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A perturbation problem in the presence of affine symmetries

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Fisica matematica. — *A perturbation problem in the presence of affine symmetries.*
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Abstract. — An approach to a local analysis of solutions of a perturbation problem is proposed when the unperturbed operator has affine symmetries. The main result is a local theorem on existence, uniqueness, and analytic dependence on a parameter.

Key words: Perturbation problems; Affine symmetries for operators; Existence theorems.

Riassunto. — *Un problema di perturbazione in presenza di simmetrie affini.* Si propone un approccio ad un'analisi locale delle soluzioni di un problema di perturbazione dove l'operatore imperturbato possiede delle simmetrie di tipo affine. Il risultato principale è un teorema locale di esistenza, unicità e dipendenza analitica di un parametro.

Preface

We consider a (perturbation) problem of the form $A(x) + \varepsilon B(x) = 0$, with A and B given operators from an open subset U of a Banach space X into a Banach space Y and ε a parameter, and suppose that the (unperturbed) operator A has affine symmetries of the type studied in Valent [4]. These symmetries are described by means of affine representations of a Lie group G on X and on Y connected by a linear mapping from X into Y.

We observe, as an example, that the perturbation problems arising in nonlinear, or linear, elastostatics when the loads depend on the unknown deformation in a general way are included in our abstract perturbation scheme.

The presence of symmetries for A leads to compatibility conditions on A, B, ε, which give rise to serious difficulties in pursuit of local existence theorems. The aim of this paper is to present a strategy for making a local analysis of solutions of the perturbation problem. To this end a crucial role is played by Lemma 4.1. In Section 5 we show that it is possible to associate to the operator B, at any pair $(\xi_0, g_0) \in U \times G$, some linear subspaces of the tangent space to G at its identity element which serve to distinguish the situations of essential singularity from those in which the singularity is apparent.

A first achievement obtained by following the ideas exposed in Sections 4 and 5 is a theorem of local existence, uniqueness, and analytic dependence on ε (see Theorem 6.1).

1. Description of an abstract setting

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces. Following Valent [4], we denote by $\mathcal{A}(X)$ the (Banach) space of continuous, affine mapping from X into itself equipped with the norm $\psi \mapsto \|\psi(0)\|_X + \sup\{\|\psi(x) - \psi(0)\|_X : \|x\|_X \leq 1\}$, and by $\mathcal{L}(X)$ the

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subspace of $\mathcal{A}(X)$ whose elements are (continuous) linear mappings. $\mathcal{A}(Y)$ and $\mathcal{L}(Y)$ have an analogous meaning. Moreover, let G be a Lie group, and let $g \mapsto \varrho_g$ and $g \mapsto \tilde{\varrho}_g$ be affine representations of G on X and on Y, respectively (*i.e.* homomorphisms of the group G into the group of invertible elements of $\mathcal{A}(X)$, and into the group of invertible elements of $\mathcal{A}(Y)$, respectively). The mappings $g \mapsto \varrho_g$ and $g \mapsto \tilde{\varrho}_g$ are supposed to be analytic; their differentials at the identity element e of the group G will be denoted by $v \mapsto R_v$ and $v \mapsto \tilde{R}_v$, respectively. For any $g \in G$, let us denote by l_g the linear part of ϱ_g and by $\tilde{l}_g$ the linear part of $\tilde{\varrho}_g$; thus $\varrho_g(x) = l_g(x) + \varrho_g(0)$ $\forall x \in X$ and $\tilde{\varrho}_g(y) = \tilde{l}_g(y) + \tilde{\varrho}_g(0)$ $\forall y \in Y$. Moreover, let $v \mapsto L_v$ and $v \mapsto \tilde{L}_v$ be the differentials at e of the mappings $g \mapsto l_g$ and $g \mapsto \tilde{l}_g$, respectively; so the mappings $v \mapsto R_v$, $v \mapsto \tilde{R}_v$, $v \mapsto L_v$, $v \mapsto \tilde{L}_v$ send the tangent space T_eG at e to the manifold G into $\mathcal{A}(X)$, $\mathcal{A}(Y)$, $\mathcal{L}(X)$, $\mathcal{L}(Y)$, respectively. It is easily seen that

$$R_v(x) = L_v(x) + R_v(0), \qquad \tilde{R}_v(y) = \tilde{L}_v(y) + \tilde{R}_v(0)$$

for all $v \in T_eG$, $x \in X$, and $y \in Y$. Finally, we set

$$\mathcal{R} = \{R_v : v \in T_eG\}, \qquad \tilde{\mathcal{R}} = \{\tilde{R}_v : v \in T_eG\},$$

and, for any $x \in X$ and $y \in Y$,

$$\mathcal{R}(x) = \{R_v(x) : v \in T_eG\}, \qquad \tilde{\mathcal{R}}(y) = \{R_v(y) : v \in T_eG\},$$

REMARK 1.1. *For any $g \in G$ we have*

$$\{R_v \circ \varrho_g : v \in T_eG\} = \{l_g \circ R_v : v \in T_eG\}. \tag{1.1}$$

Consequently,

$$u \in \mathcal{R} \Rightarrow l_{g^{-1}} \circ u \circ \varrho_g \in \mathcal{R} \qquad \forall g \in G.$$

PROOF. Let $v \in T_eG$, and for every $\lambda \in \boldsymbol{R}$ let $g_\lambda = \exp(\lambda v)$. To the curve $\lambda \mapsto g_\lambda$ in G we associate the curve $\lambda \mapsto \bar{g}_\lambda$ in G defined by putting $\bar{g}_\lambda = g^{-1} g_\lambda g$, and we consider the element $\bar{v}$ of T_eG defined by $\bar{v} = ((d/d\lambda)\bar{g}_\lambda)_{\lambda=0}$. Our proof is ended if we prove that $R_v \circ \varrho_g = l_g \circ R_{\bar{v}}$. In order to do this, it suffices to observe that, since

$$R_v = \left(\frac{d}{d\lambda} \varrho_{g_\lambda}\right)_{\lambda=0}, \qquad R_{\bar{v}} = \left(\frac{d}{d\lambda} \varrho_{\bar{g}_\lambda}\right)_{\lambda=0},$$

we have

$$R_v \circ \varrho_g = \left(\frac{d}{d\lambda} \varrho_{g_\lambda}\right)_{\lambda=0} \circ \varrho_g = \left(\frac{d}{d\lambda} (\varrho_{g_\lambda} \circ \varrho_g)\right)_{\lambda=0} = \left(\frac{d}{d\lambda} \varrho_{g_\lambda g}\right)_{\lambda=0} = \left(\frac{d}{d\lambda} \varrho_{g\bar{g}_\lambda}\right)_{\lambda=0} =$$

$$= \left(\frac{d}{d\lambda} (\varrho_g \circ \varrho_{\bar{g}_\lambda})\right)_{\lambda=0} = l_g \circ \left(\frac{d}{d\lambda} \varrho_{\bar{g}_\lambda}\right)_{\lambda=0} = l_g \circ R_{\bar{v}}.$$

The penultimate equality is true because $\varrho_g \circ \varrho_{\bar{g}_\lambda} - \varrho_g \circ \varrho_{\bar{g}_0} = l_g(\varrho_{\bar{g}_\lambda} - \varrho_{\bar{g}_0})$. ■

2. A PERTURBATION PROBLEM. ASSUMPTIONS ON THE UNPERTURBED OPERATOR

Let X, Y be real Banach spaces, U an open subset of X, $A: U \to Y$ and $B: U \to Y$ smooth operators, and ε a real parameter. We shall deal with the problem of finding $x \in U$ such that $A(x) + \varepsilon B(x) = 0$.

