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Teoria dei gruppi. — *An elementary class extending abelian-by-G groups, for G infinite.* Nota (*) di Carlo Toffalori, presentata dal Socio G. Zappa.

Abstract. — We show that for no infinite group G the class of abelian-by-G groups is elementary, but, at least when G is an infinite elementary abelian p-group (with p prime), the class of groups admitting a normal abelian subgroup whose quotient group is elementarily equivalent to G is elementary.

Key words: Elementary class of structures; Abelian-by-G group; Commutator.

Riassunto. — *Una classe elementare di gruppi abeliani-per-G, con G infinito.* Si dimostra che, per ogni gruppo infinito G, la classe dei gruppi abeliani-per-G non è elementare; tuttavia, se G è un p-gruppo abeliano elementare infinito per qualche primo p, allora la classe dei gruppi che hanno un sottogruppo normale abeliano con gruppo quoziente elementarmente equivalente a G è elementare.

Fix a group G. A group S is said to be abelian-by-G if and only if S has a normal abelian subgroup A such that the quotient group S/A is isomorphic to G. Then the conjugation in A equips the subgroup A with a canonical structure of module over the group ring $\boldsymbol{Z}[G]$. When G is finite, abelian-by-G groups closely extend abelian groups, and it turns out that, from a model theoretic point of view, most information about the abelian-by-G group S is obtained by looking at A, viewed as a $\boldsymbol{Z}[G]$-module (see [2], for instance). When G is infinite, the connection between S and A is, of course, much less immediate. But S is still a group extension (with abelian kernel) of A by G, and it is worth examining what kind of model theoretic information about S is provided by A. The present *Note* is partly concerned with this problem.

More precisely, here is the aim of this paper.

When G is finite, it is known that abelian-by-G groups are an elementary class, in other words they can be axiomatized within the first order language for groups, and even by a single sentence; this was shown in [2] when the order of G is squarefree, and in [3] in the general setting. When G is infinite, this elementarity result does not hold (see the proposition below). However one might consider alternatively, for any group G, the class $\boldsymbol{K}(G)$ of the groups S admitting a normal abelian subgroup A whose quotient group S/A is elementarily equivalent to G. When G is finite, $\boldsymbol{K}(G)$ is just the class of abelian-by-G groups, and so is elementary owing to [3]. When G is infinite, there are again several examples witnessing that $\boldsymbol{K}(G)$ may be non-elementary. However we will show that, for some infinite G (more precisely when G is an infinite abelian group of prime exponent), $\boldsymbol{K}(G)$ is elementary.

We refer to [5] for group theory, to [1] for model theory, and to [4] for model theory of modules (in particular of abelian groups).

Proposition. Let G be an infinite group. Then the class of abelian-by-G groups is not elementary.

(*) Pervenuta all'Accademia il 19 giugno 1996.

PROOF. Let A be the additive group of the group ring $\boldsymbol{Z}[G]$ where $\boldsymbol{Z}$ is the ring of integers. Regard the elements of A as sequences of integers $(a_h)_{h \in G}$, with $a_h = 0$ almost everywhere; the operation in A is the addition defined componentwise, but we prefer to adopt the multiplicative notation for A, and to reserve the additive notation for $\boldsymbol{Z}$. Let S be the semidirect product of A and G, where G acts on A in the obvious way: for $g \in G$, $a = (a_h)_{h \in G} \in A$, $a^g = (a_{hg^{-1}})_{h \in G}$. This action is faithful. Furthermore the following facts hold.

1. A is a normal subgroup of S, G is a subgroup of S, $S = A \cdot G$, and the only element in $A \cap G$ is the identity 1_G of G.

2. A is abelian, and is a maximal abelian subgroup in S; in fact, let $g \in G, g \neq 1_G$, $a, b \in A$, then $b(ag) = (ba)g$, $(ag)b = (ab^g)g = (b^g a)g$, and $g \neq 1_G$ forces $b^g \neq b$ for some $b \in A$; accordingly b and ag do not commute.

