Nicoletta Cantarini

Representations of $sl_q(3)$ at the roots of unity


Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1996_9_7_4_201_0>
Algebra. — Representations of \( \text{sl}_q(3) \) at the roots of unity. Nota (*) di Nicoletta Cantarini, presentata dal Corrisp. C. De Concini.

Abstract. — In this paper we study the irreducible finite dimensional representations of the quantized enveloping algebra \( \mathcal{U}_q(g) \) associated to \( g = \text{sl}(3) \), at the roots of unity. It is known that these representations are parametrized, up to isomorphisms, by the conjugacy classes of the group \( G = \text{SL}(3) \). We get a complete classification of the representations corresponding to the submaximal unipotent conjugacy class and therefore a proof of the De Concini-Kac conjecture about the dimension of the \( \mathcal{U}_q(g) \)-modules at the roots of 1 in the case of \( g = \text{sl}(3) \).

Key words: Enveloping algebra; Representation; Cartan matrix.

1. Introduction

In the papers [1, 3] the quantized enveloping algebra \( \mathcal{U}_q(g) \) introduced by Drinfeld [5, 6] and Jimbo [8], has been studied in the case \( q = \epsilon, \epsilon \) being an odd, primitive root of unity.

In particular it has been shown that the irreducible finite dimensional representations of \( \mathcal{U}_q(g) \) are parametrized, up to equivalence, by the conjugacy classes of the corresponding complex Lie group \( G \) with trivial center (see Section 2 for the definitions and Section 3 for the main results).

In this paper we will study the subregular representations of the quantum group \( \text{sl}_\epsilon(3) \), i.e. the irreducible representations corresponding to the unipotent conjugacy class of \( \text{SL}(3) \) of dimension 4.

The main result of this paper (see Theorem 4.8) consists in proving that any \( \text{sl}_\epsilon(3) \)-subregular module can be induced by an irreducible \( \text{sl}_\epsilon(2) \)-module in such a way that a suitable condition is satisfied (nice representation).

Hence we shall start from the construction of an induced module and study its irreducibility using a direct method (Propositions 4.4, 4.6, 4.7). In this way we shall be able to write a basis for any subregular module and to compute its dimension explicitly.

(*) Pervenuta all’Accademia l’11 luglio 1996.
2. Notations

2.1. Let \((a_{ij})\), \(i, j = 1, \ldots, n\), be a symmetric Cartan matrix and \(g\) the corresponding Lie algebra with Cartan subalgebra \(h\) and Chevalley generators \(e_i, f_i\) \((i = 1, \ldots, n)\).

Let \(Q\) be the root system associated to \((a_{ij})\), \(R\) the root lattice \(\mathcal{W}\) the Weyl group and \(\Delta = \{\alpha_1, \ldots, \alpha_n\}\) the set of simple roots. Then \(Q = Q^+ \cup Q^-\) where \(Q^+\) is the set of positive roots and \(Q^-\) is the set of negative roots.

Following Drinfeld [5, 6] and Jimbo [8] we consider the quantum group \(\mathcal{U}_q(g)\) associated to the matrix \((a_{ij})\) i.e. the associative algebra over \(C(q)\) generated by \(E_i, F_i, K_i, K_i^{-1} \((i = 1, \ldots, n)\)\) with the following relations:

\[
\begin{align*}
(2.1) & \quad K_i K_j = K_j K_i = K_{i+j}, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
(2.2) & \quad K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \\
(2.3) & \quad E_i F_j - F_j E_i = \delta_{ij} (K_i - K_i^{-1})/(q - q^{-1}), \\
(2.4) & \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \frac{1-a_{ij}}{s} \right] E_i^{1-a_{ij}-s} E_j E_i^s = 0 \quad \text{if} \ i \neq j, \\
(2.5) & \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \frac{1-a_{ij}}{s} \right] F_i^{1-a_{ij}-s} F_j F_i^s = 0 \quad \text{if} \ i \neq j.
\end{align*}
\]

Here \(\left[ \frac{1-a_{ij}}{s} \right]\) is the Gaussian binomial coefficient \(\left[ \frac{1-a_{ij}}{s} \right]_d\) with \(d = 1\).