As regards the operator A we assume that it has the symmetries expressed by the following property: *an affine representation* $g \mapsto \varrho_g$ *of a Lie group* G *on* X, *an affine representation* $g \mapsto \tilde{\varrho}_g$ *of* G *on* Y, *a one-to-one, continuous, linear mapping* $\tau \colon X \to Y$, *and an inner product* $(y_1, y_2) \mapsto y_1 \cdot y_2$ *on* Y *exist such that* $\varrho_g(U) \subseteq U$ $\forall g \in G$, *and*

$$\tilde{\varrho}_g \circ \tau = \tau \circ \varrho_g \qquad \forall g \in G \tag{2.1}$$

$$A(\varrho_g(x)) = \tilde{l}_g(A(x)) \qquad \forall g \in G \quad \text{and } \forall x \in U, \tag{2.2}$$

$$A(x) \cdot \tilde{R}_v(\tau(x)) = 0 \qquad \forall v \in T_e G \quad \text{and } \forall x \in U. \tag{2.3}$$

We emphasize the fact (proved in Valent [4, Theorem 3.2]) that, under suitable hypotheses on the operator A and the representations $g \mapsto \varrho_g$ and $g \mapsto \tilde{\varrho}_g$ of G, condition (2.3) follows from ((2.1), (2.2)).

We note that (2.1) easily get $\tilde{R}_v \circ \tau = \tau \circ R_v$ for all $v \in T_e G$, and so (2.3) takes the form

$$A(x) \cdot \tau(R_v(x)) = 0 \qquad \forall v \in T_e G \quad \text{and } \forall x \in U, \tag{2.3$'$}$$

namely the form

$$A(x) \in \mathcal{N}(x)^0 \qquad \forall x \in U, \tag{2.3$''$}$$

where $\mathcal{N}(x) = \tilde{\mathcal{R}}(\tau(x))$, *i.e.*

$$\mathcal{N}(x) = \{\tilde{\mathcal{R}}_v(\tau(x)) \colon v \in T_e G\} = \{(\tau(R_v(x))) \colon v \in T_e G\},$$

and $\mathcal{N}(x)^0$ denotes the orthogonal of $\mathcal{N}(x)$ in the Hausdorff pre-Hilbert space $(Y, \cdot)$.

We further suppose that there is $\xi_0 \in U$ such that

$$A(\xi_0) = 0, \tag{2.4}$$

$$\operatorname{Ker} A'(\xi_0) \subseteq \mathcal{R}(\xi_0), \tag{2.5}$$

$$\dim \operatorname{Ker} A'(\xi_0) \geqslant \operatorname{codim} \operatorname{Im} A'(\xi_0), \tag{2.6}$$

where $A'(\xi_0)$ denotes the differential of A at ξ_0.

From ((2.2), (2.4)) it follows without difficulty that $\operatorname{Ker} A'(\xi_0) \supseteq \mathcal{R}(\xi_0)$. Then, combining (2.2), (2.4) and (2.5), we get

$$\operatorname{Ker} A'(\xi_0) = \mathcal{R}(\xi_0). \tag{2.7}$$

On the other hand ((2.3), (2.4)) implies $A'(\xi_0)(x) \in \mathcal{N}(\xi_0)^0$ $\forall x \in X$, namely $\operatorname{Im} A'(\xi_0) \subseteq \mathcal{N}(\xi_0)^0$; then, as $\dim \mathcal{N}(\xi_0) = \dim \mathcal{R}(\xi_0) < +\infty$, it is easily seen, by using (2.5) and (2.6), that $\operatorname{Im} A'(\xi_0) = \mathcal{N}(\xi_0)^0$, and hence that

$$Y = \mathcal{N}(\xi_0) \oplus \operatorname{Im} A'(\xi_0). \tag{2.8}$$

This fact is generalized by

REMARK 2.1. *Let the hypotheses* (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) *be satisfied. Then for any* $g \in G$ *we have*

$$Y = \mathcal{N}(\varrho_g(\xi_0)) \oplus \operatorname{Im} A'(\varrho_g(\xi_0)), \tag{2.9}$$

and the subspaces $\mathcal{N}(\varrho_g(\xi_0))$ *and* $\operatorname{Im} A'(\varrho_g(\xi_0))$ *of* Y *are orthogonal for the inner product* $\cdot$ *on* Y.

PROOF. We fix $g \in G$. From (2.8) it follows that, for each $y \in Y$ there are $\xi \in X$ and $v \in T_e G$, with ξ and R_v uniquely determined, such that

$$y = A'(\xi_0)(\xi) + (\tau \circ R_v)(\xi_0).$$

Then, putting $x = \varrho_g(\xi)$, we have

$$y = A'(\xi_0)(\varrho_{g^{-1}}(x)) + (\tau \circ l_{g^{-1}} \circ (l_g \circ R_v))(\xi_0).$$

On the other hand, in view of Remark 1.1, there is $\bar{v} \in T_e G$ such that

$$l_g \circ R_v = R_{\bar{v}} \circ \varrho_g .$$

Therefore

$$y = A'(\xi_0)(\varrho_{g^{-1}}(x)) + (\tau \circ l_{g^{-1}})\,(R_{\bar{v}}(\varrho_g(\xi_0))) .$$

Now, we observe that from (2.2) it easily follows that

$$A'(\varrho_g(\xi_0)) \circ \varrho_g = \tilde{l}_g \circ A'(\xi_0).$$

Then

$$\tilde{l}_g(y) = A'(\varrho_g(\xi_0))(x) + (\tilde{l}_g \circ \tau \circ l_{g^{-1}})(R_{\bar{v}}(\varrho_g(\xi_0))),$$

namely, as $\tilde{l}_g \circ \tau \circ l_{g^{-1}} = \tau$,

$$\tilde{l}_g(y) = A'(\varrho_g(\xi_0))(x) + (\tau \circ R_{\bar{v}})(\varrho_g(\xi_0)).$$

Thus (2.9) is true, because $\tilde{l}_g$ is a bijection of Y onto itself. Finally, in order to prove that $\mathcal{N}(\varrho_g(\xi_0))$ and $\operatorname{Im} A'(\varrho_g(\xi_0))$ are orthogonal, it suffices to observe that ((2.2), (2.4)) implies $A(\varrho_g(\xi_0)) = 0$, and hence from (2.3) it follows that $A'(\varrho_g(\xi_0))(\xi)$ belongs to $\mathcal{N}(\varrho_g(\xi_0))^0$ $\forall \xi \in X$. ■

* * *

Let us fix $\bar{v}_1, \dots, \bar{v}_r \in T_e G$ such that $(R_{\bar{v}_1}, \dots, R_{\bar{v}_r})$ is a base of $\mathcal{R}$ and consider the mapping $\gamma\colon X \times Y \to \boldsymbol{R}^r$ defined by putting

$$\gamma(x,y) = \big(y \cdot \tilde{R}_{\bar{v}_j}(\tau(x))\big)_{j=1,\dots,r}, \qquad \text{namely } \gamma(x,y) = \big(y \cdot \tau(R_{\bar{v}_j}(x))\big)_{j=1,\dots,r},$$

for any $(x,y) \in X \times Y$. The mapping γ, which will have an important role, actually depends on the choice of the base $(R_{\bar{v}_1}, \dots, R_{\bar{v}_r})$ of $\mathcal{R}$, but the points where γ vanishes are independent of the choice of the base of $\mathcal{R}$; we emphasize the fact that we are interested to such points and not to the particular function γ vanishing at them. Clearly, γ is continuous, linear in y, and affine in x. We observe that from the definition of γ it immediately follows that

$$\gamma(x,y) = 0 \Leftrightarrow y \in \mathcal{N}(x)^0 .$$

Later, we shall suppose that, for $(x,y) \in X \times Y$,

$$\gamma(x,y) = 0 \Rightarrow \gamma(\varrho_g(x), \tilde{l}_g(y)) = 0 \quad \forall g \in G. \tag{2.10}$$

REMARK 2.2. *Condition* (2.10) *is satisfied provided, for* $y_1, y_2 \in Y$,

$$y_1 \cdot y_2 = 0 \Rightarrow \tilde{l}_g(y_1) \cdot \tilde{l}_g(y_2) = 0 \quad \forall g \in G. \tag{2.11}$$

PROOF. Suppose that (2.11) holds, and let $(x, y) \in X \times Y$ be such that $\gamma(x, y) = 0$, *i.e.*

$$y \cdot \tau(R_{\bar{v}_j}(x)) = 0 \qquad \forall j = 1, \dots, r,$$

where $(R_{\bar{v}_j})_{j=1,\dots,r}$ is a base of $\mathcal{R}$. Then, in view of (2.11),

$$\tilde{l}_g(y) \cdot \tilde{l}_g(\tau(R_{\bar{v}_j}(x))) = 0 \qquad \forall j = 1, \dots, r,$$

namely, by (2.1),

$$\tilde{l}_g(y) \cdot \tau(l_g(R_{\bar{v}_j}(x))) = 0 \qquad \forall j = 1, \dots, r.$$

Therefore, since in view of Remark 1.1 there are $v_1, \dots, v_r \in T_e G$ such that

$$R_{v_j} = l_g \circ R_{\bar{v}_j} \circ \varrho_{g^{-1}},$$

we have

$$\tilde{l}_g(y) \cdot \tau(R_{v_j}(\varrho_g(x))) = 0 \qquad \forall j = 1, \dots, r,$$

which means $\gamma(\varrho_g(x), \tilde{l}_g(y)) = 0$, because it is easy to see that $(R_{v_j})_{j=1,\dots,r}$, as well as $(R_{\bar{v}_j})_{j=1,\dots,r}$, is a base of $\mathcal{R}$. ■

3. Two examples from finite and linear elastostatics

Let us take out from the finite and linear elastostatics two examples of operators $A: X \to Y$ having the properties (2.1)-(2.6). As in Valent [4, Sect. 5], we take $Y = Y_1 \times Y_2$ and we make the following two choices (within Sobolev and Schauder spaces) of the Banach spaces X, Y_1, Y_2:

$$\text{(3.1)} \qquad \begin{cases} X = W^{m+2,p}(\Omega, \boldsymbol{R}^n) \\ Y_1 = W^{m,p}(\Omega, \boldsymbol{R}^n); \quad Y_2 = W^{m+1-1/p,p}(\partial\Omega, \boldsymbol{R}^n) \end{cases}$$

$$\text{(3.2)} \qquad \begin{cases} X = C^{m+2,\lambda}(\overline{\Omega}, \boldsymbol{R}^n) \\ Y_1 = C^{m,\lambda}(\overline{\Omega}, \boldsymbol{R}^n); \quad Y_2 = C^{m+1,\lambda}(\partial\Omega, \boldsymbol{R}^n), \end{cases}$$

where Ω is a smooth, bounded, open subset of $\boldsymbol{R}^n$, $\partial\Omega$ is its boundary, $1 < p \in \boldsymbol{R}$, $\lambda \in]0, 1[$, and $p(m+1) > n$. For the definitions and properties of these spaces we refer to Valent [3].

For both choices of X and Y, we take as $\tau: X \to Y$ the function defined by $\tau(x) = (x, x|_{\partial\Omega})$, and consider the inner product $\cdot$ induced on Y by the usual inner product on $L^2(\Omega, \boldsymbol{R}^n) \times L^2(\partial\Omega, \boldsymbol{R}^n)$. Furthermore, we take as U the set of elements of X that are orientation-preserving diffeomorphism of $\overline{\Omega}$ onto a subset of $\boldsymbol{R}^n$. Since X is continuously embedded in $C^1(\overline{\Omega}, \boldsymbol{R}^n)$, U is an open subset of X (see, for instance, Hirsch [1, Ch. 2, Th. 1.4]).

As a first example, let $(t, Z) \mapsto s(t, Z)$ be a given function from $\overline{\Omega} \times \boldsymbol{M}_n^+$ into $\boldsymbol{M}_n$, where $\boldsymbol{M}_n$ is the set of $n \times n$ real matrices and $\boldsymbol{M}_n^+ = \{Z \in \boldsymbol{M}_n : \det Z > 0\}$, and consider the ($n$-dimensional version of the) finite elastostatics operator $A\colon U \to Y$ defined by

$$A(x) = (-\operatorname{div} S(x),\ S(x)|_{\partial\Omega}\nu)\,, \tag{3.3}$$

where ν is the outward, unit normal to $\partial\Omega$, and $S(x)$ is the function from $\overline{\Omega}$ into $\boldsymbol{M}_n$ defined by setting

$$S(x)(t) = s(t, \partial x(t)) \tag{3.4}$$

for all $t \in \overline{\Omega}$. Here $\partial x(t)$ denotes the gradient at t of the function $x\colon \overline{\Omega} \to \boldsymbol{R}^n$. We suppose that

$$s(t, RZ) = Rs(t, Z) \qquad \forall (t, Z, R) \in \Omega \times \boldsymbol{M}_n^+ \times \boldsymbol{O}_n^+\,, \tag{3.5}$$

$$s(t, Z)Z^T \in \mathrm{Sym}_n \qquad \forall (t, Z) \in \Omega \times \boldsymbol{M}_n^+\,, \tag{3.6}$$

where Z^T is the transpose of the matrix Z, $\boldsymbol{O}_n^+$ denotes the set of $Z \in \boldsymbol{M}_n^+$ such that $Z^T = Z^{-1}$, and Sym_n denotes the set of symmetric elements of $\boldsymbol{M}_n$. In the physical context: Ω represents a reference configuration of an elastic body, the function x represents a deformation of the body, the function s is the response function for the first Piola-Kirchhoff stress, the symmetry (3.5) follows from the material frame indifference, while (3.6) is a consequence of the balance of angular momentum. On the function s we make the further two hypotheses:

$$s(t, I) = 0 \qquad \forall t \in \Omega\,, \tag{3.7}$$

$$\sum_{i,j,h,k=1}^{n} \partial_{Z_{hk}} s(t, I) Z_{hk} Z_{ij} > 0 \qquad \forall t \in \overline{\Omega} \quad \text{and } \forall Z \in \mathrm{Sym}_n \setminus \{0\}\,, \tag{3.8}$$

where I is the unit element of the ring $\boldsymbol{M}_n$. Note that $I = \partial\iota_\Omega$, where ι_Ω denotes the identity function from Ω into $\boldsymbol{R}^n$. Thus (3.7) and (3.8) concerns the behaviour of the function s at the deformation ι_Ω, namely at the reference configuration: (3.7) means that the reference configuration is unstressed, and (3.8) is usually assumed when (3.7) holds. It is not difficult to see that if (3.5) and (3.6) holds, then *the operator A defined by* ((3.3), (3.4)) *has the properties* (2.1), (2.2), (2.3) *when G is the group of isometries of $\boldsymbol{R}^n$* (*i.e.*, function from $\boldsymbol{R}^n$ onto $\boldsymbol{R}^n$ of the type $y \mapsto c + Ry$, with $c \in \boldsymbol{R}^n$ and $R \in \boldsymbol{O}_n^+$) *and ϱ_g, $\tilde{\varrho}_g$ are defined by putting*

$$\varrho_g(x) = g \circ x\,, \qquad \tilde{\varrho}_g(y_1, y_2) = (g \circ y_1, g \circ y_2)$$

for all $x \in X$, $y_1 \in Y_1$ and $y_2 \in Y_2$. (Cf. Valent [4]). Furthermore, it is possible to prove (see Valent [3, Chapter III]) that, under hypotheses (3.5)-(3.8), if Ω is of class C^{m+2} and $s \in C^{m+2}(\overline{\Omega} \times \boldsymbol{M}_n, \boldsymbol{M}_n)$ [respectively Ω of class $C^{m+2,\lambda}$ and $s \in C^{m+3}(\overline{\Omega} \times \boldsymbol{M}_n, \boldsymbol{M}_n)$], then A is a C^m-mapping when X, Y_1, Y_2 are defined by (3.1) [respectively by (3.2)], and moreover

$$\operatorname{Ker} A'(\iota_\Omega) = \mathcal{R}(\iota_\Omega)\,, \qquad \operatorname{Im} A'(\iota_\Omega) = \mathcal{N}(\iota_\Omega)^0\,,$$

whence

$$Y = \mathcal{N}(\iota_\Omega) \oplus \operatorname{Im} A'(\iota_\Omega)\,,$$

or, more in general,

$$Y = \mathcal{N}(\varrho_g(\iota_\Omega)) \oplus \operatorname{Im} A'(\varrho_g(\iota_\Omega))$$

for any isometry g of $\boldsymbol{R}^n$ (by Remark 2.1). It is interesting to observe that, for a suitable choice of the base of $\mathcal{R}$, we have

$$\gamma(x,y)(t) = \left(\int_{\Omega} y_1 + \int_{\partial\Omega} y_2\right) + \left(\int_{\Omega} x \wedge y_1 + \int_{\partial\Omega} x \wedge y_2\right) t\,, \qquad t \in \overline{\Omega}\,, \tag{3.9}$$

for any $x \in X$ and $y = (y_1, y_2) \in Y$, where $x \wedge y_1$ and $x \wedge y_2$ are pointwise defined, namely $(x \wedge y_1)(t) = x(t) \wedge y_1(t)\ \forall t \in \overline{\Omega}$ and $(x \wedge y_2)(t) = x(t) \wedge y_2(t)\ \forall t \in \partial\Omega$, with $\wedge$ the external product on $\boldsymbol{R}^n$.

The *second example* concerns the (n-dimensional version of the) linear elastostatics. In this case the operator A has the form (3.3), with $S(x)$ defined by

$$S(x) = \left(\sum_{h,k=1}^{n} s_{ijhk}\,\partial_k x_h\right)_{i,j=1,\dots,n},$$

where the s_{ijhk} are given real-valued functions defined on $\overline{\Omega}$ and x_h is the h-th component of the $\boldsymbol{R}^n$-valued function x. In Valent [4] it has been remarked that, *if*

$$s_{ijhk} = s_{hkij} = s_{jihk}\,,$$

then A has the properties (2.1), (2.2), (2.3) *when G is the tangent space at the identity function on $\boldsymbol{R}^n$ to the manifold of isometries of $\boldsymbol{R}^n$* (*i.e.*, G is the set of functions $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ of the type $g(t) = c + Wt$, $t \in \boldsymbol{R}^n$, with $c \in \boldsymbol{R}^n$ and W a skewsymmetric element of $\boldsymbol{M}_n$) *and ϱ_g, $\tilde{\varrho}_g$ are defined by putting*

$$\varrho_g(x) = x + g|_{\Omega}\,, \qquad \tilde{\varrho}_g(y_1, y_2) = (y + g|_{\Omega}, y + g|_{\partial\Omega})\,.$$

If, in addition,

$$\sum_{i,j,h,k=1}^{n} s_{ijhk}(t)\,Z_{ij}Z_{hk} > 0 \qquad \forall t \in \overline{\Omega} \qquad \text{and } \forall Z \in \mathrm{Sym}_n \setminus \{0\}\,,$$

it is known (see Valent [3]) that

$$Y = \mathcal{N}(0) \oplus \mathrm{Im}\,A'(0) \qquad (= \{(g|_{\Omega}, g|_{\partial\Omega})\colon g \in G\} \oplus \mathrm{Im}\,A)\,.$$

Finally, we observe that a suitable choice of the base of $\mathcal{R}$ leads to

$$\gamma(x,y)(t) = \left(\int_{\Omega} y_1 + \int_{\partial\Omega} y_2\right) + \left(\int_{\Omega} \iota_{\Omega} \wedge y_1 + \int_{\partial\Omega} \iota_{\partial\Omega} \wedge y_2\right) t\,, \qquad t \in \overline{\Omega}\,,$$

for all $x \in X$ and $y = (y_1, y_2) \in Y$, where $\iota_{\partial\Omega}$ denotes the identity function from $\partial\Omega$ into $\boldsymbol{R}^n$.

We observe that, in the context of elastostatics, *the operator B has the meaning of a loading operator*. Since here B is an arbitrary smooth mapping, our results applie to an arbitrary loading operator (depending on the deformation x in a quite general manner). For some concrete examples of loading operators we refer to Valent [4, Sect. 7], or Valent [3].

4. A basic lemma and preliminary remarks

Let $\xi_0 \in X$ be such that (2.4), (2.5) and (2.6) hold. In view of ((2.2), (2.4)) we have, for any $g_0 \in G$, $A(\varrho_{g_0}(\xi_0)) = 0$. Thus, for any $g_0 \in G$, the equation $A(x) + \varepsilon B(x) = 0$ is satisfied with $x = \varrho_{g_0}(\xi_0)$ and $\varepsilon = 0$. We observe that a direct application of the implicit function theorem to this equation in order to express x as a function of ε near $(\varrho_{g_0}(\xi_0), 0)$ is not possible because, by Remark 2.1, the partial differential with respect to x at $(\varrho_{g_0}(\xi_0), 0)$ of the mapping $(x, \varepsilon) \mapsto A(x) + \varepsilon B(x)$ takes its values in $\mathcal{N}(\varrho_{g_0}(\xi_0))^0$, while $A(x) + \varepsilon B(x)$ does not belong to $\mathcal{N}(\varrho_{g_0}(\xi_0))^0$. We also remark that the condition $A(x) + \varepsilon B(x) \in \mathcal{N}(\varrho_{g_0}(\xi_0))^0$, *i.e.* $\gamma(\varrho_{g_0}(\xi_0), A(x) + \varepsilon B(x)) = 0$, involves the parameter ε, while, by virtue of (2.3)″, the condition

$$A(x) + \varepsilon B(x) \in \mathcal{N}(x)^0\,, \qquad \varepsilon \neq 0\,,$$

is satisfied if and only if $B(x) \in \mathcal{N}(x)^0$ and hence does not involve ε. For this reason the following lemma will be important.

LEMMA 4.1. *For each $\bar{x} \in X$ there is a neighborhood $U_{\bar{x}}$ of $\bar{x}$ in X such that*

$$\mathcal{N}(\bar{x}) \cap \mathcal{N}(x)^0 = \{0\} \qquad \forall x \in U_{\bar{x}}\,.$$

PROOF. $\mathcal{N}(\bar{x})$ is a linear subspace of Y of (finite) dimension r; therefore it is closed. For any $x \in X$ let $\alpha(x)$ be the (continuous, linear) mapping from $\mathcal{N}(\bar{x})$ into $\boldsymbol{R}^r$ defined by putting

$$\alpha(x)(y) = \gamma(x, y)$$

for every $y \in \mathcal{N}(\bar{x})$. Note that $\alpha(\bar{x})$ is one-to-one (and hence an isomorphism of $\mathcal{N}(\bar{x})$ onto $\boldsymbol{R}^r$ for the structures of topological linear space), because the condition $\alpha(\bar{x})(y) = 0$ means $y \in \mathcal{N}(\bar{x})^0$, and so if $\alpha(\bar{x})(y) = 0$ with $y \in \mathcal{N}(\bar{x})$ then $y = 0$. It is easy to show that $x \mapsto \alpha(x)$ is a continuous mapping from X into the space of all continuous, linear mappings from $\mathcal{N}(\bar{x})$ into $\boldsymbol{R}^r$ endowed with the bounded convergence topology. Therefore, as the set of invertible elements of this space is open in it, there is a neighborhood $U_{\bar{x}}$ of $\bar{x}$ in X such that $\alpha(x)$ is one-to-one $\forall x \in U_{\bar{x}}$. Consequently, from $y \in \mathcal{N}(\bar{x}) \cap \mathcal{N}(x)^0$ with $x \in U_{\bar{x}}$ it follows $y = 0$, because the condition $y \in \mathcal{N}(x)^0$ can be written in the form $\alpha(x)(y) = 0$. ∎

Now, we observe that if $g \in G$ and $\xi = \varrho_{g^{-1}}(x)$, then from (2.2) it follows that the equality $A(x) + \varepsilon B(x) = 0$ is true if and only if

$$A(\xi) + \varepsilon\, \tilde{l}_{g^{-1}} B(\varrho_g(\xi)) = 0\,.$$

Then, with Lemma 4.1 in mind, our aim will be to see whether for some $g_0 \in G$ it occurs that for any ξ near ξ_0 in X there is an element $\hat{g}(\xi)$ of G near g_0 such that

$$\tilde{l}_{\hat{g}(\xi)^{-1}} B(\varrho_{\hat{g}(\xi)}(\xi)) \in \mathcal{N}(\xi)^0\,,$$

namely

$$\gamma(\xi, \tilde{l}_{\hat{g}(\xi)^{-1}} B(\varrho_{\hat{g}(\xi)}(\xi))) = 0\,.$$

In this case, if E_{ξ_0} denotes a topologically supplementary subspace of $\mathcal{R}(\xi_0)$ in X and p_{ξ_0} denotes the projection of Y onto its (closed) subspace $\mathcal{N}(\xi_0)^0$ ($= \operatorname{Im} A'(\xi_0)$), then setting for $(\eta, \varepsilon) \in E_{\xi_0} \times \boldsymbol{R}$

$$\Lambda(\eta, \varepsilon) = p_{\xi_0}(A(\xi_0 + \eta) + \varepsilon\, \tilde{l}_{\hat{g}(\xi_0+\eta)^{-1}} B(\varrho_{\hat{g}(\xi_0+\eta)}(\xi_0 + \eta)))\,, \tag{4.1}$$

the implicit function thorem applies to the equation $\Lambda(\xi, \varepsilon) = 0$. Indeed, Λ takes its values in $\mathcal{N}(\xi_0)^0$ and, by ((2.7), (2.8)), its partial differential with respect to η at $(0, 0)$ is a bijection from E_{ξ_0} onto $\mathcal{N}(\xi_0)^0$.

5. Subspaces of $T_e G$ associated to the operator B at any $(\xi_0, g_0) \in X \times G$

Let us define, for any $\xi \in U$, a function $M_\xi \colon G \to \mathcal{R}$ by putting

$$M_\xi(g) = \gamma(\varrho_g(\xi),\ B(\varrho_g(\xi))$$

for all $g \in G$. For any $g_0 \in G$, the differential at e of the translation $g \mapsto g g_0$ of G will be denoted by $\tilde{g}_0$; so $\tilde{g}_0$ is a continuous, linear mapping from $T_e G$ into the tangent space $T_{g_0} G$ to G at g_0.

Definition 5.1. Let $(\xi_0, g_0) \in U \times G$ and let $0 \neq v \in T_e G$. We will say that v is *critical at* (ξ_0, g_0) *for* B if the differential $M'_{\xi_0}(g_0)$ of M_{ξ_0} at g_0 vanishes at the element $\tilde{g}_0(v)$ of $T_{g_0} G$.

Definition 5.2. Let $\langle\, ,\rangle$ be the inner product on $\mathcal{R}$ carried by the inner product of $\boldsymbol{R}^r$ when the base of $\mathcal{R}$ is that one used for defining γ. For any $(\xi_0, g_0) \in U \times G$ we will denote by $\mathcal{K}_{\xi_0, g_0}(B)$ any maximal element of the set of linear subspaces $\mathcal{K}$ of $\mathcal{R}$ having the following property: «if $R_v \in \mathcal{K}$ and $\langle R_v, \gamma(x, B(x))\rangle = 0$ for every x belonging to some neighborhood of $\varrho_{g_0}(\xi_0)$ then $R_v = 0$».

In order to clarifying the meaning of the definition of the subspaces $\mathcal{K}_{\xi_0, g_0}(B)$ of $\mathcal{R}$ it is useful to observe that, if $(R_{v_1}, \dots, R_{v_r})$ is a base of $\mathcal{R}$ and $\bar{J}$ is a maximal element of the family of subsets J of $\{1, \dots, r\}$ such that the set of the functions

$$x \mapsto \langle R_{v_j}, \gamma(x, B(x))\rangle, \qquad j \in J,$$

is linearly independent on every neighborhood of $\varrho_{g_0}(\xi_0)$ in X, then $\sum_{j \in \bar{J}} \boldsymbol{R} R_{v_j}$ is a $\mathcal{K}_{\xi_0, g_0}(B)$. Conversely, any $\mathcal{K}_{\xi_0, g_0}(B)$ is of this type for a suitable choice of the base $(R_{v_1}, \dots, R_{v_2})$ of $\mathcal{R}$.

Lemma 5.3. *Let* $(\xi_0, g_0) \in X \times G$ *be such that* $M_{\xi_0}(g_0)$ *and let* T *be a linear subspace of* $T_e G$ *such that the set* $\{R_v \colon v \in T\}$ *contains some* $\mathcal{K}_{\xi_0, g_0}(B)$. *Suppose that* (2.10) *holds, that* B *is of class* C^m [*respectively is analytic at* $\varrho_{g_0}(\xi_0)$], *and that no element* $\neq 0$ *of* T *is critical for* B *at* (ξ_0, g_0). *Then neighborhoods* V_0 *of* ξ_0 *in* U *and* W_0 *of* g_0 *in* G *exist such that for each* $\xi \in V_0$ *there is a unique element* $\hat{g}(\xi)$ *of* $W_0 \cap (\exp T) g_0$ $(= W_0 \cap \cap \{(\exp v) g_0 \colon v \in T\})$ *such that*

$$\gamma(\xi, \tilde{l}_{\hat{g}(\xi)^{-1}} B(\varrho_{\hat{g}(\xi)}(\xi))) = 0\,. \tag{5.1}$$

V_0 *and* W_0 *can be chosen such that the mapping* $\xi \mapsto \hat{g}(\xi)$ *is of class* C^m [*respectively is analytic at* ξ_0].

Proof. Let $\mathcal{K} = \{R_v \colon v \in T\}$, and let π be the proiection of $\mathcal{R}$ onto $\mathcal{K}$ with respect to the inner product $\langle\, ,\rangle$. For any $(v, \xi) \in T \times U$ we set

$$\Gamma(v, \xi) = (\pi \circ \gamma)(\varrho_{(\exp v) g_0}(\xi), B(\varrho_{(\exp v) g_0}(\xi)))\,,$$

and we note that $\Gamma(v, \xi_0) = (\pi \circ M_{\xi_0})((\exp v) g_0)$. Γ is a C^m mapping from $T \times U$ into

$\mathcal{K}$: it is analytic at $(0, \xi_0)$ provided B is analytic at $\varrho_{g_0}(\xi_0)$. Since $M_{\xi_0}(g_0) = 0$ we have $\Gamma(0, \xi_0) = 0$. The differential at 0 of the mapping $v \mapsto \Gamma(v, \xi_0)$ is the mapping from T into $\mathcal{R}$

$$v \mapsto (\pi \circ M'_{\xi_0}(g_0))(\tilde{g}_0(v)).$$

As $\mathcal{K}$ contains some $\mathcal{K}_{\xi_0, g_0}(B)$, from the definition of $\mathcal{K}_{\xi_0, g_0}(B)$ it follows that there is a linear mapping $L: \mathcal{K} \to \mathcal{R}$ such that

$$\gamma(x, B(x)) = (L \circ \pi \circ \gamma)(x, B(x)) \tag{5.2}$$

for every x belonging to a suitable neighborhood U_0 of $\varrho_{g_0}(\xi_0)$ in U. Then, choosing a neighborhood W of g_0 in G such that $\varrho_g(\xi_0) \in U_0$ $\forall g \in W$, we have

$$M_{\xi_0}(g) = (L \circ \pi)(M_{\xi_0}(g)) \qquad \forall g \in W.$$

Therefore, if the differential at 0 of the mapping $v \mapsto \Gamma(v, \xi_0)$ vanishes at $v_0 \varepsilon T$ we have $M'_{\xi_0}(g_0)(\tilde{g}_0(v_0))$, which implies $v_0 = 0$ because no element $\neq 0$ of T is critical for B at (ξ_0, g_0). Thus the differential at 0 of the mapping $v \mapsto \Gamma(v, \xi_0)$ is a one-to-one (linear) mapping from T into $\mathcal{K}$; hence, as $\dim \mathcal{K} \leqslant \dim T$, we have $\dim \mathcal{K} = \dim T$ and the mapping $v \mapsto \Gamma(v, \xi_0)$ is a bijection from T onto $\mathcal{K}$. Then, in view of the implicit function theorem applied to the equation $\Gamma(v, \xi) = 0$, open neighborhoods V of ξ_0 in U and N of 0 in $T_e G$ exist such that for each $\xi \in V$ there is one and only one element $\widehat{v}(\xi)$ of $T \cap N$ such that $\Gamma(\widehat{v}(\xi), \xi) = 0$. Moreover, V and N can be chosen such that the mapping $\xi \mapsto \widehat{v}(\xi)$ is of class C^m, and analytic at ξ_0 if B is analytic at $\varrho_{g_0}(\xi_0)$. Note that the equality $\Gamma(\widehat{v}(\xi), \xi) = 0$ implies

$$(\pi \circ \gamma)\big(\varrho_{(\exp \widehat{v}(\xi))g_0}(\xi),\ B(\varrho_{(\exp \widehat{v}(\xi))g_0}(\xi))\big) = 0.$$

Hence, if V_0 is a neighborhood of ξ_0 in X contained in V and such that

$$(\exp \widehat{v}(\xi))g_0 \in W \qquad \forall \xi \in V_0,$$

then, in view of (5.2), we have

$$\gamma\big(\varrho_{(\exp \widehat{v}(\xi))g_0}(\xi),\ B(\varrho_{(\exp \widehat{v}(\xi))g_0}(\xi))\big) = 0 \qquad \forall \xi \in V_0,$$

and this implies, by (2.10),

$$\gamma\big(\xi,\ \tilde{l}_{((\exp \widehat{v}(\xi))g_0)^{-1}} B(\varrho_{(\exp \widehat{v}(\xi))g_0}(\xi))\big) = 0 \qquad \forall \xi \in V_0.$$

Therefore, to conclude the proof it suffices to set, for every $\xi \in V_0$,

$$\widehat{g}(\xi) = (\exp \widehat{v}(\xi))g_0,$$

and take as W_0 a neighborhood of g_0 contained in W and such that $v \in N$ whenever $v \in T$ and $(\exp v)g_0 \in W_0$. ■

6. Local existence, uniqueness, and analytic dependence on ε

We are now in a position to prove a local theorem on existence, uniqueness, and analytic dependence on the parameter ε for the equation $A(x) + \varepsilon B(x) = 0$.

THEOREM 6.1. *Suppose that A and B are of class C^m, that (2.1), (2.2), (2.3), (2.10) hold, that there is (ξ_0, g_0) in $U \times G$ such that (2.4), (2.5), (2.6) are satisfied and $M_{\xi_0}(g_0) = 0$, that there is a linear subspace T of $T_e G$ such that the set $\{R_v : v \in T\}$ contains some $\mathcal{K}_{\xi_0, g_0}(B)$ and no element $\neq 0$ of T is critical for B at (ξ_0, g_0). Fix a topological supplementary E_{ξ_0} of $\operatorname{Ker} A'(\xi_0)$ in X, and set $x_0 = \varrho_{g_0}(\xi_0)$ and*

$$D_{\xi_0, g_0}(T) = \bigcup_{v \in T} \varrho_{(\exp v) g_0}(\xi_0 + E_{\xi_0}).$$

Then a neighborhood V of ξ_0 in X and a neighborhood W of g_0 in G exist such that for each $\varepsilon \in \boldsymbol{R}$ with $|\varepsilon|$ sufficiently small there are a unique ξ_ε in $V \cap (\xi_0 + E_{\xi_0})$ and a unique g_ε in $W \cap (\exp T) g_0$ such that putting $x_\varepsilon = \varrho_{g_\varepsilon}(\xi_\varepsilon)$ we have

$$A(x_\varepsilon) + \varepsilon B(x_\varepsilon) = 0. \tag{6.1}$$

Consequently, if the mapping $(\xi, g) \mapsto \varrho_g(\xi)$ from $(\xi_0 + E_{\xi_0}) \times (\exp T) g_0$ onto $D_{\xi_0, g_0}(T)$ is a local homeomorphism at (ξ_0, g_0), then a neighborhood U_0 of x_0 in U exists such that for each $\varepsilon \neq 0$, with $|\varepsilon|$ sufficiently small, there is a unique x_ε in $U_0 \cap D_{\xi_0, g_0}(T)$ satisfying (6.1). The mapping $\varepsilon \mapsto x_\varepsilon$, defined also for $\varepsilon = 0$ by assuming that its value at 0 is x_0, is of class C^m in a suitable neighborhood of 0; it is analytic at 0 provided A is analytic at ξ_0 and B is analytic at x_0.

PROOF. Let V_0, W_0 and $\widehat{g}$ be as in the statement of Lemma 5.3, and let V_{ξ_0} be a neighborhood of ξ_0 in X such that $\mathcal{N}(\xi_0) \cap \mathcal{N}(\xi)^0 = \{0\}$ $\forall \xi \in V_{\xi_0}$ (see Lemma 4.1). For any $\varepsilon \in \boldsymbol{R}$ and $\eta \in E_{\xi_0}$ with $\eta + \xi_0 \in \mathring{V}_0$ we define $\Lambda(\eta, \varepsilon)$ as in (4.1). Λ is a C^m mapping from the open subset $((\mathring{V}_0 - \xi_0) \cap E_{\xi_0}) \times \boldsymbol{R}$ of the Banach space $E_{\xi_0} \times \boldsymbol{R}$ into the closed subspace $\operatorname{Im} A'(\xi_0)$ of Y. By (2.4) we have $\Lambda(0, 0) = 0$. Moreover, the differential at 0 of the mapping $\eta \mapsto \Lambda(\eta, 0)$ is $A'(\xi_0)$, which is a bijection of E_{ξ_0} onto $\operatorname{Im} A'(\xi_0)$. Therefore, in view of the implicit function theorem applied to the equation $\Lambda(\eta, \varepsilon) = 0$, there is a neighborhood V_1 of 0 in X contained in $V_0 - \xi_0$ such that for any ε with $|\varepsilon|$ sufficiently small a unique η_ε exists in $V_1 \cap E_{\xi_0}$ such that $\Lambda(\eta_\varepsilon, \varepsilon) = 0$ and the mapping $\eta \mapsto \eta_\varepsilon$ is of class C^m; this mapping is analytic at 0 provided A is analytic at ξ_0 and B is analytic at x_0, because in this case Λ is analytic at $(0, 0)$. We set $\xi_\varepsilon = \xi_0 + \eta_\varepsilon$ and observe that, since $\gamma(\xi_\varepsilon, A(\xi_\varepsilon)) = 0$ by virtue of (2.3)″, from Lemma 5.3 it follows that

$$\gamma(\xi_\varepsilon, A(\xi_\varepsilon) + \varepsilon \tilde{l}_{\widehat{g}(\xi_\varepsilon)^{-1}} B(\varrho_{\widehat{g}(\xi_\varepsilon)}(\xi_\varepsilon))) = 0,$$

namely

$$A(\xi_\varepsilon) + \varepsilon \tilde{l}_{\widehat{g}(\xi_\varepsilon)^{-1}} B(\varrho_{\widehat{g}(\xi_\varepsilon)}(\xi_\varepsilon)) \in \mathcal{N}(\xi_\varepsilon)^0.$$

Note that the condition $\Lambda(\eta_\varepsilon, \varepsilon) = 0$ means that

$$A(\xi_\varepsilon) + \varepsilon \tilde{l}_{\widehat{g}(\xi_\varepsilon)^{-1}} B(\varrho_{\widehat{g}(\xi_\varepsilon)}(\xi_\varepsilon)) \in \mathcal{N}(\xi_0).$$

Then, taking $V_1 \subseteq V_{\xi_0} - \xi_0$ and using Lemma 4.1, we deduce that

$$A(\xi_\varepsilon) + \varepsilon \tilde{l}_{\widehat{g}(\xi_\varepsilon)^{-1}} B(\varrho_{\widehat{g}(\xi_\varepsilon)}(\xi_\varepsilon)) = 0.$$

Hence, setting

$$g_\varepsilon = \widehat{g}(\xi_\varepsilon), \qquad x_\varepsilon = \varrho_{g_\varepsilon}(\xi_\varepsilon),$$

in view of (2.2) we have $A(x_\varepsilon) + \varepsilon B(x_\varepsilon) = 0$. To conclude it suffices to take $W = W_0$, and recall that g is of class C^m and it is analytic at ξ_0 provided B is analytic at x_0

7. An application to the live traction problem in finite elastostatics

Let us go back to the finite elastostatics operator introduced in Section 3, in order to make explicit a consequence of Theorem 6.1 in such a context. The loading operator B will be an arbitrary mapping from X into $Y_1 \times Y_2$, with X, Y_1, Y_2 chosen as in (3.1) or (3.2). The two components of B will be denoted by B_1 and B_2. We recall that, here, G is the group of isometries of $\boldsymbol{R}^n$ that U is the open subset of X whose elements are the orientation preserving diffeomorphisms of $\overline{\Omega}$ onto a subset of $\boldsymbol{R}^n$, and that $\varrho_g(x) = g \circ x \ \forall (x, g) \in U \times G$. We observe that $T_e G$ is the set of infinitesimal rigid deformations of $\boldsymbol{R}^n$, *i.e.* the set of functions v: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ of the type $v(t) = c + Wt$, $(t \in \boldsymbol{R}^n)$, with $c \in \boldsymbol{R}^n$ and $W \in \mathrm{Skew}_n$, that $R_v(x) = v \circ x \ \forall (x, v) \in U \times T_e G$, and that $\mathcal{R}$ can be identified with $T_e G$ through the mapping $v \mapsto R_v$. We fix a base of $\mathcal{R}$ such that $\gamma(x, y)$ has the form (3.9); thus for any $(\xi, g) \in U \times G$ and $t \in \overline{\Omega}$, we have

$$M_\xi(g)(t) = \left(\int_\Omega B_1(\varrho_g(\xi)) + \int_{\partial\Omega} B_2(\varrho_g(\xi)) \right) +$$

$$+ \left(\int_\Omega \varrho_g(\xi) \wedge B_1(\varrho_g(\xi)) + \int_{\partial\Omega} \varrho_g(\xi) \wedge B_2(\varrho_g(\xi)) \right) t\,.$$

Therefore $\mathcal{K}_{\xi_0, g_0}(B)$ can be regarded as a maximal element of the set of linear subspaces $\mathcal{K}$ of $T_e G$ having the following property: «if v: $t \mapsto c + Wt$ is an element of $\mathcal{K}$ and

$$c \cdot \left(\int_\Omega B_1(x) + \int_{\partial\Omega} B_2(x) \right) + W \cdot \left(\int_\Omega x \wedge B_1(x) + \int_{\partial\Omega} x \wedge B_2(x) \right) = 0$$

for all x belonging to some neighborhood of $g_0 \circ \xi_0$ in X, then $v = 0$».