3. For every $s \in S - A$, the centralizer of s in S has infinite index in S. In fact let $s = ag$ with $a \in A$, $g \in G$ and $g \neq 1_G$; we claim that the conjugacy class of s is infinite. For $b \in A$, $b^{-1}agb = (b^{-1}ab^g)g$ where $b^{-1}ab^g = (-b_h + a_h + b_{hg^{-1}})_{h \in G}$; fix $h \in G$ and $b_h \in \boldsymbol{Z}$; use $g \neq 1_G$, and let $b_{hg^{-1}}$ range over the integers; accordingly $-b_h + a_h + {} + b_{hg^{-1}}$ assumes infinitely many values. Hence $s = ag$ has infinitely many conjugates.

Now enlarge the first order language for groups by two 1-ary relation symbols, and in the new language consider the structure (S, A, G). Notice that the properties 1, 2 and 3 above (in particular the fact that S is the semidirect product of A and G) can be expressed in the new language by (possibly infinitely many) first order sentences. Take an elementary extension (S', A', G') of (S, A, G) such that (S', A', G') is λ-saturated for some cardinal λ greater than the power $|G|$ of G. Then A' is a normal abelian subgroup of S', but $S'/A' \simeq G'$, and so S'/A' cannot be isomorphic to G, because $|G'| = \lambda > |G|$. More generally, pick a normal abelian subgroup H' of S'. If H' is included in A', then $|S'/H'| = \lambda$, hence S'/H' cannot be isomorphic to G. Otherwise there exists some element s' in $H' - A'$; as H' is abelian, the centralizer of s' includes H'. But this centralizer has infinite index (so, index λ) in S'. Consequently H' has index λ in S', too. Hence S'/H' is not isomorphic to G. In conclusion S' is elementarily equivalent to S, but S' is not abelian-by-G. ■

DEFINITION. For every group G, $\boldsymbol{K}(G)$ is the class of the groups S such that, for some normal abelian subgroup A, S/A is elementarily equivalent to G.

We wish to deal with the following

PROBLEM. For which groups G is $\boldsymbol{K}(G)$ an elementary class (in the first order language for groups)?

REMARKS. 1. As observed before, if G is finite, then $\boldsymbol{K}(G)$ is the class of abelian-by-G groups, and so is elementary; actually a single sentence axiomatizes $\boldsymbol{K}(G)$ [3].

2. Of course, for G infinite, we can assume G countable.

3. Let us exhibit some (very familiar) examples of infinite (indecomposable) abelian groups G such that $\boldsymbol{K}(G)$ is not elementary.

(3.1) Let $G = \boldsymbol{Q}$ be the additive group of rationals. For every prime p, $\boldsymbol{Z}/p^{\infty} \oplus \boldsymbol{Q}$ is elementarily equivalent to $\boldsymbol{Z}/p^{\infty}$. But $\boldsymbol{Z}/p^{\infty} \oplus \boldsymbol{Q} \in \boldsymbol{K}(G)$, while $\boldsymbol{Z}/p^{\infty} \notin$ $\notin \boldsymbol{K}(G)$ because $\boldsymbol{Z}/p^{\infty}$ is a torsion group, and so no homomorphic image of $\boldsymbol{Z}/p^{\infty}$ can be a model of the theory of $\boldsymbol{Q}$, in particular torsionfree. So $\boldsymbol{K}(G)$ is not elementary.

(3.2) Now let $G = \boldsymbol{Z}_p$ be the localization of $\boldsymbol{Z}$ at a given prime p (recall that G is elementarily equivalent to the pure injective hull of $\boldsymbol{Z}_p$, so to its p-adic completion). Put $A = \bigoplus_n Z/p^n$ where n ranges over the positive integers. Then A is elementarily equivalent to $A \oplus \boldsymbol{Z}_p$. Moreover $A \oplus \boldsymbol{Z}_p \in \boldsymbol{K}(G)$, but $A \notin \boldsymbol{K}(G)$ because A is torsion, and any model of the theory of $\boldsymbol{Z}_p$ is torsionfree. Hence $\boldsymbol{K}(G)$ is not elementary.