2.2. Recall that the Braid group \(B_{\mathcal{W}}\) associated to \((a_{ij})\), with canonical generators \(T_i\), acts on \(\mathcal{U}_q(g)\) by automorphisms defined in [10] by:

\[
\begin{align*}
T_i K_j & = K_{\sigma_i(a_j)} K_j, \\
T_i E_j & = -F_j K_i, \quad T_i E_j = \sum_{s=0}^{-a_{ij}} (-1)^s q^{-s} E_i^{(-a_{ij}-s)} E_j E_i^s \quad \text{if} \ i \neq j, \\
T_i F_j & = -K_i^{-1} E_i, \quad T_i F_j = \sum_{s=0}^{a_{ij}} (-1)^s q^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} \quad \text{if} \ i \neq j,
\end{align*}
\]

where for each \(a \in N\) we have \(E_i^{(a)} = E_i^a / [a]!, \quad F_i^{(a)} = F_i^a / [a]!, \quad [a]! = [a] \cdots [1]\) and \([a] = (q^a - q^{-a})/(q - q^{-1})\).

Let \(\omega_0\) be the longest element in \(\mathcal{W}\) so that \(\omega_0(Q^+) = Q^-\). Chosen a reduced expression for \(\omega_0\): \(\omega_0 = s_{i_1} s_{i_2} \cdots s_{i_N}\) with \(N = |Q^+|\), we can define a convex total ordering of \(Q^+\):

\[
\beta_j = s_{i_1} \cdots s_{i_{j-1}} (\alpha_{i_j}) \quad j = 1, \ldots, N.
\]

We introduce the corresponding root vectors [10]:

\[
E_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}} E_{i_j}, \quad F_{\beta_j} = T_{i_1}^{-1} \cdots T_{i_{j-1}}^{-1} F_{i_j}, \quad j = 1, \ldots, N;
\]

then we let

\[
E^k = E_{\beta_1}^k \cdots E_{\beta_N}^k, \quad F^k = \omega E^k.
\]
for \( k = (k_1, \ldots, k_N) \in \mathbb{Z}^N_+ \), where \( \omega \) is the conjugate-linear anti automorphism of \( \mathcal{U}_q(g) \), as an algebra over \( \mathbb{C} \), defined by:

\[
\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_i) = K_i^{-1}, \quad \omega(q) = q^{-1}.
\]

It is known that \( \omega \) commutes with the action of the Braid group.

**Theorem 2.1** [9, 10]. (a) The set \( \{ F^{K_i}K_j^{m_j} E^{r_i} : k, r \in \mathbb{Z}^N_+, (m_1, \ldots, m_n) \in \mathbb{Z}^n \} \) is a basis of \( \mathcal{U}_q(g) \) over \( \mathbb{C}(q) \).

(b) For \( i < j \) one has:

\[
E^{(j-1)/2} F_i E^{(j-1)/2} F_j = \sum_{k \in \mathbb{Z}_+^N} c_k E^k
\]

where \( c_k \in \mathbb{C}(q, q^{-1}) \) and \( c_k \neq 0 \) only when \( k = (k_1, \ldots, k_N) \) is such that \( k_s = 0 \) for \( s \leq i \) and \( s \geq j \).

Now, let \( l \) be an odd integer greater than 1 and \( \varepsilon \) a primitive \( l \)-th root of 1. We denote by \( \mathcal{U}_\varepsilon \equiv \mathcal{U}_q(g) \) the algebra over \( \mathbb{C} \) obtained by specializing \( q \) to \( \varepsilon \). More precisely, let \( \mathcal{A} = \mathbb{C}[q, q^{-1}] \) and denote by \( \mathcal{A}_q \) the \( \mathcal{A} \) subalgebra of \( \mathcal{U}_q(g) \) generated by \( E_i, F_i, K_i, K_i^{-1} \) and \( (K_i - K_j^{-1})/(q - q^{-1}) \) with \( i = 1, \ldots, n \). Then \( \mathcal{U}_\varepsilon = \mathcal{A}_q / \varepsilon \mathcal{A}_q \).

