Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Giuseppe Mulone, Salvatore Rionero, Brian Straughan

Unconditional nonlinear stability in a polarized dielectric liquid

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **7** (1996), n.4, p. 241–252.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1996_9_7_4_241_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1996.

Rend. Mat. Acc. Lincei
s. 9, v. 7:241-252 (1996)

Meccanica dei fluidi. — *Unconditional nonlinear stability in a polarized dielectric liquid.* Nota di GIUSEPPE MULONE, SALVATORE RIONERO e BRIAN STRAUGHAN, presentata (*) dal Corrisp. S. Rionero.

ABSTRACT. — We derive a very sharp nonlinear stability result for the problem of thermal convection in a layer of dielectric fluid subject to an alternating current (AC). It is particularly important to note that the size of the initial energy in which we establish global nonlinear stability is not restricted whatsoever, and the Rayleigh-Roberts number boundary coincides with that found by a formal linear instability analysis.

KEY WORDS: Polarized dielectric liquid; Unconditional nonlinear stability; Generalized energy.

RIASSUNTO. — *Stabilità non lineare incondizionata in un liquido dielettrico polarizzato*. Otteniamo un risultato di stabilità non lineare incondizionata per il problema della convezione termica di un fluido dielettrico soggetto ad una corrente alternata (AC). È particolarmente importante osservare che la grandezza iniziale dell'energia rispetto a cui stabiliamo il risultato di stabilità non lineare globale non ha restrizioni e i numeri critici di Rayleigh-Roberts ottenuti coincidono con quelli trovati con l'analisi formale della instabilità lineare.

1. INTRODUCTION

The theory of nonlinear energy stability in fluid mechanics is a particularly useful tool in obtaining sharp estimates below which stability is found, see *e.g.* [24, 7, 17-19, 27]. It has had success in predicting useful stability thresholds when magnetic and electric field effects are present, see *e.g.* [12, 15-19, 27-30]. Indeed, steps toward constructing a generalized energy which works in many problems have been taken, see *e.g.* [6, 20]. However, many of the references above suffers from a serious drawback. If one desires nonlinear stability results which are sharp in the sense that the parameter threshold is close to that obtained by linear instability theory then a major sacrifice has to be made in the size of the initial data which are allowed in the stability domain. This is a particularly serious restriction. For, if R_E denotes the nonlinear stability threshold and R is a parameter such as Rayleigh number then typically $E(0) < a(R_E - R)^{\alpha}$, $a > 0$, $\alpha > 0$, for a generalized energy $E(t)$. Thus, as $R \to R_E^-$, $E(0) \to 0$, *i.e.* the radius of attraction becomes vanishingly small. In this sense, the stability so obtained is perhaps more a rigorous derivation of a linearization principle, with precisely defined constants. We should stress that when effects of a magnetic or electric field are present, or when such as rotation is present, then the nonlinearities are particularly severe, and it is amazing any rigorous nonlinear stability results may be proved at all. In our opinion, the major goal to reach now in the nonlinear energy stability theory is to remove the conditional aspect, *i.e.* the restrictions of the aforesaid type on $E(0)$, without lowering the threshold of the nonlinear stability. In this respect we recall that in recent times much effort has been devoted to producing nonstandard arguments or changing the energy functional to achieve nonlinear stability without too re-

(*) Nella seduta del 13 giugno 1996.

strictive conditions on the initial data. These techniques are presently *ad hoc*. We mention specifically convection with black body radiation, Neitzel *et al.* [14]; penetrative convection with quadratic, cubic and quintic density - temperature laws, thawing subsea permafrost, and convection in a vertical porous slot, the last three topics being reviewed in [27]. In this paper we produce what we believe is the first *truly unconditional* energy stability analysis for non-isothermal convection in a dielectric fluid subject to an AC current, cf. [26].

The problem of thermal convection in a layer of a dielectric fluid has attracted much recent attention and may be modelled from the general theories of *e.g.* Landau *et al.* [10], Muller [11], Rosenweig [22]; stability studies of this problem are contained in *e.g.* [21, 27]. Here however, we are interested in a yet more recent instability found in dielectric fluids which arises when the electric field is an alternating one of sufficiently high frequency; in this case the polarization body force is the dominant force and strongly influences the instability mechanism. For instance, when a 2mm layer of the dielectric fluid nitrobenzene is heated from below a temperature difference of 3.2 °K will give rise to thermal convection, but when an oscillatory potential difference with a root mean square value 1kV is applied across the layer then one only requires a temperature difference of 2.6 °K to see an equivalent thermal instability. The basic theory for this instability has been presented by Stiles [25] and Stiles *et al.* [26] who adopted the model of Roberts [21] to the problem at hand. The work of [25, 26] employed a constant viscosity and applied linear instability theory and weakly nonlinear stability theory respectively.

Although it is an entirely different physical problem, it is important to realize that, the linear and nonlinear analysis presented here applies equally well to the thermally heated or cooled ferromagnetic fluid problem investigated by Blennerhassett *et al.* [1].

The plan of the paper is as follows: in Section 2 we introduce the perturbation equations for convective motions in a polarized dielectric liquid between two horizontal planes. In Section 3 we first give the linear instability results obtained by Roberts [21] with the usual *normal modes* method, then we study the linear stability of the basic motion with the Lyapunov second method. We introduce a Lyapunov function $E_1(t)$ and show that – in the case of stress-free boundaries – the linear operator $\mathcal{L}$ governing the problem is symmetric in the inner product associated to the «norm» $E_1(t)$. Then, all the eigenvalues of the problem are real, *i.e.* the (strong) principle of exchange of stabilities holds and the transition from stability to instability occurs via a stationary state. Moreover the critical stability region obtained in this way coincides with the classical linear instability analysis as found by Roberts [21] and Stiles [25], *i.e.* we obtain necessary and sufficient conditions of linear stability. In particular, for any fixed Rayleigh number, the critical Roberts number of linear instability L_c coincides with the critical E_1-stability Roberts number L_{E_1}. In Section 4 we introduce another Lyapunov function $E_2(t) = E_{2\mu}(t)$ which is equivalent to $E_1(t)$ and depends on a Lyapunov parameter μ to be chosen. We consider the full nonlinear perturbation system and show a *global nonlinear stability* result of the basic motion, *i.e.* the size of the initial energy $E_2(0)$ is not restricted whatsoever. Moreover we prove that for any fixed Rayleigh number R there exists a critical Lyapunov parameter $\mu_{\text{crit}}(R)$ such that the critical nonlinear Roberts number L_{E_2} coincides with the critical linear Roberts number L_c.

2. The equations for convective motions in a polarized dielectric liquid

Let $d > 0$. We consider a dielectric liquid in a horizontal layer contained between the planes $z = 0$ and $z = d$ with constant upper and lower temperatures T_u and T_l, with $T_l > T_u$ or $T_u > T_l$. We follow the model of Stiles [25] and Stiles *et al.* [26]. In their theory the polarization effects are manifested through the body force which has the form

$$f_e = \boldsymbol{P} \cdot \nabla \boldsymbol{E} \tag{2.1}$$

(cf. [10, p. 66]) where f_e denotes the electrical body force, $\boldsymbol{P}$ is the polarization vector, $\boldsymbol{P} = \widehat{\boldsymbol{D}} - \varepsilon_0 \boldsymbol{E}$, $\boldsymbol{E}$ is the electric field, $\widehat{\boldsymbol{D}} = \varepsilon \boldsymbol{E}$ is the electrical displacement field, ε the electrical permittivity and ε_0 is the electrical permittivity of free space. There is additionally the thermal body force f_T which is assumed to be linear in the temperature field, T.

The momentum equation (Navier-Stokes) then assumes the form

$$v_{i,t} + v_j v_{j,i} = -(1/\varrho_0)p_{,i} + 2(\nu D_{ij})_{,j} + \alpha T g \delta_{i3} + P_j E_{i,j} \tag{2.2}$$

where α is the coefficient of thermal expansion, v_i, p, g are velocity, pressure and gravity fields, $D_{ij} = (v_{i,j} + v_{j,i})/2$, and standard indicial notation is used. Here we follow [25, 26] and consider ν constant and adopt a linear relationship for the electrical permittivity ε as a function of temperature. In a forthcoming paper we shall consider the case of a viscosity depending on the temperature. Stiles *et al.* [26] write the elettrical permittivity as

$$\varepsilon = \varepsilon_r(T)\varepsilon_0, \tag{2.3}$$

where the relative permittivity is

$$\varepsilon_r = \varepsilon_r^0 + \frac{d\varepsilon_r}{dT}\Big|_{T_l}(T - T_l), \tag{2.4}$$

with $\varepsilon_r^0 = \varepsilon_r(T_l)$.

In addition to equation (2.2) the model is completed with the continuity and balance of energy equations

$$v_{i,i} = 0, \tag{2.5}$$

and

$$T_{,t} + v_i T_{,i} = k \Delta T, \tag{2.6}$$

and the Maxwell equations

$$\widehat{D}_{i,i} = 0, \qquad \nabla \times \boldsymbol{E} = 0\,. \tag{2.7}$$

$(2.7)_2$ implies the existence of an electrical potential field Φ such that $\boldsymbol{E} = \nabla \Phi$.

The stationary solution whose stability we investigate is

$$\bar{\boldsymbol{v}} = 0, \quad \bar{T} = -\beta z + T_l, \quad \beta = (T_l - T_u)/d, \tag{2.8}$$

and this gives rise to an electric field which is given by

$$\bar{\boldsymbol{E}} = E_0(1 + \beta z\, d \ln \varepsilon_r / dT)\, \boldsymbol{k} = (\varepsilon_r^0 E_0 / (\varepsilon_r^0 - k_0 \beta z))\, \boldsymbol{k}, \tag{2.9}$$

where $\boldsymbol{k} = (0, 0, 1)$, and $k_0 = d\varepsilon_r/dT|_{T_l}$. The polarization vector is then,

$$\bar{\boldsymbol{P}} = \varepsilon_0(\varepsilon_r - 1)\bar{\boldsymbol{E}} = \varepsilon_0 \varepsilon_r^0 E_0 (1 - (\varepsilon_r^0 - k_0 \beta z)^{-1})\, \boldsymbol{k}, \tag{2.10}$$

and the pressure field $\bar{p}$ is obtained from equation (2.2).

In the above an overbar denotes the steady state.

The perturbation equations for the perturbation fields of velocity u_i, pressure π, temperature θ and electrical potential ϕ are then, see [26, p. 3275],

$$\begin{cases} (1/Pr)(u_{i,t} + u_j u_{i,j}) = -\pi_{,i} + \Delta u_i + (R + L)\,\theta \delta_{i3} - L\phi_{,z}\delta_{i3} - L\theta\phi_{,iz} \\ u_{i,i} = 0 \\ \theta_{,t} + u_i\, \theta_{,i} = w + \Delta\theta \\ \Delta\phi = \theta_{,z} \end{cases} \tag{2.11}$$

where a subscript z denotes differentiation with respect to z, $u_i = (u, v, w)$ and the positive numbers P_r, L are the Prandtl, and Roberts numbers, R is the Rayleigh number. The Roberts number (electric Rayleigh number)

$$L = (\partial\varepsilon/\partial T)^2 E_0^2 \beta^2 d^4 / \varepsilon k \eta \tag{2.12}$$

is defined in terms of the electric permittivity ε of the liquid (which depends linearly on the temperature T), the gradient of temperature β, the root mean square electric field strength E_0 in the absence of gradient of temperature β, the gap d between the horizontal plates, the thermal diffusivity k and the shear viscosity η. The Prandtl and the Rayleigh numbers are defined in the usual way.

These equations hold in the three-dimensional planar region $z \in (0, 1)$. We suppose u_i, π, θ, ϕ are sufficiently smooth and satisfy a plane tiling pattern in the x, y directions so they define a perturbation cell V over the lateral boundaries of which their contributions are equal and cancel out in the ensuing integrations by parts.

The initial conditions which we use are

$$u_i(\boldsymbol{x}, 0) = u_{0i}(\boldsymbol{x}), \quad \theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}), \quad \phi(\boldsymbol{x}, 0) = \phi_0(\boldsymbol{x}), \tag{2.13}$$

with u_{0i}, θ_0 and ϕ_0 which have the same regularity hypotheses of u_i, θ and ϕ and satisfy the equations $u_{0i,i} = 0$, $\Delta\phi_0 = \theta_{0,z}$.

The boundary conditions on $z = 0$, $z = 1$ which we adopt are

$$u_z = v_z = w = \theta = \phi_z = 0, \qquad \text{on } z = 0, \quad z = 1 \tag{2.14}$$

which correspond to two free surfaces, cf. [21]. This may be perceived to not be too serious a restriction since in some experiments a «wetting agent» is added to stop the fluid coating the wall, cf. [23], and then the ferromagnetic or dielectric fluid will «see» free boundaries. The precise nature of boundary conditions is a matter of contention as Kaloni [8] points out and for now we adopt (2.14). In order to assure the uniqueness of the basic motion, we also assume the «average conditions» (see [9]):

$$\int_V u\,dx = 0, \qquad \int_V v\,dx = 0, \qquad \int_V \phi\,dx = 0. \tag{2.15}$$

Moreover we consider the case $T_u > T_l$, *i.e.* $R > 0$.

3. Linear stability and the symmetric operator

The linear instability of the basic motionless state (2.8)-(2.9) has been studied by Roberts [21] (see also [25]). He considers the linear version of system (2.11):

$$\begin{cases} (1/Pr)\, u_{i,t} = -\pi_{,i} + \Delta u_i + (R+L)\,\theta\delta_{i3} - L\phi_{,z}\delta_{i3}\,, \\ u_{i,i} = 0\,, \\ \theta_{,t} = w + \Delta\theta\,, \\ \Delta\phi = \theta_{,z}\,, \end{cases} \tag{3.1}$$

and he writes the equation for the marginal stationary state:

$$\begin{cases} \Delta u_i + (R+L)\,\theta\delta_{i3} - L\phi_{,z}\delta_{i3} = \pi_{,i}\,, \\ u_{i,i} = 0\,, \\ w + \Delta\theta = 0\,, \\ \Delta\phi = \theta_{,z}\,. \end{cases} \tag{3.2}$$

Then, he uses the *normal modes method* with wave number a^2 and he finds, for any given Rayleigh number, the *critical Roberts number* L_c above which there is linear instability.

In the following tab. I we give the critical Roberts numbers L_c and the critical wave numbers a_c^2 for different values of the Rayleigh numbers R.

Now we study the linear stability of the basic motionless state with the Lyapunov second method by introducing a Lyapunov function $E_1(t)$. We prove that the linear operator associated to the *norm* E_1 is symmetric, then we find necessary

TABLE I. – *Critical Roberts and wave numbers for assigned Rayleigh numbers.*

R	a_c^2	L_c
0	π^2	1558.54
100	9.215	1355.12
200	8.517	1143.78
300	7.774	922.6
400	6.990	688.83
500	6.181	438.81
600	5.378	167.7
650	4.991	22.45
675.511	4.935	0

and sufficient conditions of linear stability: the critical values L_c found by Roberts coincide with those obtained with the E_1-Lyapunov method.

First we add to the equations (2.11) the evolution equation of the potential ϕ of the electric field. From (2.11) we easily have:

$$\Delta\phi_{,t} = -u_{i,z}\theta_{,i} - u_i\Delta\phi_{,i} + w_{,z} + \Delta\Delta\phi. \tag{3.3}$$

Here we employ the linear part of equation (3.3):

$$\Delta\phi_{,t} = w_{,z} + \Delta\Delta\phi. \tag{3.4}$$

To the system (3.1), (3.3) we add the boundary conditions (2.14) and

$$\Delta\phi_{,z} = 0 \qquad \text{on} \qquad z = 0, \qquad z = 1. \tag{3.5}$$

The last boundary conditions are consequence of (2.11) and (2.14). In fact, following Chandrasekhar [2, chap. II], by equation $(2.11)_3$ and (2.14) we obtain the boundary conditions $\theta_{zz} = 0$ on $z = 0$, $z = 1$. Then we take the derivative with respect to z of the equation $(2.11)_4$ and use this last boundary conditions to get (3.5).

Now we introduce the space $\tilde{\mathcal{H}}$ of the admissible functions,

$\tilde{\mathcal{H}} = \{\boldsymbol{u}, \theta, \phi$, sufficiently smooth, periodic in the x and y directions, satisfying the conditions $(2.11)_2$, $(2.11)_4$, (2.14), and (3.5)$\}$,

and we denote by $\mathcal{H}$ the closure of $\tilde{\mathcal{H}}$ with respect to the norm $[\|\boldsymbol{u}\|^2 + \|\theta\|^2 + \|\nabla\phi\|^2]^{1/2}$, where $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product in $L^2(V)$.

We consider the Lyapunov function

$$E_1(t) = [P_r^{-1}\|\boldsymbol{u}\|^2 + (R + L)\|\theta\|^2 - L\|\nabla\phi\|^2]/2, \tag{3.6}$$

and use $(3.1)_4$ to see that with the aid of the boundary conditions (2.14) and integrations by parts we get

$$(\Delta\phi, \phi) = -\|\nabla\phi\|^2 = (\theta_{,z}, \phi) = -(\theta, \phi_{,z}). \tag{3.7}$$

Thus, from the Cauchy-Schwarz inequality we obtain

$$\|\nabla\phi\| \leqslant \|\theta\| \tag{3.8}$$

and then for $(\boldsymbol{u}, \theta, \phi) \in \mathcal{H}$ it follows that

$$\begin{cases} E_1(t) \geqslant \dfrac{1}{2}(P_r^{-1}\|\boldsymbol{u}\|^2 + R\|\theta\|^2) \\ \|\nabla\phi\| \leqslant \|\theta\| \end{cases}$$

i.e. $E_1(t)$ is *positive definite* function of u_i, θ, ϕ.

Now we write the equations $(3.1)_1$, $(3.1)_3$, (3.4) in the form

$$\mathcal{A}U_{,t} = \mathcal{L}U \tag{3.9}$$

with $U = (u_i, \theta, \phi)$, $\mathcal{A} = \mathrm{diag}(P_R^{-1}, 1, \Delta)$ and

$$\mathcal{L} = \begin{bmatrix} \Pi\Delta & (R+L)\Pi\delta_{i3} & -L\delta_{i3}\Pi\dfrac{\partial}{\partial z} \\ \delta_{i3} & \Delta & 0 \\ \delta_{i3}\dfrac{\partial}{\partial z} & 0 & \Delta\Delta \end{bmatrix}$$

where Π is the projector operator on the space of the divergence-free vectors. Then, it is easy to see that $\mathcal{L}$ is symmetric with respect to the inner product associated to the norm $E_1(t)$.

In fact, for any $U_1 = (u_{1i}, \theta_1, \phi_1)$ and $U_2 = (u_{2i}, \theta_2, \phi_2)$, by virtue of the boundary conditions, we easily have

$$(3.10)\qquad \begin{aligned}(\mathcal{L}U_1, U_2) &= \int_V [\Pi(\Delta u_{1i})u_{2i} + (R+L)\Pi(\theta_1\delta_{i3})u_{2i} - L\Pi(\delta_{i3}\phi_{1,z})u_{2i} + \\ &+(R+L)w_1\theta_2 + (R+L)\Delta\theta_1\theta_2 + Lw_{1,z}\phi_2 + L(\Delta\Delta\phi_1)\phi_2]\,dV = \\ &= \int_V [(\Delta u_{1i})u_{2i} + (R+L)\theta_1 w_2 - L\phi_{1,z}w_2 + \\ &+(R+L)w_1\theta_2 + (R+L)\Delta\theta_1\theta_2 + Lw_{1,z}\phi_2 + L(\Delta\Delta\phi_1)\phi_2]\,dV = \\ &= \int_V [u_{1i}\Delta u_{2i} + (R+L)\theta_2 w_1 - L\phi_{2,z}w_1 + \\ &+(R+L)w_2\theta_1 + (R+L)\theta_1\Delta\theta_2 + Lw_{2,z}\phi_1 + L\phi_1\Delta\Delta\phi_2]\,dV = \\ &= (U_1, \mathcal{L}U_2).\end{aligned}$$

The symmetry of $\mathcal{L}$ implies (cf. [3-5]) that the *critical linear stability* parameters (*critical Roberts number* L_c) obtained by Roberts [21] and Stiles [25] with the usual «normal modes method» *coincide* with the *critical* E_1*-Lyapunov linear stability parameter* L_{E_1}.

The evolution equation of $E_1(t)$ is given by

$$(3.11)\qquad \begin{aligned}\dot{E}_1(t) &= 2(R+L)(\theta, w) - 2L(\phi_{,z}, w) - [\|\nabla \boldsymbol{u}\|^2 + (R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2] \leqslant \\ &\leqslant (m_1 - 1)[\|\nabla \boldsymbol{u}\|^2 + (R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2]\end{aligned}$$

where

$$(3.12)\qquad m_1 = m_1(R, L) = \max_{\mathcal{H}_1} \frac{2(R+L)(\theta, w) - 2L(\phi_{,z}, w)}{\|\nabla \boldsymbol{u}\|^2 + (R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2}$$

and $\mathcal{H}_1$ is the space:

$$\mathcal{H}_1 = \{(\boldsymbol{u}, \theta, \phi) \in \mathcal{H}, \text{ satisfying the conditions (2.15) and such that } (\boldsymbol{u}, \theta, \phi) \neq (\boldsymbol{0}, 0, 0)\}.$$

Concerning the existence of the maximum (3.12), let us notice that $(3.1)_4$ implies

$$(3.13)\qquad \|\Delta\phi\| \leqslant \|\nabla\theta\|.$$

Therefore, on setting

$$(3.14)\qquad \alpha_1 = R + L \qquad \beta_1 = (1 + L)/R,$$

for any ϕ and θ in $\mathcal{H}_1$, we have

$$(3.15)\qquad \begin{aligned}(R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2 &\leqslant \alpha_1[\|\nabla\theta\|^2 + L\|\Delta\phi\|^2] \leqslant \\ &\leqslant \alpha_1\beta_1[(R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2],\end{aligned}$$

and then the existence of the maximum can be proved as in Rionero [19]. (An easy calculation shows that *at the criticality*, *i.e.* for $m_1 = 1$, the Euler-Lagrange equations of this maximum problem coincide with (3.2)).

From (3.11), there follows

$$\dot{E}_1(t) \leq -(1-m_1)D_1 \tag{3.16}$$

where

$$D_1 = \|\nabla \boldsymbol{u}\|^2 + (R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2 . \tag{3.17}$$

Then, the condition

$$m_1(R, L) < 1, \tag{3.18}$$

which can be shown to be equivalent to

$$L < L_c, \tag{3.19}$$

because of the Poincaré inequalities

$$\pi^2\|\boldsymbol{u}\|^2 \leq \|\nabla\boldsymbol{u}\|^2, \qquad \pi^2\|\theta\|^2 \leq \|\nabla\theta\|^2, \qquad \pi^2\|\nabla\phi\|^2 \leq \|\Delta\phi\|^2, \tag{3.20}$$

implies

$$\begin{aligned}\dot{E}_1(t) &\leq -(1-m_1)D_1 \leq -(1-m_1)/\beta_1[\|\nabla\boldsymbol{u}\|^2 + \|\nabla\theta\|^2 + L\|\Delta\phi\|^2] \leq \\ &\leq -2\pi^2\sigma(1-m_1)/\beta_1[P_r^{-1}\|\boldsymbol{u}\|^2 + \|\theta\|^2 + L\|\nabla\phi\|^2] \leq \\ &\leq (-2\pi^2\sigma(1-m_1)/\alpha_1\beta_1)E_1,\end{aligned} \tag{3.21}$$

where

$$\sigma = \min(1, P_r). \tag{3.22}$$

From (3.21) we easily obtain the following theorem:

THEOREM 3.1. *For any given Rayleigh number* $R \in [0, 657.511]$, *the condition* $L < L_c$ *implies exponential linear stability according to the inequality*

$$E_1(t) \leq E_1(0)\exp\{-\eta_0(1-m_1)t\}, \tag{3.23}$$

with

$$\eta_0 = -2\pi^2\sigma(1-m_1)/\alpha_1\beta_1 . \qquad \blacksquare \tag{3.24}$$

4. UNCONDITIONAL NONLINEAR STABILITY

In the previous Section we have seen that for any fixed Rayleigh number R the condition $L < L_c = L_{E_1}$ assures linear stability with respect to the Lyapunov function E_1.

Now we consider the following Lyapunov function

$$E_2(t) = E_{2\mu}(t) = [P_r^{-1}\|\boldsymbol{u}\|^2 + \mu\|\theta\|^2 + L\|\nabla\phi\|^2]/2, \tag{4.1}$$

where μ is a positive parameter that will be chosen later.

From the inequality (3.8) it is easy to verify that the Lyapunov function $E_2(t)$ is *equivalent* to $E_1(t)$ in $\mathcal{H}_1$. In fact, for any $\boldsymbol{u}$, ϕ, θ in $\mathcal{H}_1$ we have

$$E_1(t) \leq \alpha_\mu E_2(t) \leq \alpha_\mu\beta_\mu E_1(t), \tag{4.2}$$

with α_μ, β_μ given by

$$\alpha_\mu = \max\left((R+L)/\mu, 1\right) \qquad \beta_\mu = \max\left((\mu+L)/R, 1\right). \tag{4.3}$$

We observe that, because of (4.2), the basic motion is *linear stable* also with respect to $E_2(t)$ and, by Theorem 3.1 and (4.2) it follows

$$E_2(t) \leqslant \alpha_\mu \beta_\mu E_2(0) \exp\{-\eta_0(1-m_1)t\} \tag{4.4}$$

for any Roberts number $L < L_c$.

Now we return to the full nonlinear system (2.11), (3.3)

$$\begin{cases} (1/Pr)(u_{i,t} + u_j u_{i,j}) = -\pi_{,i} + \Delta u_i + (R+L)\theta\delta_{i3} - L\phi_{,z}\delta_{i3} - L\theta\phi_{,iz}, \\ u_{i,i} = 0, \\ \theta_{,t} + u_i\theta_{,i} = w + \Delta\theta, \\ \Delta\phi = \theta_{,z}, \\ \Delta\phi_{,t} = -u_{i,z}\theta_{,i} - u_i\Delta\phi_{,i} + w_{,z} + \Delta\Delta\phi, \end{cases} \tag{4.5}$$

with the initial conditions (2.13) and boundary conditions (2.14), (3.5). From $(4.5)_4$ and the boundary conditions we have

$$-(\theta\phi_{,iz}, u_i) + (u_{i,z}\theta_{,i}, \phi) + (u_i\Delta\phi_{,i}, \phi) = 0, \tag{4.6}$$

which implies the evolution equation for the Lyapunov function $E_2(t)$

$$\begin{aligned} \dot{E}_2(t) = (R+L+\mu)(w,\theta) - L(\theta\phi_{,iz}, u_i) + L(u_{i,z}\theta_{,i}, \phi) + \\ + L(u_i\,\Delta\phi_{,i}, \phi) - [\|\nabla \boldsymbol{u}\|^2 + L\|\Delta\phi\|^2 + \mu\|\nabla\theta\|^2]. \end{aligned} \tag{4.7}$$

From this it follows the inequality

$$\dot{E}_2(t) \leqslant (m_2 - 1)[\|\nabla \boldsymbol{u}\|^2 + L\|\theta_{,z}\|^2 + \mu\|\nabla\theta\|^2], \tag{4.8}$$

where

$$m_2 = \max_{\mathcal{H}_1}\left((R+L+\mu)(w,\theta)/(\|\nabla \boldsymbol{u}\|^2 + L\|\theta_{,z}\|^2 + \mu\|\nabla\theta\|^2)\right). \tag{4.9}$$

We notice that also in this case the Rionero theorem [19] holds true. Obviously $m_2 = m_2(R, L, \mu)$ and at the *criticality*

$$m_2 = m_2(R, L, \mu) = 1.$$

Thus, for any given Rayleigh number the *critical Roberts number* L_{E_2} will depend on μ, $L_{E_2} = L_{E_2}(\mu)$.

Now we solve the maximum problem (4.9).

The Euler-Lagrange equations of the maximum problem (4.9) are

$$\begin{cases} 2m_2\,\Delta u_i + (R+L+\mu)\,\theta\delta_{i3} = \psi_{,i}, \\ (R+L+\mu)w + 2m_2(L\theta_{zz} + \mu\Delta\theta) = 0. \end{cases} \tag{4.10}$$

By taking the third component of the double *curl* of $(4.10)_1$, from (4.10) we have the system

$$\begin{cases} 2m_2\,\Delta\Delta w + (R+L+\mu)\,\Delta_1\theta = 0, \\ (R+L+\mu)w + 2m_2(L\theta_{zz} + \mu\Delta\theta) = 0, \end{cases} \tag{4.11}$$

where $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, with the boundary conditions

$$w = \Delta w = \theta = 0 \qquad \text{on} \qquad z = 0, \qquad z = 1.$$

From (4.11) a simple calculation gives

$$(R + L + \mu)^2 \Delta_1 w - 4m_2^2 (L\Delta\Delta w_{zz} + \mu\Delta\Delta\Delta w) = 0\,. \tag{4.12}$$

Following the standard *normal modes* method (see Chandrasekhar [2]) we can choose $w(x, y, z) = W(z) \cos(a_1 x + a_2 y)$. The boundary conditions require that w and all its even derivatives vanish at the boundaries. From this it follows that the required solutions must be $W(z) = C \sin n\pi z$ where C is a constant and n is an integer. Substitution of this solution in (4.12) leads to the characteristic equation

$$(R + L + \mu)^2 a^2 - 4m_2^2 [L(n^2\pi^2 + a^2)^2 n^2\pi^2 + \mu(n^2\pi^2 + a^2)^3] = 0\,, \tag{4.13}$$

where $a^2 = a_1^2 + a_2^2$ is the wave number. Equation (4.13) gives

$$m_2(a^2, n^2, \mu) = (R + L + \mu)a/(2\sqrt{L(n^2\pi^2 + a^2)^2 n^2\pi^2 + \mu(n^2\pi^2 + a^2)^3})\,.$$

The maximum with respect to n^2 of $m_2(a^2, n^2, \mu)$ is assumed for $n^2 = 1$. Hence,

$$m_2 = \min_{\mu} \max_{a^2} ((R + L + \mu)a/(2\sqrt{L(\pi^2 + a^2)^2\pi^2 + \mu(\pi^2 + a^2)^3}))\,. \tag{4.14}$$

We observe that the stability condition $m_2 < 1$ is equivalent to $0 \leqslant L < L_{E_2}$, where

$$L_{E_2} = \max_{\mu} \min_{a^2} \{ -(R + \mu) + 2\pi^2(\pi^2 + a^2)^2 a^{-2} + \tag{4.15}$$

$$+ 2(\pi^2 + a^2)a^{-1}\sqrt{\pi^4(\pi^2 + a^2)^2 a^{-2} - \pi^2(R + \mu) + \mu(\pi^2 + a^2)}\,\}\,.$$

By solving numerically (4.15) (for example one can use the method given in the Appendix of the book of Straughan [27]) we shall obtain the critical Roberts numbers.

In the following tab. II we give the critical Roberts numbers L_{E_2}, the wave numbers a_c^2 and the critical Lyapunov parameter μ_{crit} for different values of the Rayleigh numbers R.

REMARK 4.1. We underline that in the interval $R \in [0, 657.511]$ we have: $L_{E_2} = L_{E_1} = L_c$, *i.e.* the coincidence between the linear and nonlinear critical Roberts numbers. ■

TABLE II. – *Critical Roberts, wave and μ numbers for assigned Rayleigh numbers.*

R	a_c^2	μ_{crit}	L_{E_2}
0	π^2	0	1558.54
100	9.215	53.55	1355.12
200	8.517	115.88	1143.78
300	7.774	190.43	922.6
400	6.990	282.37	688.83
500	6.181	399.17	438.81
600	5.378	550.59	167.7
650	4.991	642.63	22.45
675.511	4.935	657.511	0

Now from (4.8) the condition $m_2 < 1$ or, which is the same, $L < L_{E_2} = L_c$, will assure *unconditional nonlinear exponential* stability, *i.e.* the size of the initial energy $E_2(0)$ is not restricted whatsoever. Then, the following theorem holds.

THEOREM 4.1. *For any given Rayleigh number $R \in [0, 657.511]$, the condition $L < L_c$ implies unconditional nonlinear exponential stability according to the inequality*

$$E_2(t) \leqslant E_2(0) \exp\{-2\pi^2(1-m_2)t\}. \qquad \blacksquare$$

REMARK 4.2. The evolution equation of $E_1(t)$ is

$$\begin{aligned}(4.16) \qquad \dot{E}_1(t) &= 2(R+L)(\theta, w) - 2L(\phi_{,z}, w) - L(\theta\phi_{,iz}, u_i) - L(u_{i,z}\theta_{,i}, \phi) - \\ &-L(u_i\Delta\phi_{,i}, \phi) - [\|\nabla \boldsymbol{u}\|^2 + (R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2] = \\ &= 2(R+L)(\theta, w) - 2L(\phi_{,z}, w) - 2L(\theta\phi_{,iz}, u_i) - \\ &-[\|\nabla \boldsymbol{u}\|^2 + (R+L)\|\nabla\theta\|^2 - L\|\Delta\phi\|^2].\end{aligned}$$

Therefore, because of the presence of the cubic term $-2L(\theta\phi_{,iz}, u_i)$, following the previously developed energy methods one should have expected a severe restriction on the initial data and then *conditional* nonlinear stability. The role of the choice of Lyapunov function $E_2(t)$ is to give the opportunity of overcoming this hard restriction and to obtain *global nonlinear stability*.

Of course, standing the equivalence of E_1 and E_2, the condition $L < L_c$ ensures also the *unconditional nonlinear stability* of the basic motion with respect to E_1. $\blacksquare$

ACKNOWLEDGEMENTS

This research has been partially supported by the «Gruppo Nazionale della Fisica Matematica» of the Italian C.N.R. and by the Ministry for University and Scientific Research (M.U.R.S.T.) under 40% and 60% contracts. The work of B. Straughan was performed while he was a Visiting Professor of the Italian CNR at the University of Catania in September 1993.

REFERENCES

[1] P. J. BLENNERHASSETT - F. LIN - P. J. STILES, *Heat transfer through strongly magnetized ferrofluids.* Proc. Roy. Soc. London A, vol. 433, 1991, 165-177.

[2] S. CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability.* Dover, New York 1981.

[3] S. H. DAVIS, *On the possibility of subcritical instabilities.* In: Proc. IUTAM Symp. Herrenalb, Springer-Verlag, Berlin-Heidelberg-New York 1971.

[4] G. P. GALDI - S. RIONERO, *Weighted energy methods in fluid dynamics and elasticity.* Lecture Notes in Mathematics, vol. 1134, Springer-Verlag, Berlin 1985.

[5] G. P. GALDI - B. STRAUGHAN, *Exchange of stabilities, symmetry and nonlinear stability.* Arch. Rational Mech. Anal., vol. 89, 1985, 211-228.

[6] G. P. GALDI - B. STRAUGHAN, *A nonlinear analysis of the stabilizing effect of rotation in the Bénard problem.* Proc. Roy. Soc. London A, vol. 402, 1985, 257-283.

[7] D. D. JOSEPH, *Stability of Fluid Motions.* Vols. I, II, Spinger Tracts in Natural Philosophy, vol. 27-28, 1976.

[8] P. N. KALONI, *Some remarks on the boundary conditions for magnetic fluids.* Int. J. Engng. Sci., vol. 30, 1992, 1451-1457.

[9] P. Kloeden - R. Wells, *An explicit example of Hopf bifurcation in fluid mechanics.* Proc. Roy. Soc. London A, vol. 390, 1983, 293-320.

[10] L. D. Landau - E. M. Lifshitz - L. P. Pitaevskii, *Electrodynamics of continuous media.* 2nd ed., Pergamon Press, Oxford 1984.

[11] I. Muller, *Thermodynamics.* Pitman, Boston - London - Melbourne 1985.

[12] G. Mulone - S. Rionero, *On the nonlinear stability of the magnetic Bénard problem with rotation.* Z. angew. Math. Mech., vol. 73, 1, 1993, 35-45.

[13] G. Mulone - S. Rionero - B. Straughan, *Stabilità non lineare incondizionata per la convezione elettro-termica in un liquido polarizzato.* Atti XII Congr. AIMETA, Napoli, vol. V, 1995, 45-50.

[14] G. P. Neitzel - M. K. Smith - M. J. Bolander, *Thermal instability with radiation by the energy method.* Int. J. Heat Mass Transfer, vol. 37, 1994, 2909-2915.

[15] Y. Qin - J. Chadam, *A nonlinear stability problem for ferromagnetic fluids in a porous medium.* Appl. Math. Letters, vol. 8, 1993, 25-29.

[16] Y. Qin - P. N. Kaloni, *Nonlinear stability problem of a ferromagnetic fluid with surface tension effect.* Eur. J. Mech. B/Fluids, vol. 13, 1994, 305-321.

[17] S. Rionero, *Sulla stabilità asintotica in media in magnetoidrodinamica.* Ann. Matem. Pura Appl., vol. 76, 1967, 75-92.

[18] S. Rionero, *Metodi variazionali per la stabilità asintotica in media in magnetoidrodinamica.* Ann. Matem. Pura Appl., vol. 78, 1968, 339-364.

[19] S. Rionero, *Sulla stabilità magnetofluidodinamica non lineare asintotica in media in presenza di effetto Hall.* Ricerche Matem., vol. 20, 1971, 285-296.

[20] S. Rionero - G. Mulone, *A nonlinear stability analysis of the magnetic Bénard problem through the Lyapunov direct method.* Arch. Rational Mech. Anal., vol. 103, 1988, 347-368.

[21] P. H. Roberts, *Electrohydrodynamic convection.* Q. Jl. Mech. Appl. Math., vol. 22, 1969, 211-220.

[22] R. H. Rosenweig, *Ferrohydrodynamics.* Cambridge Univ. Press, 1985.

[23] R. H. Rosenweig - M. Zahn - R. Shumovich, *Labyrinthine instability in magnetic and dielectric liquids.* J. Magnetism and Magnetic Materials, vol. 39, 1983, 127-132.

[24] J. Serrin, *On the stability of viscous fluid motions.* Arch. Rational Mech. Anal., vol. 3, 1959, 1-13.

[25] P. J. Stiles, *Electro-thermal convection in dielectric liquids.* Chemical Physics Letters, vol. 179, 1991, 311-315.

[26] P. J. Stiles - F. Lin - P. J. Blennerhassett, *Convective heat transfer through polarized dielectric liquids.* Physics Fluids A, vol. 5, 1993, 3273-3279.

[27] B. Straughan, *The energy method, stability, and nonlinear convection.* Appl. Math. Sci. Ser., vol. 91. Springer-Verlag, New York 1992.

[28] B. Straughan, *Stability problems in electrohydrodynamics, ferrohydrodynamics and thermoelectric magnetohydrodynamics.* In: J. F. Rodriguez - A. Sequeira (eds.), *Mathematical topics in fluid mechanics*. Pitman Res. Notes Math., Longman Press, vol. 274, 1992.

[29] B. Straughan, *Nonlinear stability for convection in a polarized dielectric liquid.* Article in Recent Advances in Mechanics of Structured Continua, ASME, vol. 160, 1993, 145-150.

[30] S. Venkatasubramanian - P. N. Kaloni, *Effects of rotation on the thermo-convective instability of a horizontal layer of ferrofluids*. Int. J. Engng. Sci., vol. 32, 1994, 237-256.

G. Mulone, S. Rionero:
Dipartimento di Matematica e Applicazioni «Renato Caccioppoli»
Università degli Studi di Napoli
Complesso Monte S. Angelo, Edificio T
Via Cintia - 80126 Napoli

B. Straughan:
Department of Mathematics
The University of Glasgow
G12 8QW Glasgow (Gran Bretagna)