Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Sobolev spaces of integer order on compact homogeneous manifolds and invariant differential operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **7** (1996), n.4, p. 219–233.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1996_9_7_4_219_0>

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1996.

Rend. Mat. Acc. Lincei
s. 9, v. 7:219-233 (1996)

Analisi matematica. — *Sobolev spaces of integer order on compact homogeneous manifolds and invariant differential operators.* Nota di CRISTIANA BONDIOLI, presentata (*) dal Socio E. Magenes.

ABSTRACT. — Let M be a Riemannian manifold, which possesses a transitive Lie group G of isometries. We suppose that G, and therefore M, are compact and connected. We characterize the Sobolev spaces $W_p^1(M)$ $(1 < p < +\infty)$ by means of the action of G on M. This characterization allows us to prove a regularity result for the solution of a second order differential equation on M by global techniques.

KEY WORDS: Compact homogeneous manifolds; Sobolev spaces; Invariant differential operators.

RIASSUNTO. — *Spazi di Sobolev di ordine intero su varietà omogenee compatte e operatori differenziali invarianti.* Sia M una varietà riemanniana, dotata di un gruppo di Lie G transitivo di isometrie. Si suppone che G, e pertanto M, siano compatti e connessi. Si caratterizzano gli spazi di Sobolev $W_p^1(M)$ $(1 < p < +\infty)$ tramite l'azione di G su M. Questa caratterizzazione permette di dimostrare tramite tecniche globali un risultato di regolarità per la soluzione di un'equazione differenziale del secondo ordine su M.

1. INTRODUCTION

This paper is in some sense a continuation of our previous papers [5, 6].

In [5] and in the first part of [6] we proved a characterization of a class of Nikol'skij spaces on a compact homogeneous manifold in terms of its isometries. This allowed us to establish in the second part of [6] a regularity result of Nikol'skij type for the solution of a non linear evolution equation on the manifold.

As the Nikol'skij spaces on domains Ω of $\mathbb{R}^n$ were defined by Nikol'skij himself by a condition involving the translation group of $\mathbb{R}^n$, so it is well known that the Sobolev spaces $W_p^1(\Omega)$ $(1 < p \leq +\infty)$ too can be characterized by means of the translations of $\mathbb{R}^n$.

These considerations led us to examine here the relations between the Sobolev spaces of integer order on a homogeneous manifold and the isometries on it.

Here we begin with the case of a compact homogeneous manifold. By compact homogeneous manifold we mean a compact Riemannian manifold M on which a Lie group G of isometries acts transitively.

We prove a characterization for the Sobolev spaces $W_p^1(M)$ by means of the elements of G, in such a way as to reflect the global character of M. This characterization seems to be meaningful in order to obtain regularity results for the solution of some PDE's on homogeneous manifolds by a direct technique. Usually (see, for instance, [2]) regularity results of the kind we consider here are obtained on manifolds by local charts arguments, that is by reducing the regularity problem on the manifold to a regularity problem in $\mathbb{R}^n$.

As an example we consider in § 8 the second order operator $\mathcal{L} = \sum_{i=1}^{q} [X_i^*]^2$, where

(*) Nella seduta del 13 giugno 1996.

the X_i^* are the vector fields on M induced by a basis $\{X_i \mid i = 1, \dots, q\}$ of the Lie algebra $\mathfrak{g}$ of G. If *the operator $\mathcal{L}$ is invariant under the group action* (see, for instance, [9, Chapter II]), we globally recover the result that if f is a function in $W_2^k(M)$, then the weak solution u of $u - \mathcal{L}u = f$ is in $W_2^{k+2}(M)$.

It would be interesting to extend this kind of considerations also to non compact manifolds. In [3] we proved a characterization of the mentioned Sobolev and Nikol′-skij spaces on $\mathbb{R}^n$ by means of the motions of $\mathbb{R}^n$, without any assumption on the compactness of the support of the involved functions. We expect that an analogous characterization holds also for non compact homogeneous manifolds and could help us to extend to these manifolds the regularity result of this paper as well as the one of [6].

2. Notation

We use standard notations: by a smooth function we always mean a C^∞ function. G stands for a q-dimensional connected Lie group, whose Lie algebra we denote by $\mathfrak{g}$; then exp: $\mathfrak{g} \to G$ has the usual meaning. If dg is a right-invariant Haar measure on G and $1 < p < +\infty$, it is clear what the function spaces $L_p(G)$ mean. The letter M is reserved for an n-dimensional connected oriented C^∞ manifold equipped with a smooth Riemannian metric. If $\mathcal{A}$ is an atlas compatible with the orientation and $\{y_1, \dots, y_n\}$ is the coordinate system corresponding to the chart $(U, \psi) \in \mathcal{A}$, we denote by $\sqrt{|\gamma|}dy_1 \wedge \dots \wedge dy_n$ the Riemannian volume element, *i.e.* $|\gamma|$ stands for the absolute value of the determinant of the metric matrix; moreover, dm denotes the corresponding Riemannian measure. Then also in this case the meaning of $L_p(M)$ is clear.

3. Preliminaries

In this section we collect some well known results that we will use later on.

Let us first recall some basic facts on homogeneous manifolds. For references, see, for instance, [9, Chapter I, 1]; [11, Chapter IV, § 17].

Let G be a Lie group and M be a smooth manifold. Suppose that G is a Lie transformation group of M, which acts transitively on M. Then M is called a homogeneous space of G or simply a homogeneous manifold. Let $K = \{g \in G \mid g \cdot o = o\}$ be the isotropy subgroup at a point o of M. Then M and $G/K = \{gK \mid g \in G\}$ are diffeomorphic, so we will write M or G/K indifferently.

Let π be the natural map from G to G/K. If φ is a smooth function on G/K, then $\varphi \circ \pi$ is a smooth, right K-invariant function on G. For the sequel we will denote $\varphi \circ \pi$ with $\mathcal{R}\varphi$. Conversely, if Φ is a right K-invariant, smooth function on G, then $\mathcal{P}\Phi$ defined on G/K by $\mathcal{P}\Phi(gK) = \Phi(g)$ is a smooth function on G/K.

If G is connected and K is compact, then the homogeneous manifold $M = G/K$ admits a G-invariant Riemannian metric. By the G-invariance of a Riemannian metric we mean that for every $g \in G$, the transformation $\tau_g \colon G/K \to G/K$, $\tau_g(g_1K) = gg_1K$ is an isometry.

Let $dg_K = dm$ be the corresponding invariant Riemannian measure on $G/K = M$ (which is unique up to a constant factor). We suppose that the right invariant measures dg on G and dk on K are suitably normalized so that

$$(3.1)\qquad \begin{cases} \displaystyle\int_G F(g)\,dg = \int_{G/K}\left(\int_K F(gk)\,dk\right)dg_K\,, & F \in C_c(G)\,, \\ \displaystyle\int_K dk = 1\,. \end{cases}$$

We will use the same symbols $\mathcal{R}\varphi$, $\mathcal{P}\Phi$ as above also for functions in $L_p(M)$ and $L_p(G)$.

Moreover, denoting by L_a the left translations on G, let us consider for an arbitrary element X in $\mathfrak{g}$ the one-parameter group $\{L_{\exp tX}\,|\,t \in \mathbb{R}\}$. Then the corresponding infinitesimal generator $\widetilde{X}$ is a right invariant smooth vector field on G. We recall that for every smooth function Φ

$$\widetilde{X}\Phi(g) = \lim_{t\to 0} t^{-1}[\Phi(\exp tX\cdot g) - \Phi(g)]\,.$$

Analogously $\{t \mapsto \tau_{\exp tX}\}$ is a one-parameter group of isometries of M, whose infinitesimal generator we denote by X^*, that is for every smooth function φ

$$X^*\varphi(m) = \lim_{t\to 0} t^{-1}[\varphi(\exp tX\cdot m) - \varphi(m)]$$

(here $\cdot$ denotes the action of G on M). The map $X \mapsto X^*$ is a linear map from $\mathfrak{g}$ into the space of all smooth vector fields on M.

Finally, a differential operator D on M is said to be invariant under the action of G if $D(\varphi\circ\tau_g) = (D\varphi)\circ\tau_g$ for all smooth functions φ and for all $g \in G$.

One can easily prove the following lemma, which will be useful for the sequel.

LEMMA 3.1. *Let G and M be as before, with* $\dim G = q$ *and* $\dim M = n$. *Let* $\{X_1, \ldots, X_q\}$ *be a basis in* $\mathfrak{g}$ *and* $\{X_1^*, \ldots, X_q^*\}$ *be the corresponding vector fields on M. Then there exists an atlas* $\{(U_\alpha, \psi_\alpha)\}$ *on M satisfying the condition: for every* α, *there are n fields among* $\{X_1^*, \ldots, X_q^*\}$ *which generate the* C^∞*-module of all vector fields on* U_α. *If G is compact (and therefore M is compact too), we can suppose that the atlas consists of a finite number of local charts* $\{(U_1, \psi_1), \ldots, (U_r, \psi_r)\}$.

Now let us recall the definitions of Sobolev spaces of integer order on $\mathbb{R}^n$, on Riemannian manifolds and on Lie groups. Let $1 < p < +\infty$ and let k be a positive integer.

(I) We denote $W_p^k(\mathbb{R}^n) = \{u \in L_p(\mathbb{R}^n)\,|\,D^\alpha u \in L_p(\mathbb{R}^n) \text{ if } |\alpha| \leq k\}$
(here the derivatives must be understood in the sense of distributions), equipped with

the norm

$$\|u\|_{W_p^k(\mathbb{R}^n)} = \left(\|u\|^p_{L_p(\mathbb{R}^n)} + \sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L_p(\mathbb{R}^n)}\right)^{1/p}. \tag{3.2}$$

For references, see, for instance, [1, 12, 16].

(II) Let M be as in Section 2. For the definition of $W_p^k(M)$ in its whole generality we refer to [2, Chapter 2] and [16, Chapter 7]. Since we are interested only in compact manifolds, we define the Sobolev spaces $W_p^k(M)$ in the most natural way, namely via a finite number of local charts. More precisely, if M is compact, we fix a finite atlas $\{(U_l, \psi_l) \mid l = 1, \dots, r\}$ on M and a corresponding smooth partition of unity $\{\eta_l \mid l = 1, \dots, r\}$ satisfying: for every l, there exists an open subset V_l of U_l such that $\operatorname{supp} \eta_l \subsetneq V_l \subsetneq \overline{V}_l \subsetneq U_l$. We denote

$$W_p^k(M) = \{u \in L_p(M) \mid \text{ for every } l = 1, \dots, r,\ \eta_l u \circ \psi_l^{-1} \in W_p^k(\mathbb{R}^n)\}$$

(here we assume $\eta_l u \circ \psi_l^{-1} = 0$ outside $\psi_l(U_l)$), equipped with the norm

$$\|u\|_{W_p^k(M)} = \left(\sum_{l=1}^{r} \|\eta_l u \circ \psi_l^{-1}\|^p_{W_p^k(\mathbb{R}^n)}\right)^{1/p}. \tag{3.3}$$

Up to equivalence of the norms, the definition of $W_p^k(M)$ is independent of the choice of the atlas and the corresponding partition of unity.

(III) Let G be a q-dimensional connected Lie group. We equip the Lie algebra $\mathfrak{g}$ with a scalar product and we transfer the metric on G by means of the right translations. The corresponding Riemannian volume element generates a right invariant Haar measure on G, which we suppose satisfies (3.1).

Let $X_1, \dots, X_q$ be a fixed orthonormal basis in $\mathfrak{g}$ and let $\widetilde{X}_1, \dots, \widetilde{X}_q$ be the corresponding right invariant vector fields on G. Since later on we will need Sobolev spaces of the first order on G, we recall here only their definition.

We denote

$$W_p^1(G) = \{u \in L_p(G) \mid \text{for every } i = 1, \dots, q,\ \widetilde{X}_i u \in L_p(G)\}$$

(here the derivatives must be understood in the sense of distributions), equipped with the norm

$$\|u\|_{W_p^1(G)} = \left(\|u\|^p_{L_p(G)} + \sum_{i=1}^{q} \|\widetilde{X}_i u\|^p_{L_p(G)}\right)^{1/p}. \tag{3.4}$$

For references, see [16, Chapter 7, Section 7.6].

4. The function spaces $\mathcal{A}_p(M)$

From now on we assume that G and M have the same meaning as in § 3, with the additional condition that G, and therefore M, are compact.

In this section we define some function spaces on M by means of the compact group G, that is by means of the isometries of the compact homogeneous manifold $M = G/K$. We will denote these function spaces by $\mathcal{A}_p(M)$. We will show that, up to equivalence of the norms, the spaces $\mathcal{A}_p(M)$ are the Sobolev spaces $W_p^1(M)$. So we obtain a global characterization of the spaces $W_p^1(M)$ by means of the isometries of M.

Since G is compact, the exponential map from $\mathfrak{g}$ to G is onto.

For a function $u \in L_p(M)$ $(1 < p < +\infty)$ we define

$$\alpha_p(u) = \sup_{X \in \mathfrak{g}, X \neq 0} |X|^{-1} \|u \circ \tau_{\exp X} - u\|_{L_p(M)} . \tag{4.1}$$

Now we define the function spaces

$$\mathcal{A}_p(M) = \{u \in L_p(M) \mid \alpha_p(u) < +\infty\}$$

equipped with the norm

$$\|u\|_{\mathcal{A}_p(M)} = (\|u\|_{L_p(M)}^p + \alpha_p^p(u))^{1/p} . \tag{4.2}$$

In order to prove that $W_p^1(M) = \mathcal{A}_p(M)$ as Banach spaces, in the next sections we will give a characterization of the spaces $W_p^1(M)$ by means of the vector fields $X_1^*, \ldots, X_q^*$ induced by the basis $\{X_1, \ldots, X_q\}$ of $\mathfrak{g}$.

5. The function spaces $\mathcal{B}_p(M)$

Let $1 < p < +\infty$. We recall that $\dim G = q$ and $\dim M = n$ and that an orthonormal basis $\{X_1, \ldots, X_q\}$ is fixed in $\mathfrak{g}$. It is natural to consider the following function spaces on M (see also [17, Chapter 5, 5.7] and [14, Part III, 16]):

$$\mathcal{B}_p(M) = \Big\{v \in L_p(M) \mid \forall i = 1, \ldots, q \text{ there exists } h_i \in L_p(M) \text{ such that } \forall \varphi \in C^\infty(M) \int_M v(m) X_i^* \varphi(m)\, dm = -\int_M h_i(m) \varphi(m)\, dm\Big\} . \tag{5.1}$$

It is obvious that for $i = 1, \ldots, q$, h_i is unique and it can be easily verified that, if $v \in C^\infty(M)$, then $h_i = X_i^* v$. We will therefore denote $h_i = X_i^* v$.

If $v \in \mathcal{B}_p(M)$, we define

$$\beta_p(v) = \left(\sum_{i=1}^q \|X_i^* v\|_{L_p(M)}^p\right)^{1/p} \quad \text{and} \tag{5.2}$$

$$\|v\|_{\mathcal{B}_p(M)} = (\|v\|_{L_p(M)}^p + \beta_p^p(v))^{1/p} . \tag{5.3}$$

With respect to this norm $\mathcal{B}_p(M)$ is a Banach space.

In the next section we will show that $\mathcal{B}_p(M) = W_p^1(M)$. To this aim we prove here that the space $C^\infty(M)$ is dense in $\mathcal{B}_p(M)$. This result is standard, so we will only sketch the proof. We need in advance the following two lemmas, which can be proved by routine methods.

Lemma 5.1. *Let V be a right K-invariant function in $W_p^1(G)$ and let X be an element in $\mathfrak{g}$. Then $\tilde{X}V$ is a right K-invariant function in $L_p(G)$ satisfying $\mathcal{P}V \in \mathcal{B}_p(M)$ and $X^* \mathcal{P}V = \mathcal{P}(\tilde{X}V)$.*

LEMMA 5.2. *Let v be a function in $\mathcal{B}_p(M)$ and let X be an element in $\mathfrak{g}$. Then $\mathcal{R}v$ is a right K-invariant function in $W_p^1(G)$ satisfying $\tilde{X}(\mathcal{R}v) = \mathcal{R}(X^* v)$.*

THEOREM 5.1. *The space $C^\infty(M)$ is dense in $\mathcal{B}_p(M)$.*

PROOF. Let $\varepsilon > 0$ be fixed. If $v \in \mathcal{B}_p(M)$, then, by Lemma 5.2, $\mathcal{R}v = V \in W_p^1(G)$ and therefore there exists a function $\Psi \in C^\infty(G)$ such that $\|V - \Psi\|_{W_p^1(G)} < \varepsilon$.

We define a function Φ on G by setting $\Phi(g) = \int_K \Psi(gk)\,dk$, for every $g \in G$. Φ is a right K-invariant function in $C^\infty(G)$, satisfying

$$(5.4)\qquad (\mathcal{P}\Phi)(gK) = \int_K \Psi(gk)\,dk\,,$$

$$(5.5)\qquad X^* \mathcal{P}\Phi(gK) = \int_K \tilde{X}\Psi(gk)\,dk\,, \qquad \forall X \in \mathfrak{g}\,.$$

Recalling (3.1), by the Schwarz-Hölder inequality we obtain

$$(5.6)\qquad \|v - \mathcal{P}\Phi\|_{L_p(M)}^p \leqslant \int_{G/K} \left(\int_K |V(gk) - \Psi(gk)|^p\,dk \right) dg_K =$$

$$= \int_G |V(g) - \Psi(g)|^p\,dg = \|V - \Psi\|_{L_p(G)}^p < \varepsilon^p\,.$$

Similarly for every X_i in the fixed basis $\{X_1, \ldots, X_q\}$ of $\mathfrak{g}$ by (5.5) we have

$$\|X_i^* v - X_i^* \mathcal{P}\Phi\|_{L_p(M)}^p = \int_{G/K} |X_i^* v(gK) - \int_K \tilde{X}_i \Psi(gk)\,dk|^p\,dg_K\,.$$

Since by Lemma 5.2 $\mathcal{R}(X_i^* v) = \tilde{X}_i \mathcal{R}v = \tilde{X}_i V$, reasoning as above we obtain

$$(5.7)\qquad \|X_i^* v - X_i^* \mathcal{P}\Phi\|_{L_p(M)}^p \leqslant \|\tilde{X}_i V - \tilde{X}_i \Psi\|_{L_p(G)}^p < \varepsilon^p\,.$$

So we have proved that $\|v - \mathcal{P}\Phi\|_{\mathcal{B}_p(M)} \leqslant (1+q)^{1/p}\varepsilon$, that is $C^\infty(M)$ is dense in $\mathcal{B}_p(M)$.

We conclude this section with the following remarks.

REMARK 5.1. Let us denote $\mathcal{B}_p(M) = \mathcal{B}_p^1(M)$. Then for $k = 2, 3, 4, \ldots$ it is natural to define the spaces $\mathcal{B}_p^k$ by induction. Namely

$$(5.8)\qquad \mathcal{B}_p^k(M) = \{u \in L_p(M) \mid X_i^* u \in \mathcal{B}_p^{k-1}(M) \text{ for every } i = 1, \ldots, q\}$$

equipped with the norm

$$(5.9)\qquad \|u\|_{\mathcal{B}_p^k(M)} = \left(\|u\|_{L_p(M)}^p + \sum_{i=1}^q \|X_i^* u\|_{\mathcal{B}_p^{k-1}(M)}^p \right)^{1/p}.$$

Note that for $p = 2$ $\mathcal{B}_2(M) = \mathcal{B}_2^1(M)$ is a Hilbert space with respect to the scalar product

$$(u, v)_{\mathcal{B}_2(M)} = \int_M u(m) v(m)\, dm + \sum_{i=1}^{q} \int_M X_i^* u(m) X_i^* v(m)\, dm . \tag{5.10}$$

The same remark holds for $k > 1$ (and $p = 2$).

REMARK 5.2. Let us notice that in [15] the author defines Sobolev spaces of integer order on symmetric manifolds in a way similar to the approach we have exposed in this section.

Symmetric manifolds are particular homogeneous manifolds. On them one has at his disposal the geodesic symmetries, which are particular isometries. So in [15] geodesic symmetries are used to construct suitable vector fields, which give rise to the definition of Sobolev spaces.

6. $\mathcal{B}_p(M) = W_p^1(M)$

In this section we show that the function spaces $\mathcal{B}_p(M)$ introduced in § 5 are, up to equivalence of the norms, the classical Sobolev spaces $W_p^1(M)$, whose definition was recalled in § 3. Since we have not found references to this result, for the sake of completeness we expose here the proof of it.

THEOREM 6.1. *There exist two positive constants c_1 and c_2 such that for every smooth function φ on M the following inequalities hold*

$$c_1 \|\varphi\|_{W_p^1(M)} \leqslant \|\varphi\|_{\mathcal{B}_p(M)} \leqslant c_2 \|\varphi\|_{W_p^1(M)} . \tag{6.1}$$

PROOF. Let us prove the first inequality. Since G is compact, we select a finite atlas $\{(U_1, \psi_1), \dots, (U_r, \psi_r)\}$ on M satisfying the conditions of Lemma 3.1. Let $\{\eta_1, \dots, \eta_r\}$ be a corresponding partition of unity (*i.e.*, $\operatorname{supp} \eta_l \subseteq V_l \subseteq \overline{V}_l \subseteq U_l$, $l = 1, \dots, r$, where V_l is an open set in U_l). We denote with $y_{l1}, \dots, y_{ln}$ the coordinate system associated with the local chart (U_l, ψ_l).

Let $\varphi \in C^\infty(M)$; we consider the smooth function $\eta_l \varphi$ supported in $\overline{V}_l$ ($l = 1, \dots, r$). Then by definition we have

$$\int_M \left| \frac{\partial}{\partial y_{li}} (m)(\eta_l \varphi) \right|^p dm = \int_{\psi_l(\overline{V}_l)} |D_i(\eta_l \varphi \circ \psi_l^{-1})|^p \cdot \sqrt{\Gamma(x)}\, dx$$

where Γ denotes $|\gamma \circ \psi_l^{-1}|$. Since $\Gamma(x)$ never vanishes, there is a constant $c_0 > 0$ such that for every $l = 1, \dots, r$ and for every $x \in \psi_l(\overline{V}_l)$, $\sqrt{\Gamma(x)} \geqslant c_0^{-1}$ and therefore

$$\|D_i(\eta_l \varphi \circ \psi_l^{-1})\|_{L_p(\mathbb{R}^n)}^p \leqslant c_0 \int_M \left| \frac{\partial}{\partial y_{li}} (m)(\eta_l \varphi) \right|^p dm . \tag{6.2}$$

By Lemma 3.1 there are n vectors $X_{l1}, \dots, X_{ln}$ in the basis of $\mathfrak{g}$ (which – we recall – is supposed to be orthonormal with respect to the fixed scalar product) and there are n

smooth functions $b^i_{l1}, \ldots, b^i_{ln}$ defined on U_l such that

$$\frac{\partial}{\partial y_{li}}(m) = \sum_{j=1}^{n} b^i_{lj}(m) X^*_{lj}(m).$$

Therefore

$$\int_M \left| \frac{\partial}{\partial y_{li}}(m)(\eta_l \varphi) \right|^p dm \leq \int_M \left[\sum_{j=1}^{n} |b^i_{lj}(m) X^*_{lj}(m)(\eta_l \varphi)| \right]^p dm \leq$$

$$\leq n^{p-1} \sum_{j=1}^{n} \int_M |b^i_{lj}(m)|^p \cdot |X^*_{lj}(m)(\eta_l \varphi)|^p \, dm .$$

Let

$$B_l = \max_{i=1,\ldots,n} \ \max_{j=1,\ldots,n} \ \max_{m \in \overline{V}_l} |b^i_{lj}(m)|$$

and let us denote $\int_M \left| \frac{\partial}{\partial y_{li}}(m)(\eta_l \varphi) \right|^p dm$ by (1).

So it follows

$$(1) \leq n^{p-1} B_l^p \sum_{j=1}^{n} \int_M |\eta_l X^*_{lj}(m) \varphi + \varphi X^*_{lj}(m) \eta_l|^p \, dm \leq$$

$$\leq (2n)^{p-1} B_l^p \left\{ \sum_{j=1}^{n} \int_M |X^*_{lj}(m) \varphi|^p \, dm + \sum_{j=1}^{n} \int_M |\varphi(m)|^p \, |X^*_{lj}(m) \eta_l|^p \, dm \right\}.$$

Let

$$N_l = \max_{j=1,\ldots,n} \ \max_{m \in \overline{V}_l} |X^*_{lj}(m) \eta_l| .$$

We obtain

$$(1) \leq (2n)^{p-1} B_l^p \left\{ N_l^p \cdot n \|\varphi\|^p_{L_p(M)} + \sum_{j=1}^{n} \|X^*_{lj} \varphi\|^p_{L_p(M)} \right\}.$$

Now let $B = \max_{l=1,\ldots,r} B_l$ and $N = \max_{l=1,\ldots,r} N_l$. We have

$$(1) \leq (2n)^{p-1} B^p \left\{ N^p \cdot n \|\varphi\|^p_{L_p(M)} + \sum_{j=1}^{q} \|X_j^* \varphi\|^p_{L_p(M)} \right\}$$

and therefore by (6.2) we conclude that for every $l = 1, \ldots, r$ and every $i = 1, \ldots, n$

$$\|D_i(\eta_l \varphi \circ \psi_l^{-1})\|^p_{L_p(\mathbb{R}^n)} \leq c \left[\|\varphi\|^p_{L_p(M)} + \sum_{j=1}^{q} \|X_j^* \varphi\|^p_{L_p(M)} \right]$$

with $c = c_0(2n)^{p-1} B^p (nN^p + 1)$. So summing up over i and l we conclude

$$\sum_{l=1}^{r} \sum_{i=1}^{n} \|D_i(\eta_l \varphi \circ \psi_l^{-1})\|^p_{L_p(\mathbb{R}^n)} \leq nrc \{ \|\varphi\|^p_{L_p(M)} + [\beta_p(\varphi)]^p \} . \tag{6.3}$$

Reasoning as in (6.2) we also have that for every $l = 1, \ldots, r$

$$\|\eta_l \varphi \circ \psi_l^{-1}\|^p_{L_p(\mathbb{R}^n)} \leq c_0 \int_M |\eta_l(m)\varphi(m)|^p \, dm \leq c_0 \|\varphi\|^p_{L_p(M)} . \tag{6.4}$$

So adding $\sum_{l=1}^{r} \|\eta_l \varphi \circ \psi_l^{-1}\|^p_{L_p(\mathbb{R}^n)}$ to both sides of (6.3) we obtain

$$\|\varphi\|^p_{W^1_p(M)} \leq nr(c + c_0)\|\varphi\|^p_{\mathcal{B}_p(M)} ,$$

thus the desired inequality is proved.

The second inequality can be proved in the same way. We express locally every vector field X_i^* as a linear combination of the vector fields $\partial/\partial y_{l1}, \ldots, \partial/\partial y_{ln}$ with smooth coefficients and we proceed as in the first part. We only notice that in this case the choice of an atlas verifying the conditions of Lemma 3.1 is not needed.

Since $C^\infty(M)$ is dense in both the Banach spaces $\mathcal{B}_p(M)$ and $W^1_p(M)$, (6.1) leads to the following

Theorem 6.2. *$\mathcal{B}_p(M) = W^1_p(M)$ not only as sets, but also as Banach spaces;* i.e., *there exist two positive constants c_1 and c_2 such that for every $u \in W^1_p(M)$ the following inequalities hold*

$$c_1\|u\|_{W^1_p(M)} \leq \|u\|_{\mathcal{B}_p(M)} \leq c_2\|u\|_{W^1_p(M)} . \tag{6.5}$$

7. A characterization of the space $W^1_p(M)$

In this section we prove the characterization of the spaces $W^1_p(M)$ by means of the spaces $\mathcal{A}_p(M)$.

Lemma 7.1. *Let φ be a smooth function. Then*

$$\alpha_p(\varphi) \leq q^{1/p'}\beta_p(\varphi) \tag{7.1}$$

(where $1/p + 1/p' = 1$).

Proof. Let $Y \in \mathfrak{g}$, with $|Y| = 1$. Since

$$\varphi(\exp Y \cdot m) - \varphi(m) = \int_0^1 (Y^*\varphi)(\exp tY \cdot m)\, dt ,$$

by the Schwarz-Hölder inequality we have

$$|\varphi(\exp Y \cdot m) - \varphi(m)|^p \leq \int_0^1 |(Y^*\varphi)(\exp tY \cdot m)|^p \, dt$$

and therefore, since $\exp tY$ is an isometry

$$(7.2)\qquad \|\varphi \circ \tau_{\exp Y} - \varphi\|^p_{L_p(M)} \leq \int_0^1 dt \int_M |(Y^* \varphi)(\exp tY \cdot m)|^p \, dm =$$

$$= \int_M |Y^* \varphi(m)|^p \, dm = \|Y^* \varphi\|^p_{L_p(M)} .$$

Now we express Y by means of the fixed orthonormal basis $\{X_1, \ldots, X_q\}$, that is $Y = \sum_{j=1}^{q} b_j X_j$, where the b_j's are real numbers with $|b_j| \leq 1$. So from (7.2) we deduce

$$(7.3)\qquad \|\varphi \circ \tau_{\exp Y} - \varphi\|^p_{L_p(M)} \leq \|Y^* \varphi\|^p_{L_p(M)} \leq \left(\sum_{i=1}^{q} \|X_i^* \varphi\|_{L_p(M)}\right)^p \leq q^{p-1} \sum_{i=1}^{q} \|X_i^* \varphi\|^p_{L_p(M)} .$$

If X is now an arbitrary vector in $\mathfrak{g}$ different from 0, we set $Y = |X|^{-1} \cdot X$ and from (7.2) we obtain

$$\|\varphi \circ \tau_{\exp X} - \varphi\|^p_{L_p(M)} \leq |X|^p \, \|Y^* \varphi\|^p_{L_p(M)} .$$

So, for $X \in \mathfrak{g}$, $X \neq 0$, (7.3) leads to

$$|X|^{-1} \cdot \|\varphi \circ \tau_{\exp X} - \varphi\|_{L_p(M)} \leq q^{1/p'} \left(\sum_{i=1}^{q} \|X_i^* \varphi\|^p_{L_p(M)}\right)^{1/p}$$

and so the lemma is proved.

Now we are able to prove the announced characterization.

THEOREM 7.1. *A function u of $L_p(M)$ is in $W_p^1(M)$ if and only if $\alpha_p(u) < +\infty$. Moreover, there are two positive constants C_1 and C_2 such that for every $u \in W_p^1(M)$*

$$(7.4)\qquad C_1 \|u\|_{W_p^1(M)} \leq \|u\|_{\mathfrak{a}_p(M)} \leq C_2 \|u\|_{W_p^1(M)} .$$

PROOF. Let $u \in W_p^1(M)$; we recall that in Theorem 6.2 we proved the existence of a positive constant c_2 such that $\|u\|_{\mathcal{B}_p(M)} \leq c_2 \|u\|_{W_p^1(M)}$.

The previous lemma implies that for every smooth function φ

$$\|\varphi\|_{\mathfrak{a}_p(M)} \leq q^{1/p'} \|\varphi\|_{\mathcal{B}_p(M)} \leq q^{1/p'} c_2 \|\varphi\|_{W_p^1(M)} .$$

Since the space $C^\infty(M)$ is dense in $W_p^1(M)$, we obtain that every u in $W_p^1(M)$ satisfies

$$\|u\|_{\mathfrak{a}_p(M)} \leq C_2 \|u\|_{W_p^1(M)} ,$$

that is, one part of the theorem is proved with $C_2 = q^{1/p'} c_2$.

Now let us suppose that $u \in L_p(M)$ satisfies $\alpha_p(u) < +\infty$. This means that for every $X \in \mathfrak{g}$ we have

$$(7.5)\qquad \|u \circ \tau_{\exp X} - u\|_{L_p(M)} \leq \alpha_p(u) |X| .$$

Let φ be a smooth function on M. Since for every $X \in \mathfrak{g}$

$$\int_M [u(\exp X \cdot m) - u(m)]\, \varphi(m)\, dm = \int_M u(m)[\varphi(\exp(-X)\cdot m) - \varphi(m)]\, dm\,,$$

we obtain that for every X_i in the orthonormal basis of $\mathfrak{g}$ and for every t in $\mathbb{R}$

$$\left| \int_M u(m)[\varphi(\exp tX_i \cdot m) - \varphi(m)]\, dm \right| \leq \|u \circ \tau_{\exp(-tX_i)} - u\|_{L_p(M)} \|\varphi\|_{L_{p'}(M)}\,.$$

So (7.5) leads to

$$\left| \int_M u(m) t^{-1} [\varphi(\exp tX_i \cdot m) - \varphi(m)]\, dm \right| \leq \alpha_p(u) \|\varphi\|_{L_{p'}(M)}$$

and letting $t \to 0$ we obtain

$$\left| \int_M u(m) X_i^* \varphi(m)\, dm \right| \leq \alpha_p(u) \|\varphi\|_{L_{p'}(M)}\,. \tag{7.6}$$

Formula (7.6) shows that the functional «$\varphi \mapsto \int_M u(m) X_i^* \varphi(m)\, dm$» can be uniquely extended to a linear continuous functional on $L_{p'}(M)$. So there exists a unique function $h_i \in L_p(M)$ satisfying:

$$\begin{cases} \displaystyle\int_M u(m) X_i^* \varphi(m)\, dm = -\int_M h_i(m)\, \varphi(m)\, dm\,, & \forall \varphi \in C^\infty(M)\,, \\ \|h_i\|_{L_p(M)} \leq \alpha_p(u)\,, \quad \forall i = 1, \ldots, q\,. \end{cases} \tag{7.7}$$

This means that $u \in \mathcal{B}_p(M) = W_p^1(M)$ with $X_i^* u = h_i$ and therefore

$$\beta_p(u) \leq q^{1/p} \alpha_p(u)\,. \tag{7.8}$$

Recalling Theorem 6.2 we have proved Theorem 7.1 with, for instance, $C_1 = c_1 q^{-1/p}$.

8. A regularity result

In this section we will consider a linear differential equation on the compact homogeneous manifold G/K, connected with the second order operator $\mathcal{L} = \sum_{i=1}^{q} (X_i^*)^2$.

The proof we give here is suggested by the classical technique used in the case of PDE's in $\mathbb{R}^n$ and nowadays called «method of the translations» (see [13, § 4]).

Let us suppose that $\mathcal{L}$ is invariant under the isometries of G; as already said, this means that for every $X \in \mathfrak{g}$ and for every smooth function φ the equality $(\mathcal{L}\varphi) \circ \tau_{\exp X} = \mathcal{L}(\varphi \circ \tau_{\exp X})$ holds. Such a situation occurs, for instance, if we fix a scalar product in $\mathfrak{g}$ which is AD_G-invariant. Moreover, if M is a rank one symmetric manifold, then $\mathcal{L}$ is, up to a multiplicative constant, the Laplace-Beltrami operator of M.

From now on let $p = 2$. For a better understanding of the proof, we will distinguish the spaces $\mathcal{A}_2(M)$, $\mathcal{B}_2(M)$, $W_2^1(M)$ and their corresponding norms. We recall that

$\mathcal{B}_2(M) = \mathcal{B}_2^1(M)$ is a Hilbert space with respect to the scalar product defined in (5.10).

Let $f \in L_2(M)$. For the sake of simplicity, let us only consider the following problem:

$$\text{(P)} \qquad u - \mathcal{L}u = f.$$

We call u a weak solution of (P) if u satisfies

$$(u, v)_{\mathcal{B}_2(M)} = \int_M f(m)v(m)\,dm \qquad \forall v \in \mathcal{B}_2(M).$$

From the Riesz representation theorem for functionals on Hilbert spaces, we know that there exists one and only one weak solution u of (P) with $\|u\|_{\mathcal{B}_2(M)} \leqslant \|f\|_{L_2(M)}$. Now let us prove the following regularity result.

THEOREM 8.1. *If $f \in L_2(M)$, the weak solution u of* (P) *belongs to $W_2^2(M)$ and there exists a constant $C_1 > 0$ independent of f such that $\|u\|_{W_2^2(M)} \leqslant C_1\|f\|_{L_2(M)}$. Moreover, if $f \in W_2^k(M)$, u belongs to $W_2^{k+2}(M)$ and there exists a constant $C_2 > 0$ such that $\|u\|_{W_2^{k+2}(M)} \leqslant C_2\|f\|_{W_2^k(M)}$.*

PROOF. Let X be any element in $\mathfrak{g}$. The definition of the space $\mathcal{B}_2(M)$, the density of $C^\infty(M)$ in $\mathcal{B}_2(M)$ and the invariance property of $\mathcal{L}$ with respect to the isometries $\tau_{\exp X}$ allow us to conclude that if u is the weak solution of problem (P), then $u \circ \tau_{\exp X}$ is the weak solution of problem (P) with f substituted with $f \circ \tau_{\exp X}$ on the right hand side of (P).

For $X \neq 0$ in $\mathfrak{g}$ and for $h \in L_2(M)$, let us denote

$$D_X h = |X|^{-1}[h \circ \tau_{\exp X} - h].$$

From the above result we obtain that for every $X \neq 0$ in $\mathfrak{g}$

$$(D_X u, v)_{\mathcal{B}_2(M)} = \int_M D_X f(m)v(m)\,dm \tag{8.1}$$

holds for every $v \in \mathcal{B}_2(M)$.

Since

$$\int_M D_X f(m)v(m)\,dm = \int_M f(m)D_{-X}v(m)\,dm,$$

let us choose $v = D_X u \in \mathcal{B}_2(M)$ in (8.1). We obtain

$$\begin{aligned}(8.2) \qquad \|D_X u\|^2_{\mathcal{B}_2(M)} &\leqslant \|f\|_{L_2(M)}\|D_{-X}(D_X u)\|_{L_2(M)} = \\ &= \|f\|_{L_2(M)}\|D_X(D_X u)\|_{L_2(M)} \leqslant \|f\|_{L_2(M)}\,\alpha_2(D_X u) \leqslant \|f\|_{L_2(M)}\|D_X u\|_{\mathcal{A}_2(M)}.\end{aligned}$$

Now recalling the equivalence of the norms in $\mathcal{B}_2(M)$ and $\mathcal{A}_2(M)$, we have that there exists a constant $c > 0$ such that

$$\|D_X u\|_{\mathcal{A}_2(M)} \leqslant c\|f\|_{L_2(M)} \tag{8.3}$$

and therefore, in particular

$$\sup_{Y \in \mathfrak{g},\, Y \neq 0} |Y|^{-1} \|(D_X u) \circ \tau_{\exp Y} - D_X u\|_{L_2(M)} \leqslant c\|f\|_{L_2(M)} . \tag{8.4}$$

Since

$$\|(D_X u) \circ \tau_{\exp Y} - D_X u\|_{L_2(M)} =$$

$$= \sup_{\varphi \in C^\infty(M),\, \varphi \neq 0} \left\{ \|\varphi\|^{-1}_{L_2(M)} \int_M [(D_X u) \circ \tau_{\exp Y}(m) - D_X u(m)]\, \varphi(m)\, dm \right\},$$

let us explicitly write the last integral.

For every non zero elements X, Y in $\mathfrak{g}$ we have

$$(|X|\,|Y|)^{-1} \int_M [(D_X u) \circ \tau_{\exp Y}(m) - D_X u(m)]\, \varphi(m)\, dm = \tag{8.5}$$

$$= (|X|\,|Y|)^{-1} \int_M u(m)[\varphi(\exp(-Y)\exp(-X)\cdot m) +$$

$$- \varphi(\exp(-Y)\cdot m) - \varphi(\exp(-X)\cdot m) + \varphi(m)]\, dm .$$

Now let us choose $X = tX_1$ with $t > 0$ in (8.5). Recall that $|X_1| = 1$. From (8.4) we obtain that for every $Y \neq 0$ in $\mathfrak{g}$, for every $t > 0$ and for every $\varphi \in C^\infty(M)$

$$t^{-1} \int_M u(m)[D_{-Y}\varphi(\exp(-tX_1)m) - D_{-Y}\varphi(m)]\, dm \leqslant c\|f\|_{L_2(M)} \|\varphi\|_{L_2(M)} \tag{8.6}$$

holds. Letting $t \to 0^+$, it follows that

$$\int_M u(m)(-X_1^* D_{-Y}\varphi)(m)\, dm = \int_M X_1^* u(m)\{|-Y|^{-1}[\varphi(\exp(-Y)\cdot m) - \varphi(m)]\}\, dm =$$

$$= \int_M |Y|^{-1}[X_1^* u(\exp Y\cdot m) - X_1^* u(m)]\, \varphi(m)\, dm \leqslant c\|f\|_{L_2(M)} \|\varphi\|_{L_2(M)} .$$

Therefore we have $\|D_Y(X_1^* u)\|_{L_2(M)} \leqslant c\, \|f\|_{L_2(M)}$ and the arbitrariness of Y allows us to conclude that

$$\alpha_2(X_1^* u) \leqslant c\, \|f\|_{L_2(M)} . \tag{8.7}$$

Now we proceed in the same way for every $i = 2, \ldots, q$. Consequently we deduce that $X_i^* u$ is in $\mathcal{A}_2(M)$ for $i = 1, \ldots, q$. Substituting in (8.7) X_1^* with X_i^* and recalling formula (7.8) we have

$$\beta_2(X_i^* u) \leqslant q^{1/2} c\, \|f\|_{L_2(M)} . \tag{8.8}$$

Since

$$\|u\|^2_{\mathcal{B}_2^2(M)} = \|u\|^2_{\mathcal{B}_2^1(M)} + \sum_{i=1}^{q} \beta_2^2(X_i^* u)$$

we finally obtain

$$\|u\|_{\mathcal{B}_2^2(M)} \leqslant (1 + c^2 q^2)^{1/2} \|f\|_{L_2(M)} . \tag{8.9}$$

Therefore the first part of the theorem is proven.

Now let us suppose $f \in W_2^1(M)$ and let u be the corresponding weak solution of (P). If $i \in \{1, \dots, q\}$ and $\varphi \in C^\infty(M)$, again by the invariance of $\mathcal{L}$ with respect to the isometries of G, we obtain

$$\sum_{j=1}^{q} \int_M X_j^* u(m) X_j^* X_i^* \varphi(m)\, dm = -\int_M u(m) \mathcal{L}(X_i^* \varphi)(m)\, dm =$$

$$= -\int_M u(m) X_i^* (\mathcal{L}\varphi)(m)\, dm = -\sum_{j=1}^{q} \int_M X_j^* X_i^* u(m) X_j^* \varphi(m)\, dm .$$

Therefore it follows that

$$(X_i^* u, \varphi)_{\mathcal{B}_2(M)} = -\int_M u(m) X_i^* \varphi(m)\, dm - \sum_{j=1}^{q} \int_M X_j^* u(m) X_j^* X_i^* \varphi(m)\, dm =$$

$$= -(u, X_i^* \varphi)_{\mathcal{B}_2(M)} = -\int_M f(m) X_i^* \varphi(m)\, dm = \int_M X_i^* f(m) \varphi(m)\, dm .$$

Consequently, from the first part of the theorem we obtain that for every $i = 1, \dots, q$ $X_i^* u \in \mathcal{B}_2^2(M)$ and that $\|X_i^* u\|_{\mathcal{B}_2^2(M)} \leq (1 + c^2 q^2)^{1/2} \|X_i^* f\|_{L_2(M)}$.

Therefore we have proven that if $f \in W_2^1(M)$ there exists a constant $C > 0$, independent of f, such that $\|u\|_{W_2^3(M)} \leq C \|f\|_{W_2^1(M)}$. Reasoning as above, by induction on k, we obtain the desired result.

By way of completeness let us conclude this section with the following two remarks.

Remark 8.1. In the very wide-ranging paper [14] sharp regularity results are proven for second order operators on a manifold, involving Sobolev spaces, Bessel-potential spaces and Lipschitz spaces (which in our nomenclature are called Nikol'skij spaces).

By the extent of the argument treated in [14] the approach is clearly different from ours.

Remark 8.2. The manifolds we consider here are, like every compact Riemannian manifold, also spaces of homogeneous type according to the definition of R. R. Coifman and G. Weiss [8, p. 68]. Spaces of homogeneous type are currently being intensively studied (see, for instance, [4 and the references therein]). However our approach is different: as in [5], here we focalize our attention on the group action. The definition we adopt agrees with the notion of homogeneous space as given in [9] and is in accordance, for example, with the point of view of Chapter 3 in the survey paper [10].

This research was partially supported by MURST and by «Programma Vigoni 1996».

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