Denote by \( Z_\varepsilon \) the center of \( \mathcal{U}_\varepsilon \). It is known [1] that \( E^l_{\alpha}, F^l_{\alpha} (\alpha \in \mathbb{Q}^+) \), \( K^l_i \) \((i = 1, \ldots, n) \) lie in \( Z_\varepsilon \). Let \( Z_0 \) be the subalgebra of \( Z_\varepsilon \) generated by these elements and denote by \( Z_0^- \), \( Z_0^0 \), \( Z_0^+ \) the subalgebras of \( Z_0 \) generated by \( F^l_{\alpha}, K^l_i \) and \( E^l_{\alpha} \) respectively, with \( \alpha \in \mathbb{Q}^+ \), \( j = 1, \ldots, n \). Then

\[
Z_0 = Z_0^- \otimes Z_0^0 \otimes Z_0^+.
\]

**Lemma 2.2** [1]. The algebra \( \mathcal{U}_\varepsilon \) is a free \( Z_0 \)-module on the basis \( \{ F^{K_i}K_j^{m_j} E^{r_i} : k = (k_1, \ldots, k_N), r = (r_1, \ldots, r_N) \in \mathbb{Z}^N_+, m_i \in \mathbb{Z}, 0 \leq k_i < l, 0 \leq r_i < l, 0 \leq m_i < l \} \).

### 3. Basic construction and main results

Let \( G \) be the connected complex Lie group with Lie algebra \( g \) and trivial center. Let \( T \) be the maximal torus of \( G \) corresponding to the Cartan subalgebra \( b \) of \( g \), \( U_- \) and \( U_+ \) the maximal unipotent subgroups of \( G \) corresponding to \( Q^- \) and \( Q^+ \) respectively, \( B_- = TU_- \) and \( B_+ = TU_+ \) Borel subgroups.

In this section we will recall the correspondence between the equivalence classes of the irreducible finite-dimensional representations of the quantized enveloping algebra \( \mathcal{U}_\varepsilon(g) \) and the conjugacy classes of the group \( G \), and we will collect the main results concerning this correspondence.

#### 3.1. Definition 3.1

If \( A \) is an associative algebra by \( \text{Spec} A \) we denote the set of the equivalence classes of the irreducible, finite dimensional representations of \( A \).
Remark. Using Schur's lemma one can consider the canonical map

\[ X: \text{Spec } \mathcal{U}_e \to \text{Spec } Z_e, \]

\[ \sigma \mapsto \lambda_\sigma, \]

where \( \sigma \) is an irreducible representation of \( \mathcal{U}_e \) on a vector space \( V \) such that

\[ \sigma(z)(v) = \lambda_\sigma(z)v \quad \forall z \in Z_e, \forall v \in V. \]

Proposition 3.2 [4].

1) The map \( X: \text{Spec } \mathcal{U}_e \to \text{Spec } Z_e \) is surjective;

2) the points of \( \text{Spec } Z_e \) parametrize the semisimple \( l^N \)-dimensional representations of \( \mathcal{U}_e \);

3) if \( \lambda \in \text{Spec } Z_e \), \( X^{-1}(\lambda) \) is the set of the irreducible components of the representation parametrized by \( \lambda \).

Corollary 3.3. Any finite dimensional irreducible \( \mathcal{U}_e \)-module has dimension less than or equal to \( l^N \).

Consider now the following sequence of canonical maps [3]:

\[ \varphi: \text{Spec } \mathcal{U}_e \to \text{Spec } Z_e \to Z_0 \to G. \]

Here \( \tau \) is induced by the inclusion \( Z_0 \subset Z_e \); it is finite with fibers of order less than or equal to \( l^n \) which are completely described in [1, 2]. The map \( \tau \) is constructed as follows: define

\[ \pi^-: \text{Spec } Z_0^- \to U_- \quad \text{and} \quad \pi^+: \text{Spec } Z_0^+ \to U_+ \]

respectively by the elements \( \exp(y_{\beta_n}f_{\beta_n}) \cdots \exp(y_{\beta_1}f_{\beta_1}) \) of \( U_-(Z_0^-) \) and

\[ \exp(T_0(y_{\beta_n})T_0(f_{\beta_n}) \cdots \exp(T_0(y_{\beta_1})T_0(f_{\beta_1})) \) of \( U_+(Z_0^+) \), where \( T_0 = T_i \cdots T_{i_n}, y_\alpha = (e^{1/2(a,a)} - e^{-1/2(a,a)})\beta_\alpha \) (\( \alpha \in Q_+ \)), and \( f_\alpha \) are root vectors in \( g \) defined by formulas analogous to (2.6), through the action of \( B_{\omega} \) on \( g \) introduced by Tits [11]:

\[ T_i = (\exp ad f_i)(\exp ad e_i)(\exp ad f_i). \]