(3.3) A similar argument shows that $\boldsymbol{K}(\boldsymbol{Z})$ is not elementary. In fact, when p ranges over the primes, and n over the positive integers

$$\bigoplus_{p,n} \boldsymbol{Z}/p^n \equiv \left(\bigoplus_{p,n} \boldsymbol{Z}/p^n\right) \oplus \boldsymbol{Z}$$

$\left(\text{both groups are elementarily equivalent to } \bigoplus_p \left(\bigoplus_n \boldsymbol{Z}/p^{\infty} \oplus \boldsymbol{Z}_p\right)\right)$; the latter group is in $\boldsymbol{K}(\boldsymbol{Z})$ and the former is not.

(3.4) Finally let $G = \boldsymbol{Z}/p^{\infty}$ for some prime p. Then $\boldsymbol{Q}/\boldsymbol{Z}$ is isomorphic to $\boldsymbol{Z}/p^{\infty}$, whence $\boldsymbol{Q}$, as well as $\boldsymbol{Z} \oplus \boldsymbol{Q}$, is in $\boldsymbol{K}(\boldsymbol{Z}/p^{\infty})$. But $\boldsymbol{Z} \oplus \boldsymbol{Q}$ is elementarily equivalent to $\boldsymbol{Z}$, and $\boldsymbol{Z}$ is not in $\boldsymbol{K}(\boldsymbol{Z}/p^{\infty})$ because any homomorphic image of $\boldsymbol{Z}$ is either finite or isomorphic to $\boldsymbol{Z}$, and in no case can be elementarily equivalent to $\boldsymbol{Z}/p^{\infty}$.

However we wish to show now that, for some infinite groups G, the class $\boldsymbol{K}(G)$ is elementary.

THEOREM. Let p be a prime, G be an infinite elementary abelian p-group. Then $\boldsymbol{K}(G)$ is elementary.

PROOF. First let us fix some notation. For every group S, let S^p be the subgroup generated by the p-th powers in S, and S' be the derived subgroup of S. Both S^p and S' are normal in S. In particular $S^p \cdot S'$ is a normal subgroup of S, too. We will divide our proof in two steps.

STEP 1. For every group S, $S \in \boldsymbol{K}(G)$ if and only if S satisfies the following conditions:

(*i*) $S^p \cdot S'$ is abelian;

(*ii*) $S^p \cdot S'$ has infinite index in S.

STEP 2. There are (infinitely many) sentences in the first order language for groups such that a group S satisfies all these sentences if and only if (*i*) and (*ii*) hold in S.

PROOF OF STEP 1. Let A be a normal abelian subgroup of S such that S/A is elementarily equivalent to G. Then S/A is an infinite elementary abelian p-group. Since S/A is abelian, S' is a subgroup of A. Since S/A is an elementary p-group, S^p is a subgroup of A (in fact, for every $s \in S$, $s^p A = (sA)^p = A$, so $s^p \in A$). In particular both S' and S^p are abelian, moreover commutators and p-th powers commute in S. Hence $S^p \cdot S'$ is abelian, and (i) holds. Furthermore A includes even $S^p \cdot S'$, and so $S/(S^p \cdot S')$ is infinite because projects itself onto S/A. Accordingly, also (ii) holds.

Conversely, assume (i) and (ii). (i) says that $S^p \cdot S'$ is abelian. As $S^p \cdot S'$ contains S', the quotient group $S/(S^p \cdot S')$ is abelian. As $S^p \cdot S'$ contains S^p, $S/(S^p \cdot S')$ is a p-group of exponent p. In conclusion, $S/(S^p \cdot S')$ is an elementary abelian p-group. (ii) implies that $S/(S^p \cdot S')$ is infinite; so $S/(S^p \cdot S')$ is elementarily equivalent to G, and $S \in \boldsymbol{K}(G)$ by means of $A = S^p \cdot S'$.

Proof of Step 2. It is easy to write down a first order sentence saying that in S two commutators, or two p-th powers, or a commutator and a p-th power commute. So (i) can be expressed in the required way. Now we have to deal with (ii).

Lemma. Let S be an abelian-by-H group, where H is a finite group of order n, and H is abelian. Then every element in S' can be expressed as the product of at most $3n^2$ commutators.

Proof of the Lemma. Let A be a normal abelian subgroup of S such that the quotient group S/A is isomorphic to H. Since H is abelian, S' is a (normal) subgroup of A. Let $x_1, \dots, x_n$ be a set of representatives for the cosets of A in S, then every element s of S decomposes in a unique way as $s = a \cdot x_j$, where a is in A, and $1 \leq j \leq n$. Take $a, b \in A$, $1 \leq i, j \leq n$, then $[ax_j, bx_i] = [ax_j, x_i][ax_j, b]^{x_i} = [a, x_i]^{x_j}[x_j, x_i][a, b]^{x_j x_i}[x_j, b]^{x_i}$, whence, using A abelian, $[ax_j, bx_i] = [a, x_i]^{x_j}[x_j, x_i][x_j, b]^{x_i}$ (here, the exponents denote conjugation). Moreover, if a and b are in A and $1 \leq i \leq n$, then $S' \subseteq A$ implies

$$[a, x_i][b, x_i] = a^{-1} x_i^{-1} a [x_i^{-1}, b] x_i = a^{-1} x_i^{-1} [x_i^{-1}, b] a x_i = [ba, x_i] = [ab, x_i],$$

and, similarly, $[x_i, a][x_i, b] = [x_i, ab]$. Since S' is abelian, it follows that any product of commutators in S can be expressed as a product of $[a, x_i]^{x_j}$, $[x_i, x_j]$, $[x_j, b]^{x_i}$ when i and j range over $\{1, \dots, n\}$, for a suitable choice of a and b in A, and so, in conclusion, as the product of at most $3n^2$ commutators.

This concludes the proof of the Lemma. Now let us come back to Step 2. Assume (i), hence $S^p \cdot S'$ abelian. If $S^p \cdot S'$ has a finite index $\leq n$ in S, then, owing to the Lemma, the following sentence α_n is true in S

$$\forall v_0 \dots \forall v_n \exists w \exists u_0 \dots \exists u_{3n^2-1}$$

$$\left(\bigwedge_{i<3n^2} \text{«}u_i \textit{ is a commutator}\text{»} \wedge \bigvee_{i<j\leq n} v_i v_j^{-1} = w^p \prod_{i<3n^2} u_i \right).$$

In fact any element of $S^p \cdot S'$ can be expressed as the product of a unique p-th power in S with at most $3n^2$ commutators, because, for a and b in S, $a^p b^p = (ab)^p \bmod S'$. Conversely, if S satisfies α_n for some positive integer n, then the index of $S^p \cdot S'$ in S is $\leq n$. It follows that, when (i) holds, $S/(S^p \cdot S')$ is infinite if and only if S satisfies $\neg \alpha_n$ for all positive integers n. This translates (ii) (modulo (i)) into infinitely many first order sentences, and accomplishes the required axiomatization of $\boldsymbol{K}(G)$. ■

REMARK. Notice that a simpler approach can be followed when $p = 2$. In fact, in this case, for every group S, $S \in \boldsymbol{K}(G)$ if and only if S^2 is abelian and S/S^2 is infinite.

The direction from the right to the left is a consequence of the fact that S/S^2 is a group of exponent 2, and hence is an elementary abelian 2-group. The other direction is implicit in the previous proof. «S/S^2 infinite» can be expressed by (infinitely many) first order sentences owing to the Oger result [3] saying that, in an abelian-by-H group S with H finite of order n and exponent m, every element of S^m can be expressed as the product of at most $2n - 1$ m-th powers in S.

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