We remark that a topological supplementary E_{ι_Ω} of $\mathcal{R}(\iota_\Omega)$ $[= \mathrm{Ker}\, A'(\iota_\Omega)]$ in X is

$$\left\{ x \in X \colon \int_\Omega x = 0\,, \ \int_\Omega \partial x \in \mathrm{Sym}_n \right\}.$$

For a proof of this fact we refer to Valent [3, Ch. III, Sect. 1]. Finally we point out the following result which can easily deduced from Lemma 4.4 in [3, Ch. V]: *the mapping* $(\xi, g) \mapsto \varrho_g(\xi)$ *is a homeomorphism of the set of those elements* $(\xi, (\exp v) g_0)$ *of* $(\iota_\Omega + E_{\iota_\Omega}) \times (\exp T) g$ *such that* $\int_\Omega \partial\xi$ *is positive definite onto the set of elements* x *of* X

such that $\det \int_\Omega \partial x > 0$ *and the rigid deformation of* $\boldsymbol{R}^n$

$$t \mapsto \left(\int_\Omega (x - R_x R_{g_0}^T g_0) \right) + (R_x R_{g_0}^T) t$$

belongs to $\exp T$; here R_x is the element of $\boldsymbol{O}_n^+$ such that the matrix $R_x^T \int_\Omega x$ is symmetric and positive definite, and R_{g_0} has an analogous meaning.

We are now in a position to state a consequence of Theorem 6.1 for the general live traction problem in finite elastostatics; in order to simplify the statement we take $\xi_0 = \iota_\Omega$.

THEOREM 7.1. *Let* $A \colon X \to Y_1 \times Y_2$ *be defined by* ((3.3), (3.4)), *with* X, Y_1, Y_2 *as in* (3.1), *let* Ω *be of class* C^{m+2} *and* $s \in C^{m+2}(\overline{\Omega} \times \boldsymbol{M}_n^+, \boldsymbol{M}_n)$ *satisfying* (3.5), (3.6), (3.7), (3.8). *Let* $B \colon X \to Y_1 \times Y_2$ *of class* C^m *and let* g_0 *be an isometry of* $\boldsymbol{R}^n$ *such that*

$$\int_\Omega B_1(g_0) + \int_{\partial\Omega} B_2(g_0) = 0\,, \qquad \int_\Omega g_0 \wedge B_1(g_0) + \int_{\partial\Omega} g_0 \wedge B_2(g_0) = 0\,.$$

Suppose that there is a linear subspace T *of* $T_e G$ *containing some* $\mathcal{K}_{\iota_\Omega, g_0}(B)$ *and such that no element* $\neq 0$ *of* T *is critical for* B *at* (ι_Ω, g_0). *Then for each* $\varepsilon \neq 0$ *with* $|\varepsilon|$ *small enough there is one and only one deformation* x_ε *of* $\overline{\Omega}$ *belonging to a suitable neighborhood of* $g_0|_{\overline{\Omega}}$ *in* $W^{m+2,p}(\Omega, \boldsymbol{R}^n)$ *that satisfy* (6.1) *and such that the rigid deformation*

$$t \mapsto \left(\int_\Omega (x_\varepsilon - R_{x_\varepsilon} R_{g_0}^T g_0) \right) + (R_{x_\varepsilon} R_{g_0}^T) t$$

of $\boldsymbol{R}^n$ *belongs to* $\exp T$. *The mapping* $\varepsilon \mapsto x_\varepsilon$, *defined also for* $\varepsilon = 0$ *by assuming that its value at* 0 *is* $g_0|_{\overline{\Omega}}$, *is of class* C^m *in a suitable neighborhood of* 0.

REMARK 7.2. In the statement of Theorem 7.1, if $T = T_e G$ then (7.1) obviously belongs to $\exp T$, and thus we have local uniqueness near $g_0|_{\overline{\Omega}}$ for the traction problem $A(x) + \varepsilon B(x) = 0$.

REMARK 7.3. In the very particular case when B is a constant operator (dead loading operator) T cannot be equal to $T_e G$ because each constant function $t \mapsto c$ from $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ is critical for B at (ι_Ω, g_0). Therefore T must be a space of infinitesimal rigid rotations, *i.e.* mappings from $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ of the type $t \mapsto Wt$, with $W \in \mathrm{Skew}_n$. Moreover, when $n = 3$, one can prove (see Valent [3, Ch. V]) that *the infinitesimal rigid rotation* $t \mapsto Wt$ *is critical for* B *at* (ι_Ω, g_0) *if and only if the axis of* W *is an «axis of equilibrium» for the loading at* $g_0|_{\overline{\Omega}}$,

REMARK 7.4. A simple example of live traction problem to which Theorem 7.1 trivially applies is the elastic *«balloon problem»* with zero body forces. (A local existence theorem for this problem was obtained by Le Dret [2]). In this case $\Omega = \Omega_e \backslash \Omega_i$ with Ω_e

and Ω_i open subsets of $\boldsymbol{R}^n$ and $\overline{\Omega}_i \subset \Omega_e$; moreover

$$(7.2) \qquad \begin{cases} B_1(x) = 0 & \text{in } \Omega\,, \\ B_2(x)(t) = -\pi_e(x)\operatorname{cof}\partial x(t)\,\nu(t) & \forall t \in \partial\Omega_e\,, \\ B_2(x)(t) = -\pi_i(x)\operatorname{cof}\partial x(t)\,\nu(t) & \forall t \in \partial\Omega_i\,, \end{cases}$$

where $\operatorname{cof}\partial x(t)$ is the matrix of cofactors of the matrix $\partial x(t)$, and π_e, π_i are smooth $\boldsymbol{R}^+$-valued functionals of the deformation x. It is not difficult to verify that $M_{\iota_\Omega}(g) = 0$ $\forall g \in G$; therefore $\mathcal{K}_{\iota_\Omega, g_0}(B) = \{0\}$ $\forall g_0 \in G$, and any element $\neq 0$ of $T_e G$ is critical for B at (ι_Ω, g_0). It follows that, for all $g_0 \in G$, $\{0\}$ is the only linear subspace T of $T_e G$ containing $\mathcal{K}_{\iota_\Omega, g_0}(B)$ and such that no element $\neq 0$ of T is critical for B at (ι_Ω, g_0). Then, a consequence of Theorem 7.1 is the following: *if the operator A is as in the statement of Theorem* 7.1 *and B is defined by* (7.2), *then, fixed arbitrarily an isometry $g_0 \colon t \mapsto c_0 + R_0 t$ of $\boldsymbol{R}^n$, for each $\varepsilon \neq 0$ with $|\varepsilon|$ small enough there is one and only one deformation x_ε of $\overline{\Omega}$ belonging to a suitable neighborhood of $g_0|_{\overline{\Omega}}$ in $W^{m+2,p}(\Omega, \boldsymbol{R}^n)$ such that $A(x) + \varepsilon B(x) = 0$ and $R_{x_\varepsilon} = R_0$, $\int_\Omega x_\varepsilon(t)\,dt = \int_\Omega t\,dt$.*

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