a) = J_a[D(\phi)],$$

and we call $K(\tilde{v})$, $\tilde{v} \in H_a'$, the set where the inf of (3.9) is achieved:

$$(3.10) \qquad K(\tilde{v}) := \{v \in J_a^{-1}(\tilde{v}): \phi(v) = \tilde{\phi}_a(\tilde{v})\}.$$

It is easy to check that $K(\tilde{v})$ is a closed convex set; moreover it satisfies

$$(3.11) \qquad \Lambda v \cap V_a \neq \emptyset \Rightarrow v \in K(J_a v).$$

We have

PROPOSITION 3.1. *Let $(\theta, \tilde{u})$ be given as in the previous Theorem* 1; *then $\tilde{\phi}_a(\tilde{u})$ is (essen-*

tially) bounded and a function $u \in L^2(0,T;H)$ *belongs to* $U(\alpha,\phi;L,u_0)$ *if and only if*

$$u(t) \in K(\tilde{u}(t)), \qquad \text{for a.e. } t \in]0,T[. \tag{3.12}$$

In particular, the set U *of the solutions* u *of* P *is a closed convex subset of* $L^2(0,T;H)$ *satisfying*

$$u \in U \Rightarrow \phi(u(t)) = \tilde{\phi}_a(\tilde{u}(t)), \qquad \text{for a.e. } t \in]0,T[.$$

REMARK 3.2. The second assumption of (A_{L,u_0}) could be replaced by the weaker

$$\tilde{\phi}_a(J_a u_0) < +\infty. \qquad \blacksquare$$

A condition ensuring the existence is given by

THEOREM 2 (existence). *With the same assumptions of the previous Theorem, let us suppose that*

$$(A_\phi) \qquad \phi \text{ is coercive on } H\text{: } \lim_{|v|\to\infty} \phi(v) = +\infty.$$

Then there exists a solution of problem $P(\alpha,\phi;L,u_0)$ *and* $U(\alpha,\phi;L,u_0)$ *is a bounded subset of* $L^\infty(0,T;H)$.

REMARK 3.3. If (A_ϕ) hold, then $\tilde{\phi}_a$ is l.s.c. and for every $\tilde{v} \in D(\tilde{\phi}_a)$ the set $K(\tilde{v})$ is non empty and bounded in H. $\blacksquare$

Let us denote by Π_a the orthogonal projection onto H_a

$$\Pi_a : H \mapsto H_a, \qquad (\Pi_a v - v, w) = 0, \qquad \forall w \in H_a. \tag{3.13}$$

Since J_a restricted to H_a is the usual Riesz isomorphism between H_a and H_a' and $J_a \circ \Pi_a = J_a$, the knowledge of $\tilde{u} \in H_a'$ is equivalent to the knowledge of the projection $\Pi_a u \in H_a$, which is therefore uniquely determined by the data of the problem.

Of course, if V_a is dense in H, also u is uniquely determined; one could add the further condition

$$u(t) \in H_a, \qquad \text{for a.e. } t \in]0,T[\tag{3.14}$$

in order to fix the solution, but this requirement may be not satisfied in general. Nevertheless, there is a simple *compatibility* condition, which allows (3.14):

COROLLARY. *Besides* $(A_{\Lambda,\alpha,\phi;L,u_0})$, *let us assume that*

$$(A_{\text{comp}}) \qquad \forall v \in H\text{:} \qquad \phi(\Pi_a v) \leq \phi(v).$$

Then there exists a unique solution (θ, u) *of problem* $P(\alpha,\phi;L,u_0)$ *which satisfies* (3.14), *too*; *it is also continuous with respect to the weak topology of* H *and it satisfies the initial condition* $u(0) = \Pi_a u_0$.

REMARK 3.4. It is interesting to note that (A_{comp}) is equivalent to

$$v \in H, \qquad \Lambda v \in H_a \Rightarrow \Lambda v = \Lambda \Pi_a v. \qquad \blacksquare \tag{3.15}$$

Now we vary the functionals and the data according to a parameter ε going to 0 and we want to study the dependence of the solution (θ, u) on ε. So we are given

$$\mathfrak{a}_\varepsilon, \phi_\varepsilon, L_\varepsilon, u_{0,\varepsilon}, \qquad \varepsilon \in [0, \varepsilon_0],$$

and we set correspondingly

$$V_\varepsilon := V_{\mathfrak{a}_\varepsilon}, \qquad \Lambda_\varepsilon := \partial\phi_\varepsilon, \qquad \tilde{\phi}_\varepsilon := (\widetilde{\phi_\varepsilon})_{\mathfrak{a}_\varepsilon}, \qquad J_\varepsilon := J_{\mathfrak{a}_\varepsilon}, \qquad \text{and so on}.$$

If $(\theta_\varepsilon, u_\varepsilon)$ is a solution of $\mathrm{P}(\mathfrak{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$, $\varepsilon \geqslant 0$, we look for general conditions on the data in order to obtain the convergence of $(\theta_\varepsilon, u_\varepsilon)$ to (θ_0, u_0) with respect to a suitable topology. Of course we have to impose some kind of continuity property for the data at $\varepsilon = 0$; we recall the definition of the convergence in the sense of Mosco (see *e.g.* [3, sect. 3.4]).

Definition 3.5. *Let* $\mathbb{H}$ *be an Hilbert space; we say that a family of functions* $F_\varepsilon : \mathbb{H} \mapsto]-\infty, \infty]$ *M-converges to* $F_0 : \mathbb{H} \mapsto]-\infty, +\infty]$, *as* ε *goes to* 0, *if the following two conditions are satisfied*:

$$F_0(v) \leqslant \liminf_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon), \quad \textit{for every family } v_\varepsilon \text{ weakly } \textit{convergent to } v \textit{ in } \mathbb{H},$$

$$\forall v \in \mathbb{H},\ \forall \varepsilon > 0, \quad \exists v_\varepsilon \in \mathbb{H}: \quad \lim_{\varepsilon \to 0} v_\varepsilon = v \text{ strongly } \textit{in } \mathbb{H}, \quad F_0(v) = \lim_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon).$$

In this case we write $F_0 = M\text{-}\lim_{\varepsilon \to 0} F_\varepsilon$.

It is well known [3] that this notion is well adapted to describe the convergence of convex variational functionals and it is strictly related to Γ-convergence and to the graph-convergence of the respective subdifferentials (11). We shall assume that

$$(LIM_{\mathfrak{a},\Lambda,\phi}) \qquad \begin{cases} \mathfrak{a}_\varepsilon \text{ and } \phi_\varepsilon \ M\text{-converge to } \mathfrak{a}_0 \text{ and } \phi_0 \text{ respectively as } \varepsilon \to 0, \\ \mathfrak{a}_\varepsilon, \Lambda_\varepsilon, \phi_\varepsilon \text{ satisfy } (A_{\mathfrak{a},\Lambda,\phi}) \textit{ uniformly} \text{ in }]0, \varepsilon_0]. \end{cases}$$

In order to make precise the kind of convergence of the functionals L_ε (which belongs to varying dual spaces) we state the following

Definition 3.6. *Let* $\mathbb{H}$ *be an Hilbert space,* $\mathfrak{a}_\varepsilon : \mathbb{H} \mapsto [0, +\infty]$ *be a family of generalized quadratic forms M-converging to* $\mathfrak{a}_0$, *and* $\mathbb{V}_\varepsilon$ *be the domain of* $\mathfrak{a}_\varepsilon$. *We say that a family of linear functionals* $\mathcal{L}_\varepsilon \in \mathbb{V}_\varepsilon'$ *strongly converges to* $\mathcal{L}_0 \in \mathbb{V}_0'$ *if*

$$\lim_{\varepsilon \to 0} \langle \mathcal{L}_\varepsilon, v_\varepsilon \rangle = \langle \mathcal{L}_0, v_0 \rangle,$$

for every choice of $v_\varepsilon \in \mathbb{V}_\varepsilon$ *such that*

$$v_\varepsilon \rightharpoonup v_0 \textit{ in } \mathbb{H} \textit{ and } \sup_{\varepsilon > 0} \mathfrak{a}_\varepsilon(v_\varepsilon) < +\infty.$$

(11) More precisely, a family of functionals F_ε (as in definition 3.5) M-converges to F_0 if and only if it Γ-converges to F_0 both in the *strong* and in the *weak* topology of $\mathbb{H}$. If the functionals F_ε are l.s.c., normalized (*i.e.* $F_\varepsilon(0) = 0$), and convex, then they M-converge to F_0 iff the subdifferentials ∂F_ε G-converge to ∂F_0.

Accordingly to this definition, we shall assume that

(LIM_L) $\quad L_\varepsilon \in L^2(0, T; V'_\varepsilon)$ strongly converges to $L_0 \in L^2(0, T; V'_0)$, (12)

and

(LIM_{u_0}) $\quad J_\varepsilon u_{0,\varepsilon} \in V'_\varepsilon$ strongly converges to $J_0 u_{0,0} \in V'_0$, $\quad \sup\limits_{\varepsilon>0} \phi_\varepsilon(u_{0,\varepsilon}) < +\infty$.

We have

THEOREM 3. *Let us assume that* $(LIM_{\mathfrak{a}, \Lambda, \phi; L, u_0})$ *hold and let us denote by* $(\theta_\varepsilon, u_\varepsilon)$ *a solution of* $\mathrm{P}(\mathfrak{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$. *Then*

$$u_\varepsilon \text{ is uniformly bounded in } L^\infty(0, T; H), \tag{3.16}$$

$$\theta_\varepsilon \rightharpoonup \theta_0 \text{ in } L^2(0, T; H), \quad \lim_{\varepsilon \to 0} \int_0^T \mathfrak{b}(\theta_\varepsilon - \theta_0)\, dt = 0, \tag{3.17}$$

and for every $L^\infty(0, T; H)$*-weak* cluster point* u_0 *of* u_ε, *as* $\varepsilon \to 0$, (θ_0, u_0) *is a solution of* $\mathrm{P}(\mathfrak{a}_0, \phi_0; L_0, u_{0,0})$. *Moreover, if* (A_{comp}) *is satisfied for every* $\varepsilon \geqslant 0$ *and* u_ε *is the weakly continuous solution belonging to* H_ε (cf. the previous corollary), *then*

$$\Pi_0 u_\varepsilon(t) \rightharpoonup u_0(t) \text{ in } H, \quad \forall t \in [0, T]. \tag{3.18}$$

Finally, if

$$u_{0,0} \in H_0, \quad \lim_{\varepsilon \to 0} \phi_\varepsilon(u_{0,\varepsilon}) = \phi_0(u_{0,0}) \tag{3.19}$$

also holds, then

$$\lim_{\varepsilon \to 0} \int_0^T \mathfrak{a}_\varepsilon(\theta_\varepsilon(t))\, dt = \int_0^T \mathfrak{a}_0(\theta_0(t))\, dt. \tag{3.20}$$

REMARK 3.7. In order to obtain (3.20) it would sufficient the weaker (cf. Remark 3.2)

$$\lim_{\varepsilon \to 0} \tilde{\phi}_\varepsilon(J_\varepsilon u_{0,\varepsilon}) = \tilde{\phi}_0(J_0 u_{0,0}), \tag{3.21}$$

instead of (A_{comp}) and (3.19).

4. PROOFS OF THE ABSTRACT THEOREMS

First of all we rewrite (3.5) as

$$\int_0^T \{-(u, v_t) + a(\theta, v) + b(\theta, v)\}\, dt = (u_0, v(0)) + \int_0^T \{\langle L, v\rangle + b(\theta, v)\}\, dt. \tag{4.1}$$

Now, replacing a by $a + b$, we can always assume that

$(A_{\mathfrak{ab}})$ $\quad 0 \leqslant \mathfrak{b}(u) \leqslant \mathfrak{a}(u), \quad \mathfrak{a}(u) \geqslant \alpha |u|^2, \quad \forall u \in V_a,$

(12) We are implicitely considering the usual extension (cf. [6, Prop. 2.16]) of a quadratic form on $V_\varepsilon \subset H$ to the time-dependent vectors of $L^2(0, T; V_\varepsilon) \subset L^2(0, T; H)$. This extension does not affect the M-convergence properties.

if we consider, instead of P, the following more general formulation, depending on the parameter $\lambda \in \mathbb{R}$.

PROBLEM $P_\lambda(\mathfrak{a}, \phi; L, u_0)$. *Given*

$$L \in L^2(0, T; V_a') \text{ and } u_0 \in H,$$

find $\theta \in L^2(0, T; V_a)$ *and* $u \in L^2(0, T; H)$ *such that*

$$\theta(t) \in \Lambda u(t) \quad \text{for a.e. } t \in]0, T[\tag{4.2}$$

and

$$\int_0^T \{-(u, v_t) + a(\theta, v)\}\, dt = (u_0, v(0)) + \int_0^T \{\langle L, v\rangle + \lambda b(\theta, v)\}\, dt \tag{4.3}$$

for any choice of $v \in H^1(0, T; V_a)$ *with* $v(T) = 0$.

We are going to prove the analogous of Theorem 1 for P_λ under the hypotheses $(A_{\mathfrak{ab}, \Delta; L, u_0})$.

We fix some notation.

NOTATION 4.1. *We denote by* $A: V_a \mapsto V_a'$ *the linear isomorphism*

$$\widehat{v} = Av \Leftrightarrow \langle \widehat{v}, w\rangle = a(v, w), \qquad \forall v, w \in V_a. \tag{4.4}$$

When it is possible, the superscripts $\tilde{}$, $\hat{}$ *will denote the images in* H_a', V_a' *of the correspondent vectors via* J_a *and* A, *respectively.* $|\cdot|_{\mathfrak{a}}$ *is the dual norm of* H_a':

$$|\widetilde{v}|_{\mathfrak{a}} := \min\{|v| : J_a v = \widetilde{v}\} = \min\{\langle \widetilde{v}, w\rangle : w \in H_a, |w| \leq 1\} \tag{4.5}$$

and $a'(\cdot, \cdot)$ *the dual scalar product on* V_a', *with the associated square norm* $\mathfrak{a}'(\widehat{v}) := = a'(\widehat{v}, \widehat{v})$

$$a'(\widehat{v}, \widehat{w}) := \langle \widehat{v}, w\rangle = \langle \widehat{w}, v\rangle = a(v, w). \tag{4.6}$$

We extend $\widetilde{\phi}_a$ *to* V_a' *by setting* $\widetilde{\phi}_a(\widetilde{v}) = +\infty$ *if* $\widetilde{v} \in V_a' \setminus H_a'$, *and we call*

$$\widetilde{\Lambda}: V_a' \mapsto 2^{V_a'}, \qquad \widetilde{\Lambda} := \partial_{a'} \widetilde{\phi}_a, \tag{4.7}$$

its subdifferential with respect to the scalar product a'. $b_a: V_a' \times V_a' \mapsto \mathbb{R}$ *is the bilinear form*

$$b_a(\widehat{v}, \widehat{w}) := b(v, w), \qquad \forall \widehat{v}, \widehat{w} \in V_a', \tag{4.8}$$

$B: H \mapsto H$ *and* $B_a: V_a' \mapsto V_a'$ *are the linear operators associated to* b *and* b_a:

$$(Bv, w) := b(v, w), \quad \forall v, w \in H; \quad a'(B_a \widehat{v}, \widehat{w}) := b_a(\widehat{v}, \widehat{w}), \quad \forall \widehat{v}, \widehat{w} \in V_a'. \tag{4.9}$$ ■

We briefly recall the (easy to prove) properties of $K(\cdot)$, we mentioned in the previous section:

LEMMA 4.2. *For every* $\widetilde{v} \in D(\widetilde{\phi}_a) \subset H_a'$ *the set* $K(\widetilde{v}) \subset H$ *defined by* (3.10) *is closed and convex; it is also bounded and non-empty if* (A_ϕ) *holds and in this case we have for every* $r \geq 0$

$$\widetilde{\phi}_a(\widetilde{v}) \leq r \Rightarrow |\widetilde{v}|_{\mathfrak{a}} \leq \omega(r), \qquad \sup_{v \in K(\widetilde{v})} |v| \leq \omega(r), \tag{4.10}$$

where $\omega(\cdot)$ *is the «modulus of continuity» of* ϕ *at infinity*

$$\omega: [0, +\infty[\mapsto [0, +\infty[, \qquad \omega(r) := \sup\{|v| : v \in H, \phi(v) \leq r\}. \tag{4.11}$$

The first step consists in the following result

THEOREM 4.3. *The function* $\tilde{\phi}_a$ *defined in* (3.9) *is proper, convex, and its subdifferential* $\tilde{\Lambda}$ *with respect to the scalar product* a' *satisfies*

$$\widehat{w} \in \tilde{\Lambda}\tilde{v} \Rightarrow w \in \Lambda v, \qquad \forall v \in K(\tilde{v}), \tag{4.12}$$

and

$$w \in V_a, \qquad w \in \Lambda v \Rightarrow \widehat{w} \in \tilde{\Lambda}\tilde{v}, \qquad v \in K(\tilde{v}), \tag{4.13}$$

where, following 4.1, $\widehat{w}$ *and* w *are related by* $\widehat{w} = Aw$. *Moreover, if* (A_ϕ) *holds, then* $\tilde{\phi}_a$ *is also l.s.c. in* V_a' *and coercive with respect to the norm of* H_a'. *In particular* $\tilde{\Lambda}$ *is a maximal monotone operator in* V_a'.

The *proof* is a series of simple verifications.

- $\tilde{\phi}_a$ is convex: if $\tilde{u}, \tilde{v} \in D(\tilde{\phi}_a)$ and we choose, for a fixed $\varepsilon > 0$,

$$u \in J_a^{-1}(\tilde{u}), \qquad v \in J_a^{-1}(\tilde{v}), \qquad \text{with } \phi(u) \leq \tilde{\phi}_a(\tilde{u}) + \varepsilon, \qquad \phi(v) \leq \tilde{\phi}_a(\tilde{v}) + \varepsilon,$$

then for every $\tau \in [0, 1]$ we have $J_a(\tau u + (1-\tau)v) = \tau\tilde{u} + (1-\tau)\tilde{v}$ so that

$$\tilde{\phi}_a(\tau\tilde{u} + (1-\tau)\tilde{v}) \leq \phi(\tau u + (1-\tau)v) \leq$$
$$\leq \tau\phi(u) + (1-\tau)\phi(v) \leq \tau\tilde{\phi}_a(\tilde{u}) + (1-\tau)\tilde{\phi}_a(\tilde{v}) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude.

- $\tilde{\phi}_a$ is proper: $\tilde{\phi}_a(0) = 0$.
- (4.12): by the definition of subdifferential we know

$$a'(\widehat{w}, \tilde{z} - \tilde{v}) \leq \tilde{\phi}_a(\tilde{z}) - \tilde{\phi}_a(\tilde{v}), \qquad \forall \tilde{z} \in D(\tilde{\phi}_a). \tag{4.14}$$

Let us set $w := A^{-1}\widehat{w} \in V_a$ and let us choose $v \in K(\tilde{v})$; recalling (3.7) and (4.6), for every $z \in H$, with $J_a z = \tilde{z}$, we have

$$(w, z - v) = \langle w, \tilde{z} - \tilde{v}\rangle = a'(\widehat{w}, \tilde{z} - \tilde{v}). \tag{4.15}$$

Combining with (4.14), since $\tilde{\phi}_a(\tilde{v}) = \phi(v)$ and $\tilde{\phi}_a(\tilde{z}) \leq \phi(z)$, we deduce

$$(w, z - v) \leq \tilde{\phi}_a(\tilde{z}) - \tilde{\phi}_a(\tilde{v}) \leq \phi(z) - \tilde{\phi}_a(\tilde{v}) = \phi(z) - \phi(v), \qquad \forall z \in H,$$

i.e. $w \in \Lambda v$.

- (4.13): we know that $w \in V_a$ satisfies

$$(w, z - v) \leq \phi(z) - \phi(v), \qquad \forall z \in H,$$

and, by (4.15), for every $z \in H$ we have

$$a'(\widehat{w}, \tilde{z} - \tilde{v}) \leq \phi(z) - \phi(v), \qquad \tilde{z} := J_a z. \tag{4.16}$$

Choosing $J_a z = \tilde{z} = \tilde{v}$ we get $v \in K(\tilde{v})$; keeping $\tilde{z}$ fixed and taking the infimum with respect to $z \in J_a^{-1}(\tilde{z})$ we get

$$a'(\widehat{w}, \tilde{z} - \tilde{v}) \leq \tilde{\phi}_a(\tilde{z}) - \phi(v), \qquad \forall \tilde{z} \in H_a';$$

recalling that $\phi(v) \geqslant \widetilde{\phi}_a(\widetilde{v})$ we find

$$a'(\widehat{w}, \widetilde{z} - \widetilde{v}) \leqslant \widetilde{\phi}_a(\widetilde{z}) - \widetilde{\phi}_a(\widetilde{v}), \qquad \forall \widetilde{z} \in H_a' .$$

- $\widetilde{\phi}_a$ is coercive, if (A_ϕ) holds: it follows from (4.10).
- $\widetilde{\phi}_a$ is l.s.c. if (A_ϕ) holds: we choose a sequence $\widetilde{v}_n \in V_a'$ converging to v with $\widetilde{\phi}_a(\widetilde{v}_n) \leqslant r$. By the previous Lemma, there exist $v_n \in K(\widetilde{v}_n) \subset H$ such that

$$J_a v_n = \widetilde{v}_n , \qquad \phi(v_n) = \widetilde{\phi}_a(\widetilde{v}_n) \leqslant r , \qquad |v_n| \leqslant \omega(r) .$$

We can extract a subsequence (still denoted by v_n) weakly convergent to v in H. By the (weakly) lower semicontinuity of ϕ we deduce $\phi(v) \leqslant r$ and by the continuity and the linearity of J_a we have $J_a v = \widetilde{v}$. We conclude that $\widetilde{\phi}_a(\widetilde{v}) \leqslant \phi(v) \leqslant r$. ∎

Theorem 4.3 shows that we can associate to a solution (θ, u) of P_λ the new couple $(\widehat{\theta}, \widetilde{u})$, with $\widehat{\theta} := A\theta$, $\widetilde{u} := J_a u$. This change of unknowns allows us to rewrite P_λ as a perturbation of the evolution equation in V_a' associated to the subdifferential operator $\widetilde{\Lambda}$.

THEOREM 4.4. *Suppose that* (θ, u) *is a solution of* $\mathrm{P}_\lambda(\mathfrak{a}, \phi; L, u_0)$. *Then* $(\widehat{\theta}, \widetilde{u})$ *solves the following abstract Cauchy problem* $\widetilde{\mathrm{P}}_\lambda(\mathfrak{a}, \phi; L, u_0)$

$$(4.17) \qquad \begin{cases} \text{Find } \widetilde{u} \in L^2(0,T; H_a') \cap H^1(0,T; V_a') \text{ and } \widehat{\theta} \in L^2(0,T;V_a') \text{ such that} \\ \widetilde{u}_t(t) + \widehat{\theta}(t) - \lambda B_a \widehat{\theta}(t) = L(t), \qquad \widehat{\theta}(t) \in \widetilde{\Lambda}\widetilde{u}(t), \qquad \text{for a.e. } t \in]0,T[, \\ \widetilde{u}(0) = \widetilde{u}_0 = J_a u_0 . \end{cases}$$

Conversely, if $(\widehat{\theta}, \widetilde{u})$ *is the solution of* $\widetilde{\mathrm{P}}_\lambda(\mathfrak{a}, \phi; L, u_0)$, *the corresponding* P *is solved by every couple* (θ, u) *satisfying*

$$(4.18) \qquad \theta := A^{-1}\widehat{\theta} \text{ and } u \in L^2(0,T;H) \text{ with } u(t) \in K(\widetilde{u}(t)), \qquad \text{for a.e. } t \in]0,T[.$$

PROOF. By setting $\widehat{v} := Av$ and recalling (4.9), we see that (4.3) is equivalent to

$$(4.19) \qquad \int_0^T \{-a'(\widetilde{u}, \widehat{v}_t) + a'(\widehat{\theta}, \widehat{v})\}\, dt = a'(\widetilde{u}_0, \widehat{v}(0)) + \int_0^T a'(L + \lambda B_a \widehat{\theta}, \widehat{v})\, dt$$

for any choice of $\widehat{v} \in H^1(0,T;V_a')$ with $\widehat{v}(T) = 0$. (4.19) is a weak formulation of the differential equation and the initial condition of (4.17). Applying (4.12) and (4.13), we conclude. ∎

By the general theory of nonlinear evolution equation governed by subdifferential operators in Hilbert spaces (see [6, Thm. 4.6]) we deduce

COROLLARY 4.5. *When* (A_ϕ) *holds, together to* $(A_{\mathfrak{a}\mathfrak{b}, \Lambda; L, u_0})$, *there exists a unique solution of problem* $\widetilde{\mathrm{P}}_0(\mathfrak{a}, \phi; L, u_0)$. ∎

In order to prove the uniqueness result of Theorem 1 we need the following a priori estimate.

LEMMA 4.6. *Let* $(\widehat{\theta}^j, \widetilde{u}^j)$, $j = 1, 2$, *be solutions of* $\widetilde{\mathrm{P}}_0(\mathfrak{a}, \phi; L^j + \lambda B_a \widehat{\sigma}^j, u_0^j)$ *respectively, with*

$$L^j,\ u_0^j \text{ satisfying } (A_{L,u_0}) \text{ and } \widehat{\sigma}^j \in L^2(0, T; V_a');$$

then, if $(A_{\mathfrak{a}\mathfrak{b},A})$ *hold and* $\gamma := \lambda^2 + 1$, *we have*

$$(4.20)\qquad \left.\begin{array}{l} \sup\limits_{t\in[0,T]} e^{-\gamma t}\mathfrak{a}'(\widetilde{u}^1 - \widetilde{u}^2) \\ 2\displaystyle\int_0^T e^{-\gamma t}\mathfrak{b}_\mathfrak{a}(\widehat{\theta}^1 - \widehat{\theta}^2)\,dt \end{array}\right\} \leqslant$$

$$\leqslant \mathfrak{a}'(\widetilde{u}_0^1 - \widetilde{u}_0^2) + \int_0^T e^{-\gamma s}[\mathfrak{b}_\mathfrak{a}(\widehat{\sigma}^1 - \widehat{\sigma}^2) + \mathfrak{a}'(L^1 - L^2)]\,dt\,.$$

PROOF. We apply the standard monotonicity arguments to (4.17): taking the difference of the two equations satisfied by $\widetilde{u}^j$ and multiplying (with respect to the dual scalar product a') by the difference of the corresponding solutions and the weight $2e^{-\gamma t}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\{e^{-\gamma t}\mathfrak{a}'(\widetilde{u}^1 - \widetilde{u}^2)\} + \gamma e^{-\gamma t}\mathfrak{a}'(\widetilde{u}^1 - \widetilde{u}^2) + 2e^{-\gamma t}\mathfrak{b}_\mathfrak{a}(\widehat{\theta}^1 - \widehat{\theta}^2) \leqslant \\
&\qquad \leqslant 2e^{-\gamma t}[\lambda b_a(\widehat{\sigma}^1 - \widehat{\sigma}^2, \widetilde{u}^1 - \widetilde{u}^2) + a'(L^1 - L^2, \widetilde{u}^1 - \widetilde{u}^2)] \leqslant \\
&\qquad \leqslant e^{-\gamma t}[\mathfrak{b}_\mathfrak{a}(\widehat{\sigma}^1 - \widehat{\sigma}^2) + \lambda^2\mathfrak{b}_\mathfrak{a}(\widetilde{u}^1 - \widetilde{u}^2) + \mathfrak{a}'(\widetilde{u}^1 - \widetilde{u}^2) + \mathfrak{a}'(L^1 - L^2)] \leqslant \\
&\qquad\qquad \leqslant e^{-\gamma t}[\mathfrak{b}_\mathfrak{a}(\widehat{\sigma}^1 - \widehat{\sigma}^2) + \mathfrak{a}'(L^1 - L^2) + (\lambda^2 + 1)\mathfrak{a}'(\widetilde{u}^1 - \widetilde{u}^2)]
\end{aligned}$$

where we used the easy bound (cf. $(A_{\mathfrak{a}\mathfrak{b}})$ and (4.8))

$$(4.21)\qquad \mathfrak{b}_\mathfrak{a}(\widetilde{v}) = \mathfrak{b}(A^{-1}\widetilde{v}) \leqslant \mathfrak{a}(A^{-1}\widetilde{v}) = \mathfrak{a}'(\widetilde{v}), \qquad \forall \widetilde{v} \in V_a'\,.$$

Integrating in time we conclude. ∎

We can now conclude the *proof of Theorem 1*. In fact, if (θ^j, u^j), $j = 1, 2$, are two solutions of problem $\mathrm{P}_\lambda(\mathfrak{a}, \phi; L, u_0)$, we know that

$$(4.22)\qquad (\widehat{\theta}^j, \widetilde{u}^j) \text{ solves problem } \widetilde{\mathrm{P}}_0(\mathfrak{a}, \phi; L + \lambda B_a \widehat{\theta}^j, u_0)\,.$$

By Lemma 4.6 we deduce

$$\widetilde{u}^1 = \widetilde{u}^2\,, \qquad \mathfrak{b}_a(\widehat{\theta}^1 - \widehat{\theta}^2) = 0\,,$$

and also, due to the positivity of b_a,

$$B_a\widehat{\theta}^1 = B_a\widehat{\theta}^2\,.$$

Finally from the differential equation (4.17) we read

$$\widehat{\theta}^1(t) = \widehat{\theta}^2(t)\,, \qquad \text{for a.e. } t \in]0, T[\,;$$

the conclusion follows now by Theorem 4.4. ∎

We establish another estimate.

LEMMA 4.7. *Let us assume* $(A_{\mathfrak{a}\mathfrak{b},A;L,u_0})$ *and let* (θ, u) *be a solution of problem*

$P_\lambda(\mathfrak{a}, \phi; L, u_0)$. *We have*

$$\left.\begin{array}{r}\operatorname*{ess-sup}_{s\in]0,T[}\phi(u(s))\\ (1/2)\int_0^T \mathfrak{a}(\theta)\,dt\end{array}\right\} \leq \phi(u_0) + (1/2)\int_0^T \mathfrak{a}'(L)\,dt + \lambda\int_0^T \mathfrak{b}(\theta)\,dt. \tag{4.23}$$

PROOF. We take the a'-scalar product of the differential equation (4.17) with $\widehat{\theta}(t)$ and we integrate between 0 and $s \leq T$, obtaining

$$\int_0^s a'(\widetilde{u}_t, \widehat{\theta})\,dt + \int_0^s \mathfrak{a}'(\widehat{\theta})\,dt = \int_0^s a'(L + \lambda B_a \widehat{\theta}, \widehat{\theta})\,dt. \tag{4.24}$$

We call $\overline{\phi}_a$ the lower semicontinuous envelope of $\widetilde{\phi}_a$ in V_a' (cf. [18, Ch. I, 2.2; 12, Ch. 3]):

$$\overline{\phi}_a(v) := \lim_{\varrho\to 0} \inf\{\widetilde{\phi}_a(w) : \mathfrak{a}'(w - v) \leq \varrho^2\}, \qquad \forall v \in V_a'. \tag{4.25}$$

Of course $\partial_{a'}\overline{\phi}_a$ is a maximal monotone operator which is related to $\widetilde{A}$ by [18, p. 20]

$$\widetilde{u} \in V_a', \qquad \widetilde{A}(\widetilde{u}) \neq \emptyset \Rightarrow \widetilde{A}(\widetilde{u}) = \partial_{a'}\overline{\phi}_a(\widetilde{u}), \qquad \overline{\phi}_a(\widetilde{u}) = \widetilde{\phi}_a(\widetilde{u}). \tag{4.26}$$

Since in (4.24) $\widehat{\theta} \in \widetilde{A}(\widetilde{u}) = \partial_{a'}\overline{\phi}_a(\widetilde{u})$, a.e. in $]0, T[$, the first integral is equal to (see [17, Lemma 2.2])

$$\overline{\phi}_a(\widetilde{u}(s)) - \overline{\phi}_a(\widetilde{u}(0)), \qquad \forall s \in]0, T]. \tag{4.27}$$

Applying (4.26) again, we deduce that, for a.e. $s \in]0, T[$,

$$\widetilde{\phi}_a(\widetilde{u}(s)) + \int_0^s \mathfrak{a}(\theta)\,dt = \overline{\phi}_a(u_0) + \int_0^s (\langle L, \theta\rangle + \lambda\mathfrak{b}(\theta))\,dt$$

and, by Schwarz inequality, we get (4.23), since for a.e. $s \in]0, T[$ it is $u(s) \in K(\widetilde{u}(s))$ and consequently $\phi(u(s)) = \widetilde{\phi}_a(\widetilde{u}(s))$. ■

Lemma 4.6 and 4.7 allow us to prove the following existence result.

THEOREM 4.8. *Let us assume that* $(A_{\mathfrak{ab}, A; L, u_0})$ *and* (A_ϕ) *hold; then there exists a unique solution* $(\widehat{\theta}, \widetilde{u})$ *of the perturbed problem* $\widetilde{P}_\lambda(\mathfrak{a}, \phi; L, u_0)$, *which also satisfies*

$$\widetilde{\phi}_a(\widetilde{u}) \in L^\infty(0, T). \tag{4.28}$$

PROOF. We already noticed that if (A_ϕ) holds besides the other assumptions $(A_{\mathfrak{ab}, A; L, u_0})$, then $\widetilde{P}_0$ admits a unique solution. Thanks to (4.22), it is natural to look for the solution of $\widetilde{P}_\lambda$ as the limit of a standard fixed-point iterative technique. If we choose $\widehat{\theta}^0 \in L^2(0, T; V_a')$ and we define by induction

$$(\widehat{\theta}^{n+1}, \widetilde{u}^{n+1}) := \text{the solution of } \widetilde{P}_0(\mathfrak{a}, \phi; L + \lambda B_a \widehat{\theta}^n, u_0),$$

it is easy to see that this sequence converges to the solution of $\widetilde{P}_\lambda(\mathfrak{a}, \phi; L, u_0)$; in fact, by Lemma 4.6, $\widetilde{u}^n$ and $B_a\widehat{\theta}^n$ are Cauchy sequences in $L^\infty(0, T; V_a')$ and $L^2(0, T; V_a')$

respectively, whereas

$$\int_0^T \mathfrak{b}_a(\widehat{\theta}^n)\,dt = \int_0^T \mathfrak{b}(\theta^n)\,dt \text{ is uniformly bounded}.$$

The estimate (4.23) of Lemma 4.7 holds uniformly and proves that the sequence $\widehat{\theta}^n$ is bounded in $L^2(0,T;V_a')$ and therefore $\widetilde{u}^n$ is bounded in $H^1(0,T;V_a')$ by (4.17). From the uniqueness of the limit point of $\widetilde{u}^n$ in $L^\infty(0,T;V_a')$ and the standard monotonicity arguments, we can pass to the limit in (4.17); (4.28) follows by the first of (4.23). ■

Also the *proof of Theorem* 2 *and its Corollary* is almost completed; invoking Theorem 4.4, it remains to show that there exists u satisfying (4.18), $\widetilde{u}$ being given by the previous Theorem 4.8.

Since (A_ϕ) holds, from (4.12) and Lemma 4.2 we know that

$$\theta(t) := A^{-1}\widehat{\theta}(t) \in D(\Lambda^{-1}), \qquad \text{for a.e. } t \in]0,T[,$$

so that the minimal selection

$$u^*(t) \in \Lambda^{-1}(\theta(t)) \colon |u^*(t)| = \min\{|v| : v \in \Lambda^{-1}\theta(t)\}$$

is clearly measurable (cf. [6, Prop. 2.6(*iii*)]); since $\widetilde{\phi}_a(\widetilde{u})$ is bounded, taking into account (4.13) and (4.10) we deduce that u^* belongs to $L^\infty(0,T;H)$. Finally, if (A_{comp}) holds, then

$$u \in K(\widetilde{u}) \Rightarrow \Pi_a u \in K(\widetilde{u}),$$

so that the orthogonal projection on H_a of every solution $u \in U(\mathfrak{a},\phi;L,u_0)$ is a solution again. Since H_a and H_a' are isomorphic via J_a and $J_a\Pi_a u = \widetilde{u}$ is uniquely determined and weakly continuous in H_a', we conclude. ■

Now we study the limiting behaviour of the solutions as stated in Theorem 3 and from now on we assume that $(LIM_{\mathfrak{a},\Lambda,\phi;L,u_0})$ hold.

First of all, we show some relations between M-convergence of $\mathfrak{a}_\varepsilon$, definition 3.6 and the pointwise convergence of the operators

$$\text{(4.29)} \qquad \mathfrak{R}_\varepsilon \colon H \mapsto V_\varepsilon \subset H, \qquad \mathfrak{R}_\varepsilon v := A_\varepsilon^{-1} J_\varepsilon v, \qquad \forall v \in H,$$

which obviously satisfy

$$\text{(4.30)} \qquad u = \mathfrak{R}_\varepsilon v \Leftrightarrow a_\varepsilon(u,w) = (v,w), \qquad \forall w \in V_\varepsilon.$$

LEMMA 4.9. *$\{\mathfrak{R}_\varepsilon\}_{\varepsilon\geq 0}$ is a family of uniformly bounded symmetric operators satisfying, as $\varepsilon \to 0$,*

$$\text{(4.31)} \qquad v_\varepsilon \to v_0 \text{ strongly in } H \Rightarrow \mathfrak{R}_\varepsilon v_\varepsilon \to \mathfrak{R}_0 v_0 \text{ strongly in } H,$$

and

$$\text{(4.32)} \qquad v_\varepsilon \rightharpoonup v_0 \text{ weakly in } H \Rightarrow \mathfrak{R}_\varepsilon v_\varepsilon \rightharpoonup \mathfrak{R}_0 v_0 \text{ weakly in } H.$$

Moreover, if a family $\widehat{z}_\varepsilon \in V_\varepsilon'$ strongly converges to $\widehat{z}_0 \in V_0'$ as in Definition 3.6, *then*

$$z_\varepsilon := A_\varepsilon^{-1}\hat{z}_\varepsilon \to \hat{z}_0 := A_0^{-1} z_0 \text{ in } H, \qquad \mathfrak{a}'_\varepsilon(\hat{z}_\varepsilon) \to \mathfrak{a}'_0(\hat{z}_0). \tag{4.33}$$

PROOF. From (4.30) and $(A_{\mathfrak{a}\mathfrak{b}})$ we have

$$\alpha |u|^2 \leqslant \mathfrak{a}_\varepsilon(u) \leqslant \alpha^{-1}|v|^2, \qquad \text{if } u = \mathfrak{R}_\varepsilon v.$$

Formulae (4.31) and (4.32) follow easily from [12, Corollary 13.7(*b*) and Def. 13.3].

Finally, if $\hat{z}_\varepsilon \in V'_\varepsilon$ strongly converges to $\hat{z}_0 \in V'_0$, then, for every $w_\varepsilon \rightharpoonup w$ in H, we have

$$(z_\varepsilon, w_\varepsilon) = a_\varepsilon(z_\varepsilon, \mathfrak{R}_\varepsilon w_\varepsilon) = \langle \hat{z}_\varepsilon, \mathfrak{R}_\varepsilon w_\varepsilon\rangle \to \langle \hat{z}_0, \mathfrak{R}_0 w_0\rangle = (z_0, w_0),$$

i.e. $z_\varepsilon \to z_0$ strongly in H; choosing $w_\varepsilon := z_\varepsilon$ as test function, we get

$$\mathfrak{a}'_\varepsilon(\hat{z}_\varepsilon) = \langle \hat{z}_\varepsilon, z_\varepsilon\rangle \to \langle \hat{z}_0, z_0\rangle = a'_0(\hat{z}_0). \quad \blacksquare$$

REMARK 4.10. Thanks to the linearity and to the uniform boundedness $\mathfrak{R}_\varepsilon$, the previous properties also hold for vector valued functions; *e.g.*, for every v_ε strongly converging to v_0 in $L^2(0, T; H)$ or $H^1(0, T; H)$,

$$\mathfrak{R}_\varepsilon v_\varepsilon \to \mathfrak{R}_0 v_0 \text{ in } L^2(0, T; H) \text{ or } H^1(0, T; H), \qquad \text{respectively.} \quad \blacksquare \tag{4.34}$$

Now we prove that (3.5) is stable with respect to the weak limit of solutions.

PROPOSITION 4.11. *Let us assume that* $(LIM_{\mathfrak{a}, \Lambda, \phi; L, u_0})$ *hold and let* $(u_\varepsilon, \theta_\varepsilon)$ *be a family of solutions of* $\mathrm{P}_\lambda(\mathfrak{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$; *then there exists a constant C independent of ε such that*

$$\|\phi(u_\varepsilon)\|_{L^\infty(0,T)} + \|u_\varepsilon\|_{L^\infty(0,T;H)} + \int_0^T \left(\mathfrak{a}'_\varepsilon\left(\frac{d}{dt}\tilde{u}_\varepsilon\right) + \mathfrak{a}_\varepsilon(\theta_\varepsilon) + \mathfrak{b}(\theta_\varepsilon)\right) dt \leqslant C. \tag{4.35}$$

Moreover, as $\varepsilon \to 0$, *every couple of weak cluster points* (θ_0, u_0) *of* $(\theta_\varepsilon, u_\varepsilon)$ *in* $L^2(0, T; H)$ *satisfies* (3.5) *for* $\mathrm{P}_\lambda(\mathfrak{a}_0, \phi_0; L_0, u_{0,0})$; *the correponding* $(\hat{\theta}_0, \tilde{u}_0)$ *satisfies the differential equation of* (4.17) (13).

PROOF. From definition 3.6, (LIM_{L, u_0}), and Lemma 4.9, we deduce that there exists $C_{\text{data}} > 0$ such that

$$\phi_\varepsilon(u_{0,\varepsilon}) + \mathfrak{a}'_\varepsilon(u_{0,\varepsilon}) + \int_0^T \mathfrak{a}'_\varepsilon(L_\varepsilon)\, dt \leqslant C_{\text{data}}, \qquad \forall \varepsilon \in [0, \varepsilon_0].$$

The uniformity assumption of $(LIM_{\mathfrak{a}, \Lambda, \phi})$ and (4.20) entail that the integral in $]0, T[$ of $\mathfrak{b}(\theta_\varepsilon)$ is uniformly bounded; by (4.23) of Lemma 4.7 we get (4.35).

Let us now assume that for a decreasing sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ converging to 0 we have

$$u_{\varepsilon_n} \rightharpoonup u_0, \qquad \theta_{\varepsilon_n} \rightharpoonup \theta_0, \qquad \text{in } L^2(0, T; H). \tag{4.36}$$

(13) At this level, we do not say anything about $\theta_0 \in \Lambda_0 u_0$.

We fix

$$w \in H^1(0, T; H), \qquad \text{with } w(T) = 0, \tag{4.37}$$

and we choose the test functions $v_{\varepsilon_n} := \mathfrak{R}_{\varepsilon_n} w$ in the weak formulation (3.5) of $P_\lambda(\mathfrak{a}_{\varepsilon_n}, \phi_{\varepsilon_n}; L_{\varepsilon_n}, u_{0,\varepsilon_n})$, obtaining for every $n \in \mathbb{N}$

$$\int_0^T \{-(u_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w_t) + (\theta_{\varepsilon_n}, w)\}\, dt = \tag{4.38}$$

$$= \langle \tilde{u}_{0,\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w(0)\rangle + \int_0^T \{\langle L_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w\rangle + \lambda b(\theta_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w)\}\, dt.$$

By Lemma 4.9 (cf. also 4.10) as $n \to \infty$ we get

$$\int_0^T \{-(u_0, \mathfrak{R}_0 w_t) + (\theta_0, w)\}\, dt = \langle \tilde{u}_{0,0}, \mathfrak{R}_0 w(0)\rangle + \int_0^T \{\langle L_0, \mathfrak{R}_0 w\rangle + \lambda b(\theta_0, \mathfrak{R}_0 w)\}\, dt$$

for every w satisfying (4.37). Since $\mathfrak{R}_0(H)$ is dense in V_0 (see *e.g.* [12, Prop. 12.17]), we conclude. ■

The previous Proposition does not say if θ_0 belongs to $A_0 u_0$; in order to answer this basic question, we have to work a little bit more.

Let us denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := L^2_\zeta(0, T; H) \text{ where } \zeta \text{ is the measure } d\zeta(t) := e^{-\gamma t}\, dt, \quad \gamma := \lambda^2 + 1, \tag{4.39}$$

and let us denote by A_ε again the canonical extension of A_ε to $\mathcal{H}$ (see [6, 2.1.3 and 2.3.3]). We introduce the multivalued operator $\mathcal{W}_\varepsilon : \mathcal{H} \mapsto 2^{\mathcal{H}}$ with the same domain of A_ε

$$w \in \mathcal{W}_\varepsilon(u) \Leftrightarrow \exists \theta \in A_\varepsilon u, \qquad \text{such that } w = \theta + \gamma \mathfrak{R}_\varepsilon u - \lambda \mathfrak{R}_\varepsilon B\theta. \tag{4.40}$$

Lemma 4.12. *For every $u, w \in \mathcal{H}$ such that $w \in \mathcal{W}_\varepsilon u$, there exists a unique $\theta := \Theta_\varepsilon(w, u) \in \mathcal{H}$ such that*

$$\theta \in A_\varepsilon u \text{ and } w = \theta + \gamma \mathfrak{R}_\varepsilon u - \lambda \mathfrak{R}_\varepsilon B\theta \tag{4.41}$$

as in the definition (4.40).

Proof. From (A_A) we see that

$$\theta^1, \theta^2 \in A_\varepsilon u \Rightarrow \mathfrak{b}(\theta^1 - \theta^2) = 0 \Rightarrow B\theta^1 = B\theta^2,$$

so that

$$\Theta_\varepsilon(w, u) := w - \gamma \mathfrak{R}_\varepsilon u + \lambda \mathfrak{R}_\varepsilon B\bar{\theta}, \qquad \forall \bar{\theta} \in A_\varepsilon u, \tag{4.42}$$

is well defined and belongs to $\mathcal{H}$. ■

We have

Lemma 4.13. *$\mathcal{W}_\varepsilon$ is a maximal monotone operator in $\mathcal{H}$.*

PROOF. The monotonicity is easy: if $w^i \in \mathcal{W}_\varepsilon(u^i)$ with $\theta^i := \Theta_\varepsilon(w^i, u^i)$, we have

$$(4.43)\qquad \int_0^T (w^1 - w^2, u^1 - u^2)\, d\zeta \geq$$

$$\geq \int_0^T [\gamma \mathfrak{a}_\varepsilon'(J_\varepsilon(u^1 - u^2)) + \mathfrak{b}(\theta^1 - \theta^2) - \lambda b(\theta^1 - \theta^2, \mathfrak{R}_\varepsilon(u^1 - u^2))]\, d\zeta \geq$$

$$\geq \int_0^T [(\gamma/2)\, \mathfrak{a}_\varepsilon'(J_\varepsilon(u^1 - u^2)) + (1/2)\, \mathfrak{b}(\theta^1 - \theta^2)]\, d\zeta\,.$$

In order to check the maximality, we fix $v \in \mathcal{H}$ and we have to solve the equation

$$u + \mathcal{W}_\varepsilon u \ni v\,.$$

We repeat the fixed point argument of Theorem 4.8, solving iteratively (14)

$$u^{n+1} + \theta^{n+1} + \gamma \mathfrak{R}_\varepsilon u^{n+1} = v + \lambda \mathfrak{R}_\varepsilon B \theta^n\,, \qquad \theta^{n+1} \in \Lambda_\varepsilon u^{n+1}\,. \qquad \blacksquare$$

We already said that the M-convergence of a family of convex functionals implies the *graph* convergence of the respective subdifferentials [3, Thm. 3.66]; here is the definition (see [3, 3.58]):

DEFINITION 4.14. *Let* $\mathbb{H}$ *be a Hilbert space and* $\mathfrak{A}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$, *be a family of maximal monotone (multivalued) operators of* $\mathbb{H}$. *We say that* $\mathfrak{A}_\varepsilon$ *G-converges to* $\mathfrak{A}_0$ *as* $\varepsilon \to 0$ *if for every* θ_0, $u_0 \in \mathbb{H}$ *with* $\theta_0 \in \mathfrak{A}_0 u_0$ *there exist* $\bar\theta_\varepsilon \in \mathfrak{A}_\varepsilon \bar u_\varepsilon$ *such that*

$$\lim_{\varepsilon \to 0} [\|\bar\theta_\varepsilon - \theta_0\|_{\mathbb{H}} + \|\bar u_\varepsilon - u_0\|_{\mathbb{H}}] = 0\,. \qquad \blacksquare$$

LEMMA 4.15. $\mathcal{W}_\varepsilon$ *G-converges to* $\mathcal{W}_0$ *as* $\varepsilon \to 0$.

PROOF. We fix $w_0 \in \mathcal{W}_0(u_0)$ and choose $\theta_0 := \Theta_0(w_0, u_0) \in \Lambda_0 u_0$ as suggested by Lemma 4.12. Since Λ_ε G-converges to Λ_0 as $\varepsilon \to 0$, we find by the definition

$$\bar\theta_\varepsilon \in \Lambda_\varepsilon \bar u_\varepsilon \ \text{ such that } \lim_{\varepsilon \to 0} [\|\bar\theta_\varepsilon - \theta_0\|_{\mathcal{H}} + \|\bar u_\varepsilon - u_0\|_{\mathcal{H}}] = 0\,.$$

Setting

$$(4.44)\qquad \bar w_\varepsilon := \bar\theta_\varepsilon + \gamma \mathfrak{R}_\varepsilon \bar u_\varepsilon - \lambda \mathfrak{R}_\varepsilon B \bar\theta_\varepsilon \in \mathcal{W}_\varepsilon \bar u_\varepsilon$$

we deduce that $\bar w_\varepsilon \to w_0$ strongly in $\mathcal{H}$, since the following strong convergences hold by Remark 4.10

$$\mathfrak{R}_\varepsilon \bar u_\varepsilon \to \mathfrak{R}_0 u_0\,, \qquad \mathfrak{R}_\varepsilon B \bar\theta_\varepsilon \to \mathfrak{R}_0 B \theta_0\,. \qquad \blacksquare$$

We have

PROPOSITION 4.16. *Let* $w_{\varepsilon_n} \in \mathcal{W}_{\varepsilon_n} u_{\varepsilon_n}$ *be given in* $\mathcal{H} \times \mathcal{H}$, $\{\varepsilon_n\}_{n \in \mathbb{N}}$ *being a decreasing*

(14) This equation admits a unique solution for every $n \in \mathbb{N}$, thanks to the maximal mononotonicity of Λ_ε and to the monotone and Lipschitz character of $\mathfrak{R}_\varepsilon$ in $\mathcal{H}$.

sequence going to 0, *and let us assume that*

$$(4.45)\qquad u_{\varepsilon_n} \rightharpoonup u_0\,, \quad w_{\varepsilon_n} \rightharpoonup w_0 \ \text{in } \mathcal{H}, \quad \text{as } n \to \infty\,,$$

with

$$(4.46)\qquad \limsup_{n\to\infty} \int_0^T (w_{\varepsilon_n}, u_{\varepsilon_n})\,d\zeta \le \int_0^T (w_0, u_0)\,d\zeta\,.$$

Then $w_0 \in \mathcal{W}_0 u_0$ *and setting* $\theta_{\varepsilon_n} := \Theta_{\varepsilon_n}(w_{\varepsilon_n}, u_{\varepsilon_n})$ *we have*

$$(4.47)\qquad \theta_{\varepsilon_n} \rightharpoonup \theta_0 := \Theta_0(w_0, u_0) \ \text{in } \mathcal{H}, \qquad \lim_{n\to\infty} \int_0^T \mathfrak{b}(\theta_{\varepsilon_n} - \theta_0)\,d\zeta = 0\,.$$

PROOF. The relation $w_0 \in \mathcal{W}_0 u_0$ is a standard consequence of the maximal monotonicity and the graph convergence of $\mathcal{W}_\varepsilon$. Let us fix $\theta_0 := \Theta_0(w_0, u_0) \in \Lambda_0 u_0$ and $\overline{w}_\varepsilon, \overline{\theta}_\varepsilon, \overline{u}_\varepsilon$ as in the previous Lemma (we omit for simplicity the index n); we obtain by (4.43)

$$0 \le \int_0^T \{\mathfrak{b}(\theta_\varepsilon - \overline{\theta}_\varepsilon) + \mathfrak{a}'(J_\varepsilon(u_\varepsilon - \overline{u}_\varepsilon))\}\,d\zeta \le C \int_0^T (w_\varepsilon - \overline{w}_\varepsilon, u_\varepsilon - \overline{u}_\varepsilon)\,d\zeta\,.$$

Splitting the right-hand scalar product and passing to the limit, we get by (4.45) and by (4.46)

$$\lim_{\varepsilon\to 0} \int_0^T [\mathfrak{b}(\theta_\varepsilon - \overline{\theta}_\varepsilon) + \mathfrak{a}'(J_\varepsilon(u_\varepsilon - \overline{u}_\varepsilon))]\,d\zeta = 0\,.$$

Since $\overline{\theta}_\varepsilon$ strongly converges to θ_0 in $\mathcal{H}$, we deduce the second part of (4.47) and consequently $B\theta_\varepsilon \to B\theta_0$ in $\mathcal{H}$. By (4.45) and (4.32), we can pass to the limit in the equation

$$\theta_\varepsilon = w_\varepsilon - \gamma \mathfrak{R}_\varepsilon u_\varepsilon + \lambda \mathfrak{R}_\varepsilon B\theta_\varepsilon$$

and we conclude that

$$\theta_\varepsilon \rightharpoonup w_0 - \gamma \mathfrak{R}_0 u_0 + \lambda \mathfrak{R}_0 B\theta_0 = \Theta_0(w_0, u_0) = \theta_0\,. \qquad \blacksquare$$

We conclude now the *proof of Theorem* 3. We consider a family $(\theta_\varepsilon, u_\varepsilon)$ of solutions of $\mathrm{P}_\lambda(\mathfrak{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$ and we denote by (θ_0, u_0) a weak limit point in $\mathcal{H} \times \mathcal{H}$ of a suitable weakly convergent subsequence $(\theta_{\varepsilon_n}, u_{\varepsilon_n})$, whose existence is implied by Proposition 4.11. We define

$$w_\varepsilon := \theta_\varepsilon + \gamma \mathfrak{R}_\varepsilon u_\varepsilon - \lambda \mathfrak{R}_\varepsilon B\theta_\varepsilon\,, \qquad \varepsilon \in [0, \varepsilon_0]\,,$$

and we know that

$$w_\varepsilon \in \mathcal{W}_\varepsilon u_\varepsilon\,, \qquad \theta_\varepsilon = \Theta_\varepsilon(w_\varepsilon, u_\varepsilon)\,, \qquad \text{if } \varepsilon > 0\,.$$

Since (4.45) holds, we want to show that (4.46) is satisfied, too. We take the a_ε'-scalar product of the equation (4.17) with $e^{-\gamma t} J_\varepsilon(u_\varepsilon(t))$; integrating we obtain

$$\frac{e^{-\gamma T}}{2} \mathfrak{a}_\varepsilon(\mathfrak{R}_\varepsilon u_\varepsilon(T)) + \int_0^T (w_\varepsilon, u_\varepsilon)\, d\zeta = \frac{1}{2} \mathfrak{a}'_\varepsilon(\tilde{u}_{0,\varepsilon}) + \int_0^T \langle L_\varepsilon, \mathfrak{R}_\varepsilon u_\varepsilon \rangle\, d\zeta ,$$

which also holds for $\varepsilon = 0$ by Proposition 4.11. Since $\mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n}$ is uniformly bounded in $H^1(0,T;H)$, it converges to $\mathfrak{R}_0 u_0$ in the pointwise weak topology of H; in particular

$$\liminf_{\varepsilon \to 0} \mathfrak{a}_{\varepsilon_n}(\mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n}(t)) \geq \mathfrak{a}_0(\mathfrak{R}_0 u_0(t)) , \qquad \forall t \in [0,T] .$$

Since $\mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n} \rightharpoonup \mathfrak{R}_0 u_0$ in $L^2(0,T;H)$ we have by (LIM_L)

$$\lim_{\varepsilon \to 0} \int_0^T \langle L_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n} \rangle\, d\zeta = \int_0^T \langle L_0, \mathfrak{R}_0 u_0 \rangle\, d\zeta .$$

By Lemma 4.9, we deduce

$$\mathfrak{a}'_{\varepsilon_n}(u_{0,\varepsilon_n}) \to \mathfrak{a}'_0(u_{0,0})$$

and (4.46); by Proposition 4.16 we obtain that $\theta_0 \in A u_0$ and by Proposition 4.11 we conclude that (θ_0, u_0) is a solution of $\mathrm{P}_0(\mathfrak{a}_0, \phi_0; L_0, u_{0,0})$.

Theorem 1 and (4.47) entail (3.17).

The convergence (3.18) follows by the uniform boundedness of u_ε in H, the uniqueness of the (projection of) the limit by (A_{comp}), the pointwise convergence of $\mathfrak{R}_\varepsilon u_\varepsilon$, and the injectivity of $\mathfrak{R}_0|_{H_0}$.

In order to check (3.20), we first observe that the related assumptions of Theorem 3 surely imply (3.21), so that we assume this weaker condition. Let us recall that by the definition of M-convergence 3.5 we know

$$\liminf_{\varepsilon \to 0} \int_0^T \mathfrak{a}_\varepsilon(\theta_\varepsilon)\, dt \geq \int_0^T \mathfrak{a}_0(\theta_0)\, dt .$$

Therefore, it remains to check the opposite inequality for the «lim sup». We rewrite (4.27), which now holds for every $s \in [0,T]$ thanks to (A_ϕ) (recall that $\overline{\phi}_\varepsilon = \tilde{\phi}_\varepsilon$, being $\tilde{\phi}_\varepsilon$ l.s.c.)

$$\tilde{\phi}_\varepsilon(\tilde{u}_\varepsilon(T)) + \int_0^T \mathfrak{a}_\varepsilon(\theta_\varepsilon)\, dt = \tilde{\phi}_\varepsilon(u_{0,\varepsilon}) + \int_0^T (\langle L_\varepsilon, \theta_\varepsilon \rangle + \lambda \mathfrak{b}(\theta_\varepsilon))\, dt .$$

Since

$$\lim_{\varepsilon \to 0} \tilde{\phi}_\varepsilon(\tilde{u}_{0,\varepsilon}) = \tilde{\phi}_0(\tilde{u}_{0,0})$$

by (3.21), and

$$\lim_{\varepsilon \to 0} \int_0^T (\langle L_\varepsilon, \theta_\varepsilon \rangle + \lambda \mathfrak{b}(\theta_\varepsilon))\, dt = \int_0^T (\langle L_0, \theta_0 \rangle + \lambda \mathfrak{b}(\theta_0))\, dt$$

by (LIM_L) and (3.17), our conclusion follows if we show that

$$\liminf_{\varepsilon \to 0} \tilde{\phi}_\varepsilon(\tilde{u}_\varepsilon(T)) \geqslant \tilde{\phi}_0(\tilde{u}_0(T)) . \tag{4.48}$$

To this aim, we choose $v_\varepsilon \in K(\tilde{u}_\varepsilon(T))$ and, up to a subsequence, we can assume that

$$v_\varepsilon \rightharpoonup v_0 , \qquad \mathfrak{R}_\varepsilon v_\varepsilon \rightharpoonup \mathfrak{R}_0 v_0 \text{ in } H .$$

On the other hand

$$\mathfrak{R}_\varepsilon v_\varepsilon = \mathfrak{R}_\varepsilon u_\varepsilon(T) \rightharpoonup \mathfrak{R}_0 u_0(T)$$

so that $J_0 v_0 = J_0 u_0$. We conclude

$$\liminf_{\varepsilon \to 0} \tilde{\phi}_\varepsilon(\tilde{u}_\varepsilon(T)) = \liminf_{\varepsilon \to 0} \phi_\varepsilon(v_\varepsilon) \geqslant \phi_0(v_0) \geqslant \tilde{\phi}_0(\tilde{u}_0(T)) . \qquad \blacksquare$$

5. Proofs of the theorems of sect. 1 and 2

We begin by writing a Fubini-type formula for integrable functions on Ω_2 (cf. [16, sect. 3]); the (sketches of the) proofs of this and other simple results are collected in the appendix.

Notation 5.1. *We set* (cf. (1.13) and (G2) of section 2)

$$R(x) := I - d_\Gamma(x) S(x_\Gamma) , \qquad r(x) := \det R(x) \tag{5.1}$$

and we denote by μ the measure $r \cdot \mathcal{H}^1$ and by ν the measure $r^{-1} \cdot \mathcal{L}$ on Ω_2. For every segment s_x, $x \in \Gamma$, and every $\mathcal{H}^1$-measurable function f, we have (cf. (2.9) and (2.10))

$$\int_{s_x} f \, d\mu := \int_{s_x} f(s) r(s) \, d\mathcal{H}^1(s) = \int_0^{\ell(x)} f(x_\lambda) \, d\mu_x(\lambda) , \tag{5.2}$$

whereas for every $L^1(\Omega_2)$ function g we have

$$\int_{\Omega_2} g \, d\nu := \int_{\Omega_2} g(x) \frac{dx}{r(x)} . \tag{5.3}$$

Observe that r is bounded on Ω_2 and greater than $\eta > 0$ by (G2). $\blacksquare$

Lemma 5.2. *Let f be a function of $L^1(\Omega_2)$; then for $\mathcal{H}^{N-1}$-a.e. $x \in \Gamma$, the restriction $f|_{s_x}$ is $\mathcal{H}^1$-measurable and*

$$\int_{\Omega_2} f(x) \, dx = \int_\Gamma d\mathcal{H}^{N-1}(x) \int_{s_x} f \, d\mu , \tag{5.4}$$

$$\int_{\Omega_2} f \, d\nu = \int_\Gamma d\mathcal{H}^{N-1}(x) \int_{s_x} f \, d\mathcal{H}^1 . \tag{5.5}$$

The second property we need is to characterize $L^2_n(\Omega_2)$ as the subspace of the $L^2(\Omega_2)$-functions which are constant along ($\mathcal{H}^{N-1}$-almost) every segments s_x, $x \in \Gamma$.

LEMMA 5.3. *The linear space $C^1(\overline{\Omega}_2) \cap H^1_n(\Omega_2)$ is dense in $H^1_n(\Omega_2)$ and*

$$(5.6) \qquad L^2_n(\Omega_2) = \{u \in L^2(\Omega_2) : u|_{s_x} \text{ is constant for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma\}.$$

In particular, the projection Π_n on $L^2_n(\Omega_2)$ with respect to the weighted scalar product of $L^2_{\varrho_2}(\Omega_2)$ is given by

$$(5.7) \qquad \Pi_n f(x) = \fint_{s_x} f\, d[\varrho_2 \mu] := \left(\int_{s_x} \varrho_2 \, d\mu \right)^{-1} \cdot \int_{s_x} f \varrho_2 \, d\mu, \qquad \text{for a.e. } x \in \Omega_2 .$$

We begin now the Proofs of the «concrete» theorems.

We observe that Theorem 1.5 and the related existence result of Proposition 1.1 are almost already explained in section 1: they follow by applying the abstract results 1-3 with the choices (1.21)-(1.25), and, for $\varepsilon = 0$, (1.29), (1.30). We limit us to point out the simple technical links.

- The definition of ϕ is standard (see [5, Ex. 2.8.1, 2.8.3]): we introduce a primitive of β_i

$$(5.8) \qquad j_i : \mathbb{R} \mapsto [0, +\infty[, \qquad j_i(s) := \int_0^s \beta_i(\tau)\, d\tau ,$$

and we set for every $U := (u_1, u_2) \in H$

$$(5.9) \qquad \phi(U) := \sum_i \int_{\Omega_i} j_i(u_i(x))\, \varrho_i(x)\, dx \ (^{15}).$$

- (A_ϕ) follows from (1.4).
- $(A_{A,\alpha})$ hold for the quadratic form

$$(5.10) \qquad \mathfrak{b}(\Theta) := c_\beta \, |\Theta|^2_H$$

thanks to (1.5) and to (1.3).
- (A_{L,u_0}) are trivially satisfied since $D(\phi) = H$.
- (A_{comp}) holds also for $\varepsilon = 0$. Of course, the projection Π_0 of H on H_0 is given by

$$\Pi_0(U) := (u_1, \Pi_n u_2),$$

Π_n being given by (5.7); therefore, it is sufficient to prove that

$$\int_{\Omega_2} j_2(\Pi_n u_2(x))\, \varrho_2(x)\, dx \leqslant \int_{\Omega_2} j_2(u_2(x))\, \varrho_2(x)\, dx$$

[15] β_i are Lipschitz functions thanks to (1.5), so that $D(\phi)$ coincides with H. This assumption could be avoided, following the definitions of the quoted examples of [5].

for every $u_2 \in L^2_{\varrho_2}(\Omega_2)$. By applying (5.4) and Jensen's inequality, we have

$$\int_{\Omega_2} j_2(\Pi_n u_2(x))\,\varrho_2(x)\,dx = \int_\Gamma d\mathcal{H}^{N-1}(x)\int_{S_x} j_2\left(\fint_{S_x} u_2\, d(\varrho_2\mu)\right) d(\varrho_2\mu) \leq$$

$$\leq \int_\Gamma d\mathcal{H}^{N-1}(x)\int_{S_x}\left[\fint_{S_x} j_2(u_2)\, d(\varrho_2\mu)\right] d(\varrho_2\mu) =$$

$$= \int_\Gamma d\mathcal{H}^{N-1}(x)\int_{S_x} j_2(u_2)\,\varrho_2\, d\mu = \int_{\Omega_2} j_2(u_2(x))\,\varrho_2(x)\,dx\,.$$

- (LIM_{L,u_0}) are trivial.
- $(LIM_{\alpha,A,\phi})$ is stated by Proposition 1.6; also the uniformity follows by the increasing property of α_ε.
- (1.18) correponds to (3.18).
- (3.19) is satisfied if the initial datum $u_{0,2}$ belongs to $L^2_n(\Omega_2)$. In this case, (3.20) gives the strong convergence of θ^ε_i in $L^2(0,T;H^1(\Omega_i))$, the convergence in $L^2(Q_i)$ being ensured by (3.7) and (5.10). ■

The *proof of Theorem I* is also almost complete: the following two results show the equivalence between wLP_I and the weak formulation of the system of Theorem I given by Remark 2.3.

LEMMA 5.4. *Let us set*

(5.11) $\quad L^2_R(\Omega_2;\mathbb{R}^N) := \{\boldsymbol{v} \in L^2(\Omega_2;\mathbb{R}^N) \colon R(x)\boldsymbol{v}(x) \in L^2_n(\Omega_2;\mathbb{R}^N),\ \boldsymbol{n}(x)\cdot\boldsymbol{v}(x) = 0\}$.

Then for every $u \in H^1_n(\Omega_2)$ we have $\nabla u \in L^2_R(\Omega_2;\mathbb{R}^N)$.

LEMMA 5.5. *The trace operator $u \mapsto \tilde{u} = u|_\Gamma$ is a linear isomorphism mapping $H^1_n(\Omega_2)$ onto $H^1(\Gamma)$, $L^2_n(\Omega_2)$ onto $L^2(\Gamma)$, and $L^2_R(\Omega_2;\mathbb{R}^N)$ onto $L^2(\Gamma;\mathbb{R}^N)$. Furthermore, it satisfies*

$$\nabla_\Gamma(\tilde{u}) = (\nabla u)|_\Gamma, \qquad \forall u \in H^1_n(\Omega_2)$$

and

$$\int_{\Omega_2} \varrho_2 u\, v\, dx = \int_\Gamma \hat{\varrho}_2 \tilde{u}\,\tilde{v}\, d\mathcal{H}^{N-1}, \qquad \forall u, v \in L^2_n(\Omega_2), \tag{5.12}$$

$$\begin{cases} \displaystyle\int_{\Omega_2} A_2 \nabla\theta\, \nabla v\, dx = \int_\Gamma \hat{A}_2 \nabla_\Gamma\tilde{\theta}\cdot\nabla_\Gamma\tilde{v}\, d\mathcal{H}^{N-1}, & \forall \theta, v \in H^1_n(\Omega_2), \\ \displaystyle\int_{\Omega_2} f_2(x,t)\, v(x)\, dx = \int_\Gamma \hat{f}_2(x,t)\,\tilde{v}\, d\mathcal{H}^{N-1}, & \forall v \in L^2_n(\Omega_2), \end{cases} \tag{5.13}$$

where, for $\mathcal{H}^{N-1}$-a.e. $x \in \Gamma$

$$\hat{f}_2(x,t) := \int_{S_x} f(\cdot,t)\, d\mu\,, \quad \hat{\varrho}_2(x) := \int_{S_x} \varrho_2\, d\mu\,, \quad \hat{A}(x) := \int_{S_x} R^{-1} A R^{-1}\, d\mu\,. \tag{5.14}$$ ■

We conclude this section with the *proof of Theorem II*, which we divide into three steps: first of all, we operate a rescaling of the variables in Ω_2^ε in order to write a family of problems in the fixed domains Ω_1, Ω_2. Then we apply the abstract results in a similar way as in the previous proof; finally, we employ Lemma 5.5 to come back to the final formulation on Ω_1, Γ.

STEP 1: RESCALING. Following the notation of the case II, let us operate the change of variables

$$x := G^\varepsilon(z), \qquad x \in \Omega_2^\varepsilon, \qquad z \in \Omega_2. \tag{5.15}$$

For every function $v(x)$ defined on Ω_2^ε, we denote by $v(z)$ again the composition $v \circ G^\varepsilon$, when no misunderstanding are possible. If ν denotes the measure $r^{-1}(z)\,dz$ on Ω_2 and $R_\varepsilon, r_\varepsilon$ are given by

$$R_\varepsilon(z) := I - \varepsilon d_\Gamma(z) S(z_\Gamma), \qquad r_\varepsilon(z) := \det R_\varepsilon(z), \tag{5.16}$$

standard computations show

LEMMA 5.6. *The change of variables* (5.15) *defines a linear isomorphism between* $L^2(\Omega_2^\varepsilon)$ *and* $L^2(\Omega_2)$, $H^1(\Omega_2^\varepsilon)$ *and* $H^1(\Omega_2)$; *the following formulae hold*

$$\int_{\Omega_2^\varepsilon} \varrho_2^\varepsilon(x)\, u(x)\, v(x)\, dx = \int_{\Omega_2} \check{\varrho}_2^\varepsilon(z)\, u(z)\, v(z)\, d\nu(z), \qquad \forall u,\ v \in L^2(\Omega_2^\varepsilon), \tag{5.17}$$

$$\int_{\Omega_2^\varepsilon} A_2^\varepsilon(x) \nabla_x \theta(x) \cdot \nabla_x v(x)\, dx = \int_{\Omega_2} \check{A}_2^\varepsilon(z) \nabla_z \theta(z) \cdot \nabla_z v(z)\, d\nu(z), \qquad \forall \theta,\ v \in H^1(\Omega_2^\varepsilon), \tag{5.18}$$

where ∇_x *and* ∇_z *are the gradient with respect to the variables* x *and* z *respectively,*

$$\int_{\Omega_2^\varepsilon} f_2^\varepsilon(x,t)\, v(x)\, dx = \int_{\Omega_2} \check{f}_2^\varepsilon(z,t)\, v(z)\, d\nu(z), \qquad \forall v \in L^2(\Omega_2^\varepsilon), \tag{5.19}$$

where

$$\check{\varrho}_2^\varepsilon(z) := \varepsilon r_\varepsilon(z)\, \varrho_2^\varepsilon(z), \qquad \check{f}_2^\varepsilon(z,t) := \varepsilon r_\varepsilon(z)\, f_2^\varepsilon(z,t), \tag{5.20}$$

and (*in the rescaled variable* z)

$$\check{A}_2^\varepsilon := \varepsilon r_\varepsilon PRR_\varepsilon^{-1} A_2^\varepsilon R_\varepsilon^{-1} RP + (r_\varepsilon/\varepsilon) N A_2^\varepsilon N. \tag{5.21}$$

Here, $P = P_z$ *is the tangent projection of* (2.1), *and* $N := N_z = \boldsymbol{n}(z)\boldsymbol{n}^T(z)$ *is the normal one*; *recall that* R *and* R_ε *are symmetric matrices.*

It is immediate to see that the new unknowns in the z-variable (coupled with the old ones in Ω_1) satisfy the same weak formulation of wPT in the fixed domains Ω_1, Ω_2 if we replace ϱ_2, A_2, f_2 by the corresponding functions $\check{\varrho}_2^\varepsilon, \check{A}_2^\varepsilon, \check{f}_2^\varepsilon$. We call RTP_{II}^ε this *rescaled* version of PT_{II}^ε.

STEP 2: APPLICATION OF THE ABSTRACT RESULTS. Now we can apply the abstract machinery as before, setting (cf. with (1.21))

$$H := L^2_{\varrho_1}(\Omega_1) \times L^2_\nu(\Omega_2), \tag{5.22}$$

$$a_\varepsilon(\Theta, V) := \int_{\Omega_1} A_1 \nabla\theta_1 \cdot \nabla v_1 \, dx + \int_{\Omega_2} \check{A}_2^\varepsilon \nabla\theta_2 \cdot \nabla v_2 \, d\nu(z), \tag{5.23}$$

and, for $\varepsilon > 0$, V_ε as in (1.23). Unlike case I, the operators Λ_ε are defined by

$$\Theta \in \Lambda_\varepsilon(U) \Leftrightarrow \theta_1(x) = \beta_1(u_1(x)), \qquad \theta_2(z) = \beta_2(u_2(z)/\check{\varrho}_2^\varepsilon(z)), \tag{5.24}$$

and they are the subdifferentials in H of the convex functionals

$$\phi_\varepsilon(U) := \int_{\Omega_1} j_1(u_1(x))\, dx + \int_{\Omega_2} \check{\varrho}_2^\varepsilon(z) j_2(u_2(z)/\check{\varrho}_2^\varepsilon(z))\, d\nu(z), \tag{5.25}$$

where j_i are defined by (5.8). Finally, if

$$\begin{aligned}\langle L_\varepsilon(t), V\rangle := \int_{\Omega_1} f_1(x,t) v_1(x)\, dx \, + \\ + \int_{\Gamma_1} g_1(x,t) v_1(x)\, d\mathcal{H}^{N-1}(x) + \int_{\Omega_2} \check{f}_2^\varepsilon(z,t) v_2(z)\, d\nu(z),\end{aligned} \tag{5.26}$$

then the same simple application of Lemma 5.6 gives the following statement.

LEMMA 5.7. *$(\theta_i^\varepsilon, u_i^\varepsilon)_{i=1,2}$ is the rescaled weak solution of RTP_{II}^ε if and only if the couple $(\Theta^\varepsilon, U^\varepsilon)$ given by*

$$U^\varepsilon := (u_1^\varepsilon, \check{\varrho}_2^\varepsilon u_2^\varepsilon), \qquad \Theta^\varepsilon := (\theta_1^\varepsilon, \theta_2^\varepsilon), \tag{5.27}$$

is the solution of $\mathrm{P}(a_\varepsilon, \phi_\varepsilon; L_\varepsilon, U_{0,\varepsilon})$ for the choices (5.22)-(5.26) and for the initial datum

$$U_{0,\varepsilon} := (u_{0,1}, \check{\varrho}_2^\varepsilon u_{0,2}^\varepsilon). \qquad \blacksquare$$

Now we observe that

$$\lim_{\varepsilon\to 0} r_\varepsilon(z) = 1, \qquad \lim_{\varepsilon\to 0} R_\varepsilon(z) = I, \qquad \lim_{\varepsilon\to 0} \check{\varrho}_2^\varepsilon(z) = \check{\varrho}_2(z) = \varrho_\Gamma(z_\Gamma) \tag{5.28}$$

uniformly in Ω_2. These limits lead us to set

$$\phi_0(U) := \int_{\Omega_1} j_1(u_1(x))\, dx + \int_{\Omega_2} \check{\varrho}_2(z) j_2(u_2(z)/\check{\varrho}_2(z))\, d\nu(z), \tag{5.29}$$

and, if V_0 is defined as in (1.29),

$$a_0(\Theta) := \int_{\Omega_1} A_1 \nabla\theta_1 \cdot \nabla\theta_1 \, dx + \int_{\Omega_2} \check{A}_2(z) \nabla\theta_2 \cdot \nabla\theta_2 \, d\nu(z) \qquad \forall \Theta := (\theta_1, \theta_2) \in V_0,$$

where

$$\check{A}_2(z) := P_z R(z) A_\Gamma(z_\Gamma) R(z) P_z, \qquad A_\Gamma \text{ given by } (1.36). \tag{5.30}$$

Similarly, we set

$$\check{f}_2(z,t) := f_\Gamma(z_\Gamma, t), \qquad \check{u}_{0,2}(z) := u_{0,\Gamma}(z_\Gamma) \tag{5.31}$$

and we have

$$\langle L_0(t), V\rangle := \int_{\Omega_1} f_1(x,t) v_1(x)\,dx + \\ + \int_{\Gamma_1} g_1(x,t) v_1(x)\,d\mathcal{H}^{N-1}(x) + \int_{\Omega_2} \check{f}_2(z,t) v_2(z)\,dv(z), \tag{5.32}$$

and

$$U_{0,0} := (u_{0,1}, \varrho_2(z)\,\check{u}_{0,2}(z)). \tag{5.33}$$

Again we can apply the abstract results of sect. 3; we omit the details, which are analogous to the previous calculations, thanks to the limits (5.28). We observe that the crucial role is played by the following natural result:

PROPOSITION 5.8. *For every* $0 \leqslant \varepsilon < 1$ *let us define* $\mathfrak{a}_\varepsilon$, ϕ_ε, L_ε, $U_{0,\varepsilon}$ *according to* (5.22), ..., (5.26), *and to* (5.29), ..., (5.33), *and let us assume that* (1.34), ..., (1.36) *hold. Then as* ε *goes to* 0, $\mathfrak{a}_\varepsilon$ *and* ϕ_ε *M-converge to* $\mathfrak{a}_0$ *and* ϕ_0 *in H, and* L_ε, $U_{0,\varepsilon}$ *strongly converges to* L_0, $U_{0,0}$ *accordingly to the Definition* 3.6.

PROOF. We only consider the simpler case (2.18), (2.19).

- $\mathfrak{a}_\varepsilon$ M-converges to $\mathfrak{a}_0$ on H. We check the first condition of Definition 3.5, the other one being trivial; furthermore, it is not restrictive to consider only the «Ω_2-contribution» to $\mathfrak{a}_\varepsilon$ and $\mathfrak{a}_0$, *i.e.*

$$\mathfrak{a}_{2,\varepsilon}(\theta) := \int_{\Omega_2} \check{A}_2^\varepsilon(z)\nabla_z\theta(z)\cdot\nabla_z\theta(z)\,dv(z), \qquad \forall\varepsilon \geqslant 0. \tag{5.34}$$

Let us given $\theta^\varepsilon \in H^1(\Omega_2)$, $\varepsilon > 0$, with

$$\theta^\varepsilon \rightharpoonup \theta^0 \text{ in } L^2(\Omega_2), \qquad \text{and} \qquad \liminf_{\varepsilon\to 0} \mathfrak{a}_{2,\varepsilon}(\theta^\varepsilon) < +\infty.$$

By (5.21) and (2.18) we get

$$\mathfrak{a}_{2,\varepsilon}(\theta^\varepsilon) \geqslant \alpha\eta \int_{\Omega_2} |R_\varepsilon^{-1}(z)R(z)P_z\nabla_z\theta^\varepsilon(z)|^2\,dv(z) + \frac{\alpha\eta}{\varepsilon^2}\int_{\Omega_2}\left|\frac{\partial\theta^\varepsilon}{\partial \boldsymbol{n}}\right|^2 dv(z).$$

Since $R(z)$ has a uniformly bounded inverse and $R_\varepsilon(z)$ is uniformly bounded, we deduce

$$\liminf_{\varepsilon\to 0}\|\theta^\varepsilon\|_{H^1(\Omega_2)} < +\infty, \qquad \liminf_{\varepsilon\to 0}\int_{\Omega_2}\left|\frac{\partial\theta^\varepsilon}{\partial \boldsymbol{n}}\right|^2 dv(z) = 0,$$

so that $\theta^0 \in H^1_{\boldsymbol{n}}(\Omega_2)$, $\theta^\varepsilon \rightharpoonup \theta^0$ in $H^1(\Omega_2)$.

Since

$$\boldsymbol{v}^\varepsilon := \sqrt{r_\varepsilon}R_\varepsilon^{-1}RP_z\nabla_z\theta^\varepsilon \rightharpoonup RP_z\nabla_z\theta^0 = \boldsymbol{v}^0(z) \quad \text{in } L^2(\Omega_2;\mathbb{R}^N),$$

and

$$\mathfrak{a}_{2,\varepsilon}(\theta^\varepsilon) \geq \int_{\Omega_2} A_\Gamma v^\varepsilon(z)\cdot v^\varepsilon(z)\,d\nu(z), \qquad \text{for } \varepsilon > 0,$$

we deduce

$$\liminf_{\varepsilon\to 0} \mathfrak{a}_{2,\varepsilon}(\theta^\varepsilon) \geq \liminf_{\varepsilon\to 0} \int_{\Omega_2} A_\Gamma v^\varepsilon(z)\cdot v^\varepsilon(z)\,d\nu(z) \geq \int_{\Omega_2} A_\Gamma v^0(z)\cdot v^0(z)\,d\nu(z) = \mathfrak{a}_{2,0}(\theta^0).$$

- $\phi_\varepsilon M$-converges to ϕ_0 on H. As before, we can consider the behaviour of

$$\phi_{2,\varepsilon}(u) := \varrho_\Gamma \int_{\Omega_2} r_\varepsilon(z) j_2(u^\varepsilon(z)/\varrho_\Gamma r_\varepsilon(z))\,d\nu(z),$$

where $u^\varepsilon \rightharpoonup u^0$ in $L^2(\Omega_2)$. Being j_2 of quadratic growth, we write

$$\phi_{2,\varepsilon}(u^\varepsilon) \geq \varrho_\Gamma \int_{\Omega_2} j_2(u^\varepsilon/\varrho_\Gamma r^\varepsilon(z))\,d\nu(z) - C\|u^\varepsilon\|^2_{L^2(\Omega_2)} \sup_{z\in\Omega_2} |r_\varepsilon(z) - 1|$$

for a suitable positive constant C, independent of ε. Since

$$u^\varepsilon/(\varrho_\Gamma r^\varepsilon) \rightharpoonup u^0/\varrho_\Gamma \qquad \text{in } L^2_\nu(\Omega_2),$$

we conclude by the convexity and the lower semicontinuity of j_2.

- The convergence of the data L_ε, U_ε are easy to check, since (1.34) and (1.35) entail

$$\check{f}_2^\varepsilon \to \check{f}_2 \text{ strongly in } L^2(Q_2), \qquad \check{u}_{0,2}^\varepsilon \to \check{u}_{0,2} \text{ strongly in } L^2_\nu(\Omega_2). \qquad \blacksquare$$

By the previous Proposition and Theorem 3, we deduce that the rescaled solutions $(\theta_i^\varepsilon, u_i^\varepsilon)$ of Lemma 5.7 satisfy

$$(5.35) \qquad \theta_i^\varepsilon \to \theta_i \text{ strongly in } L^2(0,T;H^1(\Omega_i))$$

and, if Π_n is now the orthogonal projection on $L^2_n(\Omega_2)$ with respect to the scalar product of $L^2_\nu(\Omega_2)$,

$$(5.36)\ u_1^\varepsilon(\cdot,t) \rightharpoonup u_1(\cdot,t),\ \ \Pi_n(\check{\varrho}_2^\varepsilon(\cdot)u_2^\varepsilon(\cdot,t)) \rightharpoonup \check{\varrho}_2(\cdot)u_2(\cdot,t), \text{ weakly in } L^2(\Omega_i),$$

for every $t\in[0,T]$, where $\Theta := (\theta_1,\theta_2)$, $U := (u_1, \check{\varrho}_2 u_2)$ are the unique solution of $\mathrm{P}(\mathfrak{a}_0, \phi_0; L_0, U_{0,0})$ satisfying (3.14), *i.e.* $\check{\varrho}_2(\cdot)u_2(\cdot,t) \in L^2_n(\Omega_2)$ for a.e. $t\in]0,T[$.

STEP 3: FORMULATION ON Γ. It is clear that (5.35) implies [16]

$$\overline{\theta}_2^\varepsilon(x) = \fint_{S_x} \theta_2^\varepsilon(z)\,d\mathcal{H}^1(z) \to \overline{\theta}_2(x) = \fint_{S_x} \theta_2(z)\,d\mathcal{H}^1(z) = \theta_2|_\Gamma(x),$$

[16] We neglect for simplicity the dependence on t. Observe that the mean value along the normal segments is not affected by the rescaling; its regularity on Γ depends on the regularity of the thikness $\ell(x)$.

strongly in $L^2(0,T;H^1(\Gamma))$. Regarding u_2, we note that, (cf. Lemma 5.3)

$$\Pi_n(\check{\varrho}_2^\varepsilon u_2^\varepsilon)(z) = \frac{\int_{s_z} u_2^\varepsilon(s)\,\check{\varrho}_2^\varepsilon(s)\,d\mathcal{H}^1(s)}{\int_{s_z} d\mathcal{H}^1(s)} = \fint_{s_z} u_2^\varepsilon(s)\,\check{\varrho}_2^\varepsilon(s)\,d\mathcal{H}^1(s) =$$

$$= \check{\varrho}_2(z)\,\bar{u}_2^\varepsilon(z) + \fint_{s_z} u_2^\varepsilon(s)[\check{\varrho}_2^\varepsilon(s) - \check{\varrho}_2(s)]\,d\mathcal{H}^1(s),$$

so that by (5.36) and (5.28) we deduce

$$\lim_{\varepsilon\to 0} \Pi_n(\check{\varrho}_2^\varepsilon u_2^\varepsilon) = \lim_{\varepsilon\to 0} \check{\varrho}_2\bar{u}_2^\varepsilon, \qquad \text{in the weak topology of } L^2(\Omega_2).$$

It follows that

$$\bar{u}_2(x) = \lim_{\varepsilon\to 0} \bar{u}_2^\varepsilon(x) = u_2\,|_\Gamma(x), \qquad \text{weakly in } L^2(\Gamma).$$

In order to write the weak formulation of the coupled system, as suggested by Remark 2.3, we observe that (5.5) and Lemma 5.4, 5.5 entail

$$\int_{\Omega_2} \check{\varrho}_2(z)\,u(z)\,v(z)\,dv(z) = \int_\Gamma \ell\varrho_\Gamma u\,v\,d\mathcal{H}^{N-1}, \qquad \forall u,\ v \in L_n^2(\Omega_2), \tag{5.37}$$

$$\int_{\Omega_2} \check{A}_2(z)\,\nabla\theta(z)\,\nabla v(z)\,dv(z) = \int_\Gamma \ell A_\Gamma \nabla_\Gamma\theta\cdot\nabla_\Gamma v\,d\mathcal{H}^{N-1}, \qquad \forall\theta,\ v\in H_n^1(\Omega_2), \tag{5.38}$$

$$\int_{\Omega_2} \check{f}_2(z,t)\,v(z)\,dv(z) = \int_\Gamma \ell(x)\,f_\Gamma(x,t)\,v(x)\,d\mathcal{H}^{N-1}(x), \qquad \forall v\in L_n^2(\Omega_2). \quad \blacksquare \tag{5.39}$$

6. Appendix

We list here some useful differential identities; we recall that $x_\lambda := x + \lambda\boldsymbol{n}(x)$ solves the Cauchy problem

$$x_0 = x, \qquad dx_\lambda/d\lambda = \boldsymbol{n}(x_\lambda). \tag{6.1}$$

Lemma 6.1. *For every point $x\in\overline{\Omega}_2$ we have*

$$S(x)\boldsymbol{n}(x) = 0, \qquad \boldsymbol{n}^T(x)\,S(x) = 0, \tag{6.2}$$

$$x_\lambda\in\overline{\Omega}_2 \Rightarrow S(x_\lambda)(I-\lambda S(x)) = S(x), \qquad (I+\lambda S(x_\lambda)) = (I-\lambda S(x))^{-1}. \tag{6.3}$$

Moreover, setting

$$R(x) := (I + d_\Gamma(x)\,S(x))^{-1} = I - d_\Gamma(x)\,S(x_\Gamma), \qquad r(x) := \det R(x), \tag{6.4}$$

we have

$$\partial r(x)/\partial\boldsymbol{n} = -(\operatorname{tr} S(x))\,r(x) = \operatorname{div}\boldsymbol{n}(x)\,r(x). \tag{6.5}$$

The Proof follows by differentiating the identity

$$\boldsymbol{n}(x_\lambda) = \boldsymbol{n}(x), \qquad \forall x,\ x_\lambda\in\Omega_2$$

and by the application of Liouville Theorem to (6.1). $\blacksquare$

LEMMA 6.2. *Let Φ be the mapping*

$$\Phi\colon \Omega_2 \mapsto \mathbb{R}^N \times \mathbb{R}, \qquad \Phi(x) := (x_\Gamma, d_\Gamma(x)). \tag{6.6}$$

Then

$$[J\Phi(x)]^2 := \det[(D\Phi(x))^T D\Phi(x)] = r^{-2}(x). \tag{6.7}$$

PROOF. We shall see that (cf. (6.3))

$$(D\Phi(x))^T D\Phi(x) = (I + d_\Gamma(x) S(x))^2. \tag{6.8}$$

Since

$$Dx_\Gamma = I - \boldsymbol{n}(x)\boldsymbol{n}^T(x) + d_\Gamma(x) S(x) = P_x + d_\Gamma(x) S(x),$$

we have for every $\boldsymbol{v} \in \mathbb{R}^N$

$$D\Phi(x)\boldsymbol{v} = (P_x\boldsymbol{v} + d_\Gamma(x) S(x)\boldsymbol{v}, N_x\boldsymbol{v}) \in \mathbb{R}^N \times \mathbb{R},$$

and

$$|D\Phi(x)\boldsymbol{v}|^2 = |P_x\boldsymbol{v}|^2 + |N_x\boldsymbol{v}|^2 + |d_\Gamma(x) S(x)\boldsymbol{v}|^2 + 2d_\Gamma(x)\boldsymbol{v}^T S(x)\boldsymbol{v} = \\ = |\boldsymbol{v} + d_\Gamma(x) S(x)\boldsymbol{v}|^2.$$

Since we are dealing with symmetric matrices, this is equivalent to (6.8). ■

PROOF OF LEMMA 5.2. Let us observe that Φ is a C^1 diffeomorphism of Ω_2 onto $\Phi(\Omega_2) \subset \Gamma \times \mathbb{R}$, which satisfies

$$\Phi(x_\lambda) = (x, \lambda), \qquad \forall x \in \Gamma,\ x_\lambda \in \Omega_2.$$

By the change of variable formula (see [19]) we write

$$\int_{\Omega_2} f(x)\,dx = \int_{\Phi(\Omega_2)} f(\Phi^{-1}(z)) \frac{1}{J\Phi(\Phi^{-1}(z))}\, d\mathcal{H}^N(z) =$$

$$= \int_\Gamma d\mathcal{H}^{N-1}(x) \int_0^{\ell(x)} f(x_\lambda) r(x_\lambda)\, d\lambda = \int_\Gamma d\mathcal{H}^{N-1}(x) \int_{s_x} f(s)\, d\mu(s). \qquad ■$$

Our aim is now to show that $L^2_{\boldsymbol{n}}(\Omega_2)$ is given by the functions of $L^2(\Omega_2)$ which are constant along $\mathcal{H}^{N-1}$-a.e. segment s_x.

For $x \in \overline{\Omega}_2$ let us define

$$\boldsymbol{b}(x) := \boldsymbol{n}(x)/r(x), \qquad \text{which satisfies } \operatorname{div} \boldsymbol{b}(x) = 0 \text{ in } \Omega_2, \qquad \boldsymbol{b}(x) = \boldsymbol{n}(x) \text{ on } \Gamma$$

by (6.5), and let us call s_x^+ the part of s_x joining x with Γ_2. We have

COROLLARY 6.3. *Let v be a C^1 function with compact support in Ω_2 and let us define*

$$\boldsymbol{v}(x) := -\boldsymbol{b}(x) \int_{s_x^+} v\, d\mu. \tag{6.9}$$

Then

$$\operatorname{div} \boldsymbol{v} = v.$$

PROOF. Being $\boldsymbol{b}$ a divergence free vector field, we have

$$\operatorname{div} \boldsymbol{v}(x) = -\frac{1}{r(x)} \frac{\partial}{\partial \boldsymbol{n}} \int_{s_x^+} v \, d\mu = \frac{1}{r(x)} v(x) r(x) = v(x). \qquad \blacksquare$$

COROLLARY 6.4. *Let* $u \in H^1_{\boldsymbol{n}}(\Omega_2)$; *then*

$$u|_{s_x} = u(x_\Gamma) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma.$$

PROOF. Let us fix v as in the previous corollary; then we calculate by the Green's formula

$$\int_{\Omega_2} u(x) v(x) \, dx = \int_{\Omega_2} u(x)(\operatorname{div} \boldsymbol{v}(x)) \, dx =$$

$$= -\int_{\Omega_2} \nabla u(x) \cdot \boldsymbol{v}(x) \, dx - \int_{\Gamma} u|_\Gamma(x) \, \boldsymbol{v}(x) \cdot \boldsymbol{n}(x) \, d\mathcal{H}^{N-1}(x),$$

since $\boldsymbol{v}$ vanishes on Γ_2. By definition of $H^1_{\boldsymbol{n}}(\Omega_2)$, we know that

$$\nabla u \cdot \boldsymbol{v} = (1/r) \nabla u \cdot \boldsymbol{n} = 0,$$

so that by (6.9)

$$\int_{\Omega_2} u(x) v(x) \, dx = \int_{\Gamma} u|_\Gamma(x) \, d\mathcal{H}^{N-1}(x) \int_{s_x} v \, d\mu = \int_{\Omega_2} u(x_\Gamma) v(x) \, dx.$$

Since v is arbitrary, we conclude. $\blacksquare$

PROOF OF LEMMA 5.3. Let $u_n \in C^1(\overline{\Omega}_2)$ be a sequence converging to $u \in H^1_{\boldsymbol{n}}(\Omega_2)$ in the strong topology of $H^1(\Omega_2)$. It is easy to check that

$$\bar{u}_n(x) = \Pi u_n(x) := \fint_{s_x} u_n \, d\mathcal{H}^1$$

is a C^1 function in $H^1_{\boldsymbol{n}}(\Omega_2)$; since the linear operator Π defined above is bounded in $H^1(\Omega_2)$ and $\Pi u = u$, we conclude. $\blacksquare$

Lemma 5.4 and 5.5 follow easily; it is sufficient to work with C^1 functions and to apply (6.4) to

$$u(x_\lambda) = u(x) \Rightarrow (I - \lambda S(x)) \nabla u(x_\lambda) = \nabla u(x).$$

We make explicit the last elementary computation for Lemma 5.6.

LEMMA 6.5. *Let* G^ε *be defined as in* (1.32), R_ε, r_ε *as in* (5.16); *then*

$$DG^\varepsilon(z)\boldsymbol{v} = R^{-1}(z) R_\varepsilon(z) P_z \boldsymbol{v} + \varepsilon N_z \boldsymbol{v}, \qquad \det DG^\varepsilon(z) = \varepsilon r_\varepsilon(z)/r(z).$$

PROOF. We know that

$$G^\varepsilon(z) = z - (1 - \varepsilon) d_\Gamma(z) \boldsymbol{n}(z),$$

so that by (6.3)

$$
\begin{aligned}
DG^{\varepsilon}(z)\,\boldsymbol{v} = I - (1-\varepsilon)\boldsymbol{n}(z)\boldsymbol{n}^T(z) + (1-\varepsilon)\,d_\Gamma(z)\,S(z) = \\
= \varepsilon N_z \boldsymbol{v} + [I + d_\Gamma(z)\,S(z) - \varepsilon d_\Gamma(z)\,S(z)]\,P_z \boldsymbol{v} = \\
= \varepsilon N_z \boldsymbol{v} + (I + d_\Gamma(z)\,S(z))[I - \varepsilon d_\Gamma(z)(I + d_\Gamma(z)\,S(z))^{-1}S(z)]\,P_z \boldsymbol{v} = \\
= \varepsilon N_z \boldsymbol{v} + (I - d_\Gamma(z)\,S(z_\Gamma))^{-1}[I - \varepsilon d_\Gamma(z)\,S(z_\Gamma)]\,P_z \boldsymbol{v}\,. \quad \blacksquare
\end{aligned}
$$

References

[1] E. Acerbi - G. Buttazzo, *Reinforcement problems in the calculus of variations.* Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 3, 1986, 273-284.

[2] D. Andreucci, *Existence and uniqueness of solutions to a concentrated capacity problem with change of phase.* Europ. J. Appl. Math., 1, 1990, 330-351.

[3] H. Attouch, *Variational Convergence for Functions and Operators.* Pitman, London 1984.

[4] T. Aubin, *Non linear Analysis on Manifolds. Monge-Ampère Equations.* Springer, New York 1982.

[5] H. Brezis, *Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations.* Proc. Symp. by the Mathematics Research Center, Madison, Wisconsin, Academic Press, New York 1971, 101-156.

[6] H. Brezis, *Opérateurs maximaux monotones et sémi-groupes de contractions dans les espaces de Hilbert.* North Holland, Amsterdam 1973.

[7] H. Brezis - L. A. Caffarelli - A. Friedman, *Reinforcement problems for elliptic equations and variational inequalities.* Ann. Mat. pura e appl., 123 (IV), 1980, 219-246.

[8] J. R. Cannon - G. H. Meyer, *On diffusion in a fractured medium.* SIAM J. Appl. Math., 20(3), 1971, 434-448.

[9] P. G. Ciarlet, *Plates and Junctions in Elastic Multi-Structures.* Masson-Springer-Verlag, Paris 1990.

[10] F. H. Clarke, *Optimization and Nonsmooth Analysis.* Wiley, New York 1983.

[11] P. Colli - J. F. Rodrigues, *Diffusion through thin layers with high specific heat.* Asymptotic Anal., 3, 1990, 249-263.

[12] G. Dal Maso, *An Introduction to Γ-Convergence.* Birkhasuer, Boston 1993.

[13] A. Damlamian, *Some results on the multi–phase Stefan problem.* Comm. P.D.E., 2, 1977, 1017-1044.

[14] A. Damlamian, *How to homogenize a nonlinear diffusion equation: Stefan's problem.* SIAM J. Math. Anal., 12, 1981, 306-313.

[15] M. C. Delfour - J. P. Zolésio, *Shape analysis via oriented distance functions.* J. Funct. Anal., 86, 1989, 129-201.

[16] M. C. Delfour - J. P. Zolésio, *A boundary differential equation for thin shells.* J. Differential Equations, to appear.

[17] E. DiBenedetto - R. E. Showalter, *Implicit degenerate evolution equations and applications.* SIAM J. Math. Anal., 12, 1981, 731-751.

[18] I. Ekeland - R. Temam, *Analyse Convexe et Problèmes Variationnels.* Dunod, Gauthier-Villars, Paris 1974.

[19] L. C. Evans - R. Gariepy, *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, CRC Press, 1992.

[20] A. Fasano - M. Primicerio - L. Rubinstein, *A model problem for heat conduction with a free boundary in a concentrated capacity.* J. Inst. Maths. Applics., 26, 1980, 327-347.

[21] D. Gilbarg - N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin 1983.

[22] H. Le Dret, *Problèmes variationnels dans les multi-domaines.* Masson, Paris 1991.

[23] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires.* Dunod, Gauthier-Villars, Paris 1969.

[24] J. L. Lions - E. Magenes, *Non Homogeneous Boundary Value Problems and Applications I, II.* Springer-Verlag, Berlin 1972.

[25] E. Magenes, *On a Stefan problem on a boundary of a domain.* In: M. Miranda (ed.), *Partial Differential Equations and Related Subjects.* Longman Scient. Techn., 1992, 209-226.

[26] E. Magenes, *Some new results on a Stefan problem in a concentrated capacity.* Rend. Mat. Acc. Lincei, s. 9, v. 3, 1992, 23-34.

[27] E. Magenes, *The Stefan problem in a concentrated capcaity.* In: P. E. Ricci (ed.), *Atti Simp. Int. «Problemi attuali dell'Analisi e della Fisica Matematica».* Dip. di Matematica, Univ. «La Sapienza», Roma 1993, 155-182.

[28] E. Magenes, *Regularity and approximation properties for the solution of a Stefan problem in a concentrated capacity.* Proc. Int. Workshop *Variational Methods, Nonlinear Analysis and Differential Equations.* E.C.I.G., Genova, 1994, 88-106.

[29] E. Magenes, *On a Stefan problem in a concentrated capacity.* In: P. Marcellini - G. Talenti - E. Vesentini (eds.), *P.D.E. and Applications.* Marcel Dekker, Inc., 1996, 237-253.

[30] E. Magenes, *Stefan problems in a concentrated capacity.* Adv. Math. Comp. and Appl., Proc. AMCA 95, N.C.C. Pubbl., Novosibirsk 1996, 82-90.

[31] U. Mosco, *Convergence of convex sets and of solutions of variational inequalitites.* Adv. in Math., 3, 1969, 510-585.

[32] U. Mosco, *On the continuity of the Young-Fenchel transformation.* J. Math. Anal. Appl., 35, 1971, 518-535.

[33] H. Pham Huy - E. Sanchez-Palencia, *Phénomènes de transmission à travers des couches minces de conductivité élevée.* J. Math. Anal. and Appl., 47, 1974, 284-309.

[34] L. Rubinstein, *The Stefan problem: Comments on its present state.* J. Inst. Maths. Applics., 24, 1979, 259-277.

[35] E. Sanchez-Palencia, *Problèmes de perturbations liés aux phénomènes de conduction à travers des couches minces de grande résistivité.* J. Math. pures et appl., 53, 1974, 251-270.

[36] E. Sanchez-Palencia, *Non Homogeneous Media and Vibration Theory.* Lect. Notes in Phys. 127, Springer, Berlin-Heidelberg-New York 1980.

[37] M. Shillor, *Existence and continuity of a weak solution to the problem of a free boundary in a concentrated capcity.* Proc. Roy. Soc. Edinburgh, Sect. A. 100, 1985, 271-280.

[38] A. Visintin, *Partial differential equations in domains with self-contact.* Rend. Sem. Mat. Univ. Padova, 81, 1989, 37-48.

G. Savaré:
Istituto di Analisi Numerica del C.N.R.
Via Abbiategrasso, 209 - 27100 Pavia

A. Visintin:
Dipartimento di Matematica
Università degli Studi di Trento
38050 Povo TN