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Geometria. — *Levi equation and evolution of subsets of* $\boldsymbol{C}^2$. Nota di Zbigniew Slodkowski e Giuseppe Tomassini, presentata (*) dal Socio E. Vesentini.

Abstract. — In this *Note* we state some results obtained studying the evolution of compact subsets of C^2 by Levi curvature. This notion appears to be the natural extension to Complex Analysis of the notion of evolution by mean curvature.

Key words: Pseudoconvex domains; Real submanifolds in complex manifolds; Parabolic equations and systems.

Riassunto. — *Equazioni di Levi ed evoluzione di sottoinsiemi di* C^2. In questa *Nota* si enunciano alcuni risultati ottenuti nello studio dell'evoluzione di sottoinsiemi compatti di C^2 secondo la curvatura di Levi. La nozione di evoluzione che qui si considera appare come la naturale estensione all'Analisi Complessa della nozione di evoluzione secondo la curvatura media.

0. Introduction

Let M be a real smooth hypersurface of C^2 defined by $\varrho = 0$. We denote by $\nu = |\partial\varrho|^{-1}(\varrho_{\bar 1}, \varrho_{\bar 2})$ the normal field along M and by

$$k_L(M) = -|\partial\varrho|^{-3} \det \begin{bmatrix} 0 & \varrho_{z_1} & \varrho_{z_2} \\ \varrho_{\bar z_1} & \varrho_{\bar z_1 z_1} & \varrho_{\bar z_1 z_2} \\ \varrho_{\bar z_2} & \varrho_{\bar z_2 z_1} & \varrho_{\bar z_2 z_2} \end{bmatrix} = \left(\delta_{\alpha\beta} - \frac{\varrho_{\bar\alpha}\varrho_\beta}{|\partial\varrho|^2}\right)\varrho_{\alpha\bar\beta}$$

the Levi curvature of M ($\varrho_\alpha = \partial\varrho/\partial z^\alpha$, $\varrho_{\bar\alpha} = \partial\varrho/\partial\bar z^\alpha$, $\alpha = 1, 2$, $|\partial\varrho|^2 = |\varrho_1|^2 + |\varrho_2|^2$).

An interesting geometric interpretation of k_L can be obtained by considering the exponential map $\exp_{z_0}: B_0 \to M$, where $z_0 \in M$ and B_0 is an open ball contained in the complex tangent line. Then $k_L(z_0)$ equals the mean curvature of the surface $\exp_{z_0}(B_0)$.

Let us consider a smooth family $\{M_t\}_{t \geq 0}$ of (smooth) hypersurfaces of a domain Ω of C^2 where $M_t = \{z \in \Omega: u(z, t) = 0\}$ and $M_t \cap M_{t'} = \emptyset$ for $t \neq t'$. Let us assume that $U = \bigcup_{t \geq 0} M_t$ is open and consider the vector field $k_L \nu$ on U. We will say that the level sets M_t *evolve according to their Levi curvature* if for $z \in M_t$, $t \geq 0$, the integral lines of $k_L \nu$, $s \mapsto z(s)$, $z(0) = z$ satisfy $z(s) \in M_{t+s}$. This turns out to be equivalent to the following assertion: u is a solution of the parabolic equation

$$u_t = \mathcal{L}(u) = (\delta_{\alpha\beta} - u_{\bar\alpha} u_\beta / |\partial u|^2)\, u_{\alpha\bar\beta}\,.$$

In the above situation we will also say that $\{M_t\}_{t \geq 0}$ is the *evolution of* M_0 *by Levi curvature*. In the light of what is preceding, this notion appears to be the natural extension to Complex Analysis of the notion of evolution of a smooth hypersurface by mean curvature [3-5].

(*) Nella seduta del 13 giugno 1996.

In general, given a compact subset $K \subset \boldsymbol{C}^2$, the zero set $\{g = 0\}$ of a continuous function $g: \boldsymbol{C}^2 \to \boldsymbol{R}$ which is constant for $|z| \gg 0$, we consider the parabolic problem corresponding to g: $u_t = \mathcal{L}(u)$ in $\boldsymbol{C}^2 \times (0, +\infty)$, $u = g$ for $t = 0$ and u is constant for $|z| + t \gg 0$. If u is a continuous weak solution of this problem (in the sense of viscosity [3-5]), we set $K_t = \{z \in \boldsymbol{C}^2 : u(z, t) = 0\}$, for $t \geq 0$. The family $\{K_t\}_{t \geq 0}$, which does not depend on g but only on K, is called the *evolution of* K (*by Levi curvature*).

In what follows we are going to give a short report of some results concerning the existence and the geometric properties of the evolution of compact subsets in $\boldsymbol{C}^2$ with special regards to boundaries of pseudoconvex domains.

1. Properties of weak solutions. Existence

1. Let $U \subset \boldsymbol{C}^2 \times (0, +\infty)$ be an open subset and $u: U \to \boldsymbol{R}$ be an upper semicontinuous function; u is said to be a *weak subsolution* of $u_t = \mathcal{L}(u)$ if, for every $\phi \in C^\infty(U)$ such that $u - \phi$ has a local maximum at (z^0, t^0), one has

$$\phi_t \leq (\delta_{\alpha\beta} - \phi_{\bar{\alpha}} \phi_\beta)\, \phi_{\alpha\bar{\beta}}$$

at (z^0, t^0) if $\partial\phi(z^0, t^0) \neq 0$ and

$$\phi_t \leq (\delta_{\alpha\beta} - \bar{\eta}^\alpha \eta^\beta)\, \phi_{\alpha\bar{\beta}}$$

for some $\eta \in \boldsymbol{C}^2$ with $|\eta| \leq 1$, if $\partial\phi(z^0, t^0) = 0$; a lower semicontinuous function $u: U \to \boldsymbol{R}$ is said to be a *weak supersolution* if, for every $\phi \in C^\infty(U)$ such that $u - \phi$ has a local minimum at (z^0, t^0), one has

$$\phi_t \geq (\delta_{\alpha\beta} - \phi_{\bar{\alpha}} \phi_\beta)\, \phi_{\alpha\bar{\beta}}$$

at (z^0, t^0) if $\partial\phi(z^0, t^0) \neq 0$ and

$$\phi_t \geq (\delta_{\alpha\beta} - \bar{\eta}^\alpha \eta^\beta)\, \phi_{\alpha\bar{\beta}}$$

for some $\eta \in \boldsymbol{C}^2$ with $|\eta| \leq 1$, if $\partial\phi(z^0, t^0) = 0$.

A *weak solution* is a continuous function which is both a weak subsolution and a weak supersolution. If u is a weak solution of $u_t = \mathcal{L}(u)$ and $\Phi: \boldsymbol{R} \to \boldsymbol{R}$ is continuous, then $\Phi(u)$ is a weak solution as well. Uniform limits on compact subsets of sequences of weak subsolutions (weak supersolutions) are weak subsolutions (weak supersolutions).

2. Let us consider the cylinder $Q = \Omega \times (0, h)$ in $\boldsymbol{C}^2 \times [0, +\infty)$, where Ω is a bounded domain of $\boldsymbol{C}^2$ and let $\Sigma = (\overline{\Omega} \times \{0\}) \cup (b\Omega \times (0, h))$. We have the following comparison principle

THEOREM 1.1. *Let $u, v \in C^0(\overline{Q})$ be respectively a weak subsolution and a weak supersolution in Q. If $u \leq v$ on $\overline{\Sigma}$ then $u \leq v$ on $\overline{Q}$.*

As a consequence we obtain

COROLLARY 1.2. *Let $u, v \in C^0(\boldsymbol{C}^2 \times [0, +\infty))$ be respectively a weak subsolution and a weak supersolution in $\boldsymbol{C}^2 \times (0, +\infty)$, and suppose that u and v are constant for $|z| + t \gg 0$. If $u \leq v$ for $t = 0$ then $u \leq v$.*

The above results permit us to use the Perron method to prove the following existence theorem

THEOREM 1.3. *Let* $g\colon C^2 \to R$ *be continuous and constant for* $|z| \gg 0$. *Then the parabolic problem corresponding to* g *has a unique weak solution.*

2. EVOLUTION OF A COMPACT SUBSET: GEOMETRIC PROPERTIES

1. Let $K \subset C^2$ be a compact subset, the zero set $\{g = 0\}$ of a continuous function $g\colon C^2 \to R$ (which is constant for $|z| \gg 0$), u a weak solution of the parabolic problem corresponding to g and $\{K_t\}_{t \geq 0}$ the evolution of K (by Levi curvature). We will also use the notation $K_t = \mathcal{E}_t^{\mathcal{L}}(K)$. Then the semigroup property

$$\mathcal{E}_{t'}^{\mathcal{L}}(\mathcal{E}_t^{\mathcal{L}}(K)) = \mathcal{E}_{t+t'}^{\mathcal{L}}(K)$$

holds true and, by definition, there exists a time t^*, called *extinction time* of K, such that $\mathcal{E}_t^{\mathcal{L}}(K) = \emptyset$ for $t > t^*$. Moreover, if K and K' are compact and $K \subseteq K'$, $\mathcal{E}_t^{\mathcal{L}}(K) \subseteq \mathcal{E}_t^{\mathcal{L}}(K')$ for all $t \geq 0$.

THEOREM 2.1. *Let* $K \subset C^2$ *be a compact subset. If* U *is a Stein neighbourhood of* K *then* $\mathcal{E}_t^{\mathcal{L}}(K) \subset U$ *for every* $t \geq 0$. *In particular*:

(*a*) *if* K *is a Stein compact then* $\mathcal{E}_t^{\mathcal{L}}(K) \subset K$ *for all* $t \geq 0$;

(*b*) *if* K *belongs to a Stein analytic subset* X, *then* $\mathcal{E}_t^{\mathcal{L}}(K) \subset X$ *for all* $t \geq 0$.

2. Let us assume now that K is the boundary Γ_0 of a bounded domain $\Omega \subset C^2$ and let $\{\Gamma_t\}_{t \geq 0}$ be its evolution. We will say that the evolution is *strictly contracting* (respectively *contracting*) if, for every $t > 0$, $\Gamma_t \subset \Omega$ (respectively $\Gamma_t \subset \overline{\Omega}$). $\{\Gamma_t\}_{t \geq 0}$ is said to be *stationary* if, for every $t \geq 0$, $\Gamma_t = \{z \in \Omega\colon v(z) = -t\}$ where v is a weak solution of the *stationary problem* associated with the evolution $\{\Gamma_t\}_{t \geq 0}$: $\mathcal{L}(v) = 1$ in Ω and $v = 0$ on $b\Omega$.

For every weak solution of the stationary problem we have $\{z \in \Omega\colon v(z) = -t\} \subseteq \Gamma_t$. Moreover

PROPOSITION 2.2. *Let* $v \in C^0(\overline{\Omega})$ *be a weak solution of the stationary problem and extend it by* 0 *on* C^2. *Then*

$$u(z,t) = \begin{cases} \min(0, v(z) + t) & \text{if } (z,t) \in \overline{\Omega} \times [0, +\infty), \\ 0 & \text{if } (z,t) \in (C^2 \setminus \overline{\Omega}) \times [0, +\infty), \end{cases}$$

is a weak solution of $u_t = \mathcal{L}(u)$ (*and* $u = v$ *for* $t = 0$).

COROLLARY 2.3. *Let* $v \in C^0(\overline{\Omega})$ *be a weak solution of the stationary problem. Set* $N_t = \{z \in \Omega\colon v(z) = -t\}$, $t \geq 0$. *Then*

(*a*) *for every* t_0, $t > 0$

$$\mathcal{E}_t^{\mathcal{L}}(N_{t_0}) = N_{t+t_0},$$

(*b*) *for every* $t \geq 0$

$$N_t \subseteq \mathcal{E}_t^{\mathcal{L}}(b\Omega) = \mathcal{E}_t^{\mathcal{L}}(N_0).$$

A partial converse of Proposition 2.2 is provided by the following

PROPOSITION 2.4. *Let u be the weak solution of the parabolic problem corresponding to g and let M be the zero set of u. If* $\Gamma_t \cap \Gamma_{t'} = \emptyset$ *for* $t \neq t'$, *then M is the compact graph of a continuous function* $v\colon X \to (-\infty, 0]$, $X \subset C^2$, *such that* $\overline{\Omega} \subseteq X$ *and* $v < 0$ *in* Ω, $v = 0$ *on* $b\Omega$. *Moreover* $v_0 = v_{|\Omega}$ *is a weak solution of the stationary problem and* Ω *is Stein.*

REMARK 2.1. The last part of this statement follows from [6].

We conjecture that if $b\Omega$ is smooth but Ω is not strictly pseudoconvex then $\Gamma_t \not\subset \overline{\Omega}$ for some t.

3. STATIONARY EVOLUTION

1. Let Ω be a bounded domain defined by $\{\varrho < 0\}$, where ϱ is smooth and strictly p.s.h. in a neighbourhood of $\overline{\Omega}$ and $d\varrho \neq 0$ on $b\Omega$.

THEOREM 3.1. *Let* $g \in C^0(b\Omega)$. *The Dirichlet problem* $\mathcal{L}(u) = 1$ *in* Ω *and* $u = g$ *on* $b\Omega$ *has a unique weak solution* $u \in C^0(\overline{\Omega})$. *If g belongs to* $C^{2,\alpha}(b\Omega)$ *then* $u \in \operatorname{Lip}(\overline{\Omega})$.

In order to prove the existence of a continuous weak solution the Perron method applies and in this case $b\Omega$ is allowed to be P-regular [6]. To obtain the Lipschitz regularity we prove that u is a uniform limit of a C^1-bounded sequence of smooth solutions of perturbed elliptic boundary problems [6].

We also have estimates of solutions. In order to state this let us denote $\lambda_1(z) \leq \lambda_2(z)$ the eigenvalues of the matrix $(\varrho_{\alpha,\bar\beta}(z))$ at $z \in \overline{\Omega}$ and set $\lambda_1 = \min \lambda_1(z)$, $\lambda_2 = \max \lambda_2(z)$ in $\overline{\Omega}$.

THEOREM 3.2. *Let* $u \in C^0(\overline{\Omega})$ *be a weak solution of* $\mathcal{L}(u) = 1$ *in* Ω. *Then the following estimate holds true*:

$$\lambda_1^{-1}\varrho(z) + \min_{b\Omega} u \leq u(z) \leq \lambda_2^{-1}\varrho(z) + \max_{b\Omega} u .$$

We finally derive the following

THEOREM 3.3. *The stationary problem* $\mathcal{L}(u) = 1$ *in* Ω *and* $u = 0$ *on* $b\Omega$ *has a unique solution* $u \in \operatorname{Lip}(\overline{\Omega})$ *such that*

$$\|\partial u\| \leq \lambda_1^{-1} \|\partial\varrho/\partial\nu\|_{b\Omega} .$$

2. Boundaries of strictly pseudoconvex domains evolve in stationary way. More generally let Ω be a bounded domain in C^2 such that $\Omega = \mathring{\overline{\Omega}}$ and W an open neighbourhood of $\overline{\Omega}$. Assume that there exists a continuous function $h\colon W \backslash \overline{\Omega} \to (0, +\infty)$ such that: h is weak subsolution of $\mathcal{L}(h) = 1$, $h(z) \to 0$ as $z \to z^0$ and $D^+h(z^0) = \limsup_{z \to z^0} |z - z^0|^{-1} h(z) = +\infty$ for every $z^0 \in b\Omega$.

All these conditions are satisfied whenever $b\Omega$ is strictly pseudoconvex.

Under these hypotheses we have

THEOREM 3.4. *Let $v \in C^0(\overline{\Omega})$ be a weak solution of the stationary problem and for every $t \geq 0$ let $\Gamma_t = \{z \in \overline{\Omega}: v(z) + t = 0\}$ and $\Omega_t = \{z \in \overline{\Omega}: v(z) + t = 0\}$. Then $\{\Gamma_t\}_{t \geq 0}$ and $\{\Omega_t\}_{t \geq 0}$ are respectively the evolution of $\Gamma_0 = b\Omega$ and $\Omega_0 = \Omega$.*

As for instantaneous disappearance we have

THEOREM 3.5. *Let $K \subset C^2$ be a compact subset, $K = \phi^{-1}(0)$ where ϕ is a non negative function on a neighbourhood U of K and such that $\mathcal{L}(\phi) \geq 1$ in U (in the weak sense). Then the extinction time t^* of K is* 0, i.e. $\mathcal{E}_t^{\mathcal{L}}(K) = \phi$ *for* $t \geq 0$.

This is the case of a compact subset of a totally real submanifold $M \subset C^2$.

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