We shall identify \( \text{Spec } Z_0^0 \) with \( T \) through the isomorphism \( R \to lR \) given by multiplication by \( l \). Now consider the map

\[ \pi: \text{Spec } Z_0 = \text{Spec } Z_0^- \times T \times \text{Spec } Z_0^+ \to G, \]

\[ \pi(a, t, b) = \pi^-(a) t^2 \pi^+(b); \]

the image of \( \pi \) is the big cell \( (U_-TU_+) \) of the group \( G \).

Theorem 3.4 [3]. There exists a canonical infinite dimensional group \( \tilde{G} \) of automorphisms of \( \mathcal{U}_e \) such that:

a) \( \tilde{G} \) stabilizes \( Z_0 \) and therefore acts on \( \text{Spec } Z_0 \):

\[ (\tilde{g}\lambda)(z) = \lambda(\tilde{g}^{-1}z), \quad \lambda \in \text{Spec } Z_0, \quad z \in Z_0, \quad \tilde{g} \in \tilde{G}; \]

b) \( X \) is an equivariant map with respect to the \( \tilde{G} \)-action;

c) the set \( F \) of fixed points of \( \tilde{G} \) in \( \text{Spec } Z_0 \) is \( (\pi)^{-1}(1) \);
d) if \( \mathcal{O} \) is the conjugacy class of a non central element of \( G \) then \( \pi^{-1}(\mathcal{O}) \) is a single \( \bar{G} \)-orbit and \( (\text{Spec} \, \mathcal{O}_0) - \mathcal{F} \) is a union of these \( \bar{G} \)-orbits.

The above theorem allows us to parametrize the equivalence classes of the irreducible \( \mathcal{U}_q(g) \)-modules by the conjugacy classes of the group \( G \). The following conjecture states the existence of a linking between the geometry of these conjugacy classes and the structure of the corresponding representations in a more precise sense:

**Conjecture [3].** If \( \sigma \in \text{Spec} \, \mathcal{U}_e \) is an irreducible representation of \( \mathcal{U}_e \) on a vector space \( V \) such that \( \varphi(\sigma) \) belongs to a conjugacy class \( \mathcal{O} \) in \( G \) then \( \dim V \) is divisible by \( \frac{1}{2} \dim \mathcal{O} / 2 \).

We recall that each conjugacy class in \( G \) has got even dimension less than or equal to \( 2N \). The above conjecture was proved in [4] in the maximal case:

**Theorem 3.5.** Any representation \( \sigma \in \text{Spec} \, \mathcal{U}_e \) such that \( \varphi(\sigma) \) lies in a regular conjugacy class of \( G \) has maximal dimension \( (= l^N) \).

From now on we consider the quantized enveloping algebra \( \mathcal{U}_q(g) \) associated to \( g = \mathfrak{sl}(n) \). Then \( \mathfrak{W} = S_n \) and \( G = SL(n) \). We will denote the Borel subgroups of \( G \) of upper and lower triangular matrices by \( B_+ \) and \( B_- \) respectively, while \( U_+ \) and \( U_- \) will be the corresponding unipotent subgroups and \( T \) the maximal torus of diagonal matrices.

**Definition 3.6.** We say that \( \sigma \in \mathcal{U}_e \) is unipotent if \( \varphi(\sigma) \) is a unipotent element in \( SL(n) \).

Take a non unipotent element \( \sigma \) in \( \text{Spec} \, \mathcal{U}_e \) and write \( m = \varphi(\sigma) = m_s m_u \) where \( m_s \) and \( m_u \) are the semisimple and unipotent part of \( m \) respectively \( (m_s \neq 1) \). Define \( T' = \text{center} (\text{centralizer}_G(m_s)) \) and put \( b' = \text{Lie} (T') \). Then \( b' \) will be a proper subalgebra of the Cartan subalgebra \( b \) of \( g \). Let \( