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On compact orbits in singular Kähler spaces

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Geometria. — *On compact orbits in singular Kähler spaces.* Nota (*) di Jörg Winkelmann, presentata dal Socio E. Vesentini.

Abstract. — For a complex solvable Lie group acting holomorphically on a Kähler manifold every closed orbit is isomorphic to a torus and any two such tori are isogenous. We prove a similar result for singular Kähler spaces.

Key words: Kähler manifold; Solvable group; Isogenous tori.

Riassunto. — *Sulle orbite compatte in spazi di Kähler singolari.* Ogni orbita chiusa di un gruppo di Lie risolubile operante olomorficamente su una varietà di Kähler compatta è isomorfa ad un toro. Inoltre due tori siffatti sono isogeni. Scopo della *Nota* è l'estensione di questi risultati noti a spazi di Kähler singolari.

1. Introduction

For a holomorphic action of a solvable complex Lie group G on a compact Kähler manifold X the following is known: *If there exists any closed orbit, then there exists a subtorus T of the Albanese variety $Alb(X)$ such that every closed orbit is a torus isogenous to T.* The principal goal of this paper is to generalize this result to possibly singular Kähler spaces. To achieve our goal, we consider an equivariant desingularization and study the behaviour of closed orbits under such a desingularization. We have to show that the projection map of the desingularization becomes a finite map after restriction to a closed orbit in $\tilde{X}$. Actually we will prove a more general result. For connected holomorphic maps between compact G-complex spaces we prove that the generic fiber dimension is always greater or equal to the fiber dimension of the same map restricted to a closed orbit. This implies in particular that if a closed orbit is contained in a fiber, then the dimension of this orbit can not exceed the generic fiber dimension. It follows that for a generically finite map (thus in particular for any desingularization) a positive-dimensional closed orbit cannot be contained in a fiber.

The main idea is to discuss the induced action of the group on the normal bundle of a closed orbit (and more generally on some kind of normal jet bundles). The philosophy is the following: If an orbit is contained in one fiber, but its dimension is larger than the generic fiber dimension, then its normal bundle should be slightly negative. On the other hand, the normal bundle of an orbit is necessarily a homogeneous vector bundle and homogeneous vector bundles over tori tend to be topologically trivial.

We now state the main results. First we need a definition.

Definition. Let X, Y be complex spaces and A a complex subspace of X. A holomorphic map $f: X \to Y$ is called *locally trivial along A* if for any $p, q \in A$ there exist open neighbourhoods U, V of p resp. q in X and a biholomorphic map $\phi: (U, p) \to (V, q)$ such that $f|_U = f|_V \circ \phi$.

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THEOREM 1. *Let G be a connected complex Lie group and X a complex G-space with a compact G-orbit Z which is biholomorphic to a torus. Assume that X is smooth in some open neighbourhood of Z. Let $f: X \to Y$ be a holomorphic map to some complex space Y. Let A be a connected component of a fiber of $f|_Z: Z \to Y$.*

Then $f: X \to Y$ is locally trivial along A.

It is informative to consider the special case where Z coincides with a fiber of f. In this case we have the following implications:

f has constant rank in a neighbourhood of Z

$\Rightarrow f$ is locally trivial along Z

$\Rightarrow f$ has constant fiber-dimension in a neighbourhood of Z.

PROPOSITION 1. *Under the assumptions of the theorem the fiber dimension of f at any point $x \in X$ is greater than or equal to the fiber dimension of the restricted map $f|_Z: Z \to Y$.*

COROLLARY. *Let X be a complex G-manifold and $f: X \to Y$ be a proper holomorphic map which is generically finite.*

Then for every fiber E and every compact G-orbit $G(x)$ isomorphic to a torus the intersection $E \cap G(x)$ is finite.

These results enable us to control the behaviour of compact orbits under desingularization. Recall that a compact complex space X is said to be of class $\mathcal{C}$ if it is bimeromorphic to a Kähler manifold. This assumption is equivalent to the existence of a holomorphic surjection from a Kähler manifold onto such a space. Spaces in class $\mathcal{C}$ are often also called *weakly Kähler*.

THEOREM 2. *Let X be a complex G-space in class $\mathcal{C}$ and $\tau: \tilde{X} \to X$ be an equivariant proper modification. Let $x \in X$ with $G(x)$ compact. Then there exists a point $\tilde{x} \in \tilde{X}$ with $\tau(\tilde{x}) = x$ and $G(\tilde{x})$ compact. The restricted map $\tau: G(\tilde{x}) \to G(x)$ is finite if G is solvable.*

It should be noted that for non-solvable groups $\tau: G(\tilde{x}) \to G(x)$ is not necessarily finite.

EXAMPLE. Let $\tilde{X}$ be a Hirzebruch-surface Σ_n with $n \geq 2$. There is a $SL_2(\mathbb{C})$-action on Σ_n with three orbits: the 0-section, the ∞-section and an open orbit. Now for $n \geq 2$ the ∞-section has self-intersection number $-n$. Hence it may be blown down to a singular point. This is an example for an $SL_2(\mathbb{C})$-equivariant desingularization $\tilde{X} \to X$ where the projection maps an one-dimensional orbit on a fixed point.

COROLLARY. *Let X be a complex G-space in class $\mathcal{C}$. Let $\tau\colon \tilde{X} \to X$ be an equivariant desingularization. Let $K_G(\tilde{X})$ resp. $K_G(X)$ denote the union of compact orbits.*
Then $\tau(K_G(\tilde{X})) = K_G(X)$.

Note that by a result of Snow [13] the sets $K_G(X)$, $K_G(\tilde{X})$ are closed analytic subsets of X resp. $\tilde{X}$.

THEOREM 3. *Let X be a complex space in class $\mathcal{C}$ and G a solvable complex Lie group acting on X.*
Then any two compact G-orbits are tori isogenous to each other and furthermore isogenous to the G-orbits in the Albanese of any equivariant desingularization of X.

It should be noted that the statement of Theorem 3 does not hold if one drops the assumption $X \in \mathcal{C}$. It is even possible that there are compact orbits of different dimensions:

EXAMPLE. Let Γ be the group of contractions on $Y = \mathbb{C}^2 \setminus \{0\}$ generated by $(z, w) \mapsto (2z, \pi w)$. The quotient Y/Γ is a Hopf surface with two curves given by $z = 0$ resp. $w = 0$. Consider the group $\mathbb{C}^*$ acting by $(z, w) \mapsto (\lambda z, w)$. The curve $w = 0$ is a one-dimensional compact orbit in X while all the points in the curve $z = 0$ are fixed points.

2. LOCAL TRIVIALITY AND FIBER DIMENSION

As preparation for Theorem 1 we begin by proving some auxiliary results on homogeneous vector bundles.

Let G be a complex Lie group acting holomorphically on a complex space X. A vector bundle E over X is called *homogeneous* (with respect to the given G-action) if the G-action on X lifts to an action on E.

Now let E be such a homogeneous vector bundle over a homogeneous manifold Z. Consider a holomorphic function $f\colon E \to \mathbb{C}$ which is linear on every fiber of E. Any such function fibers through the natural map $\lambda_E\colon E \to (\Gamma(Z, E^*))^*$. Now define a subset N of E by $N = \{v \in E\colon \lambda_E(v) = 0\}$. For a homogeneous vector bundle E over a homogeneous manifold Z this subset N is a subvectorbundle. Then, for the quotient vector bundle $F = E/N$, the map λ_F is injective on every fiber of F. Hence the dual bundle F^* is spanned by global sections. Now recall that if Z is a torus then the trivial bundle is the only homogeneous vector bundle which is spanned by global holomorphic sections [10].

Thus we obtain

LEMMA 1. *Let E be a homogeneous vector bundle over a compact complex torus Z. Then E fits into a short exact sequence of homogeneous vector bundles*

$$0 \to N \to E \to F \to 0$$

such that λ_E vanishes identically on N and thus may be pushed-forward to a map from F to $(\Gamma(Z, E^*))^*$. The bundle F is a trivial vector bundle.

Lemma 2. *Let G be a complex Lie group acting holomorphically and transitively on a compact complex torus Z, H a Lie subgroup such that $Z \simeq G/H$. Let $\varrho\colon G \to GL_n(\mathbb{C})$ be a Lie group homomorphism.*

Then $\varrho(G)$ is contained in the algebraic Zariski-closure of $\varrho(H)$.

Proof. Let $\overline{H}$ be the algebraic Zariski-closure of $\varrho(H)$. Let $M = GL_n(\mathbb{C})/\overline{H}$. By a result of Chevalley [8] there exists an equivariant embedding of M in some projective space $\mathbb{P}_N$. On the other hand the Borel Fixed Point Theorem implies that every equivariant map from a torus to a projective space must be constant. Hence $G = \overline{H}$. ∎

Proposition 2. *Let E be a homogeneous vector bundle over a compact complex torus Z. Let G be a complex Lie group acting on E such that the induced action on Z is transitive*, i.e. $Z = G/H$. *Let $x = eH \in Z$ and denote the natural representation of H in $GL(E_x)$ by ϱ. Then for every $g \in G$ there exists an element $\bar{h}$ in the algebraic Zariski-closure of $\varrho(H)$ in $GL(E_x)$ such that the following diagram commutes.*

$$\begin{array}{ccc} E_x & \xrightarrow{\ g\ } & E_{g(x)} \\ \downarrow{\scriptstyle \bar{h}} & & \downarrow{\scriptstyle \lambda_E} \\ E_x & \xrightarrow{\ \lambda_E\ } & \Gamma(Z, E^*)^* \end{array}$$

Proof. Due to Lemma 1 we may assume that E is the trivial bundle. Since Z is compact, any global trivialization $E \simeq Z \times V$ is unique and hence equivariant. Thus the G-action on E is given by a product of representations $\xi_1\colon G \to \mathrm{Aut}(Z)$ and $\xi_2\colon G \to GL(V)$. On the other hand E being trivial implies that $\Gamma(Z, E^*)^*$ is just $V^{**} \simeq V$. Thus $\lambda_E\colon E \to V$ is the projection on the second factor of $E \simeq Z \times V$. To make the diagram commutative it is hence sufficient to choose $\bar{h} \in GL(V) = GL(E_x)$ such that $\xi_2(g) = \bar{h}$. Since $\xi_2|_H = \varrho$, this is possible by Lemma 2. ∎

We now apply the above results on homogeneous vector bundles in order to study holomorphic mappings. Any holomorphic map induces a vector bundle homomorphism between the tangent bundles. For our purposes this first-order approximation does not provide sufficient information, *i.e.* we need higher-order approximation.

Recall that the tangent space at a point may be defined as $T_x(X) = (m_x/m_x^2)^*$, where m_x denotes the maximal ideal of the local ring $\mathcal{O}_x$. In this spirit let $S_x^k = (m_x/m_x^k)^*$. It is clear from this definition that S^k is a vector bundle and that every holomorphic map $f\colon X \to Y$ induces a vector bundle homomorphism $S^k(X) \to S^k(Y)$. If Z is an orbit of a complex Lie group G acting on a complex manifold X, then $S^k(X)|_Z$ will be a homogeneous vector bundle (1).

Any holomorphic automorphism of the space-germ (X, x) induces a linear automorphism of S_x^k, but not conversely.

(1) *Caveat*: These bundles S^k are not the usual jet bundles J_k as defined in *e.g.* [6, 5]. In fact they are in some sense dual to the usual jet bundles. There are natural projections $J_{k+1} \to J_k$, but we have injections $S^k \to S^{k+1}$.

LEMMA 3. *Let G_k denote the group of elements in $GL(S_x^k)$ which are induced by holomorphic automorphisms of the space-germ (X, x).*
Then G_k is an algebraic subgroup of $GL(S_x^k)$.

PROOF. Let $z_1, \ldots, z_n$ be local coordinates. Then z_i are the generators of the $\mathbb{C}$-algebra $A_k = \mathcal{O}_x / m_x^k$. It is easy to see that any algebra automorphism of A_k is induced by a holomorphic automorphism of (X, x). Furthermore the group of algebra automorphisms of A_k is a linear-algebraic subgroup of $GL(A_k)$ and stabilizes the maximal ideal. Now G_k is just the linear-algebraic group $\mathrm{Aut}(A_k)$ in its dual representation. ■

In the following we say that two holomorphic map-germs f, g from (X, x) to (Y, y) are k-equivalent ($f \sim_k g$) if the induced maps from S_x^k to S_y^k coincide. Similarly we say that two sets A, B of maps are k-equivalent if for each $f \in A$ there exists a $g \in B$ such that $f \sim_k g$ and vice versa.

PROPOSITION 3. *Let (X, x), (Y, y) be space-germs and f, g holomorphic map-germs from (X, x) to (Y, y). Assume that for all $k \in \mathbb{N}$ there exists an automorphism $\phi_k \in \mathrm{Aut}(X, x)$ such that $f \circ \phi_k \sim_k g$.*
Then there exists an automorphism $\phi \in \mathrm{Aut}(X, x)$ such that $f \circ \phi = g$.

PROOF. Define $H_k = \{h \in \mathrm{Aut}(X, x)\colon f \circ h \sim_k f\}$. The image of any H_k under the natural homomorphism $\varrho_j\colon \mathrm{Aut}(X, x) \to GL(m/m^j)^*$ is an algebraic subgroup. Obviously $\varrho_j(g) = \varrho_j(h)$ iff $g \sim_j h$. Since any descending sequence of algebraic varieties becomes stationary after finitely many steps, it follows that for all j there exists an N such that $H_N \sim_j H_l$ for all $l > N$. For each j let $N(j)$ denote a minimal such N and define $j(k) = \sup\{j\colon k \geq N(j)\}$. Then $\lim_{k \to \infty} j(k) = \infty$. Next define $\nu(k) = \min\{k, j(k)\}$. Let E_k be defined by $E_k = \{g\colon g \sim_{\nu(k)} h\ \exists h \in H_k\}$. Then $E_l \subset E_k$ and $E_l \sim_{\nu(k)} E_k$ for $l > k$. Moreover: $E_k \subset H_{\nu(k)}$. Next define $W_k = \{\phi_k \circ h\colon h \in E_k\}$. Observe that the sets W_k form a descending sequence of sets with $W_l \sim_{\nu(k)} W_k$ for $l > k$ and $f \circ \zeta \sim_{\nu(k)} g$ for $\zeta \in W_k$. We may now choose a sequence of elements $\psi_k \in W_k$ such that $\psi_k \sim_{\nu(k)} \psi_{k+1}$. Since $\lim_{k \to \infty} \nu(k) = \infty$ it follows that the ψ_k converge coefficient-wise to a formal power series ψ which fulfills the functional equation $f \circ \psi = g$.

But Artin has proved that such a functional equation always has a convergent solution ϕ if it has a formal solution [1]. ■

PROOF OF THEOREM 1. We do not lose any generality if we assume Y to be smooth, because any complex space is locally embeddable in a complex manifold. Furthermore we may assume that f is constant on Z, because it is a general fact that for every holomorphic map from a torus to an arbitrary space all the irreducible components of the fibers must be subtori.

Now consider the induced map $f_*\colon S^k(X)|_Z \to S_y^k(Y)$ with $\{y\} = F(Z)$. $S^k(X)|_Z$ is a homogeneous vector bundle on Z and $S_y^k(Y) \simeq \mathbb{C}^m$ ($m \in \mathbb{N}$). Fix $p, q \in Z$ and $g \in G$ such that $g(p) = q$. Let H by the isotropy in x and $\varrho_k\colon H \to GL(S_x^k(X))$ the natural representation. Then Proposition 2 implies that the following diagram commutes for some

h_k in the Zariski-closure of $\varrho_k(H)$:

$$\begin{array}{ccc} S_p^k(X) & \xrightarrow{g} & S_q^k(X) \\ \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle f_*} \\ S_p^k(X) & \xrightarrow{f_*} & S_y^k(Y) \end{array}$$

The group of all elements in $GL(S_p^k(X))$ induced by holomorphic automorphisms of the space germ (X, p) is an algebraic group (Lemma 2). Since this group contains $\varrho_k(H)$, it contains h_k. Thus for each k there exists an automorphism ϕ_k of the space germ (X, p) such that $F_q \circ g \circ \phi_k \sim_k F_p$. Therefore the statement of the Theorem follows from Proposition 3. ■

We now proceed toward the proof of Proposition 3.

LEMMA 4. *Let f: $X \to Y$ be a holomorphic map between complex spaces $x \in X$, and C a curve on X defined in some open neighbourhood W of x. Assume that $f|_C$ is not constant.*

Then there exist open neighbourhoods U and U' of x resp. $f(x)$ (with $U \subset W$) such that $f(U) \subset U'$ and the restricted map $\tilde{f}$: $C \cap U \to U'$ is proper. In particular $\tilde{f}(C \cap U)$ is an analytic subset of U'.

PROOF. Let t be a coordinate function on a desingularization τ: $\tilde{C} \to C$ of C. Now we can write $f \circ \tau$ locally in the form $f \circ \tau$: $t \mapsto (t^{k_1}\phi_1(t), \ldots, t^{k_n}\phi_n(t))$ with $\phi_i(0) \neq 0$. Therefore a small enough neighbourhood U' of $f(x)$ will have the property that the connected component of $C \cap f^{-1}(U')$ containing x is relatively compact in C. Taking this connected component as U yields the result. ■

PROPOSITION 4. *Let X be a complex manifold with a submanifold Z and a holomorphic map F from X to a complex manifold Y which is constant on Z. Assume that for all $x, y \in Z$ there exist open neighbourhoods U, V and a biholomorphic map ϕ: $U \to V$ such that $F|_U = F|_V \circ \phi$.*

Then the generic fiber dimension of F is greater or equal to the dimension of Z.

PROOF. Let d be the generic fiber dimension. Let Ω denote the subset of X where the fiber dimension equals d. Then Ω is Zariski-open [11]. Choose a small neighbourhood W of x and a curve C in W through x such that $C \backslash \{x\} \subset \Omega$. Furthermore choose C in such a way that $F|_C$ is not constant. By the preceding lemma we may assume that $F(x)$ has an open neighbourhood W' such that $F(C)$ is a closed analytic set in W'. Thus $A = W \cap F^{-1}(F(C \cap W))$ is a closed analytic subset of W. It consists possibly of different irreducible components. Let A_0 denote the component containing $C \backslash \{x\}$. Obviously $\dim_{\mathbb{C}}(A_0) = d + 1$. But now the assumption of the proposition implies that in some neighbourhood of x every point of Z is contained in the topological closure of A_0. Thus $Z \cap W \subset A_0$. Since A_0 is irreducible, it follows that $\dim_{\mathbb{C}}(A_0) > \dim_{\mathbb{C}}(Z)$, *i.e.* $d \geqslant \dim_{\mathbb{C}}(Z)$. ■

Proposition 1 is now an immediate consequence of the preceding proposition together with Theorem 1.

3. Compact orbits in class $\mathcal{C}$

We are now in a position to start the proof of our results on the behaviour of compact orbits under proper modifications.

Proof of Theorem 2. Let $G = R \cdot S$ be a Levi-Malcev-decomposition. A result of Borel-Remmert [2] implies that $G(x)$ is compact iff both $R(x)$ and $S(x)$ are compact. Thus we can treat R and S separately.

We first discuss R. If X is in class $\mathcal{C}$ then $\tilde{X}$ is also in class $\mathcal{C}$. Hence the connected part of the automorphism group acts compactifiably and admits a certain Zariski-topology such that every closed subgroup acts compactifiably [9, 3]. Let $\overline{R}$ be the closure of R with respect to this topology. Now $\overline{R}$ is a solvable group acting compactifiably on the $\overline{R}$-stable compact analytic set $A = \tau^{-1}(R(x))$. Hence there exists a point $\tilde{x} \in A$ with $\overline{R}(\tilde{x})$ compact. Now $\overline{R}(\tilde{x})$ must be a torus. By our Theorem 2 it follows that the restricted map $\tau\colon \overline{R}(\tilde{x}) \to R(x)$ is finite. Thus R acts transitively on $\overline{R}(\tilde{x})$.

Now for a compact orbit $G(x)$ in X, define K to be the set of all points p in $\tau^{-1}(G(x))$ such that $R(p)$ is compact. We have just proved that K is not empty. From [13] we know that K is a closed analytic subset. Note that S stabilizes K because S normalizes R. Now every action of a semisimple complex Lie group on a space in class $\mathcal{C}$ is compactifiable. It follows that S has a compact orbit in K. ■

Proof of Theorem 3. For a smooth Kähler manifold this is an immediate consequence of the results of Sommese [14]. By [3] this extends to smooth manifolds in class $\mathcal{C}$. Using Hironaka's desingularization [8] our Theorem 2 implies the statement for singular spaces. (Keep in mind that the Albanese of the desingularization is independent of the chosen desingularization). ■

Finally we note a fixed-point-theorem implied by our results.

Proposition 5. *Let X be a compact complex space in class $\mathcal{C}$ with a positive-dimensional exceptional set E* (i.e. *the set E can be blown down to a point*). *Let G be a solvable complex Lie group acting on X. Then G stabilizes E and has a fixed point in E.*

Proof. Recall that any proper map with connected fibers is equivariant for any action of any connected complex Lie group [12]. In particular the blowing down $X \to X_0$ is equivariant. It follows that E is invariant. Blowing-down E is equivariant for any connected Lie group acting on X [12]. Now the image of E is a fixed point. Thus the statement follows from Theorem 2. ■

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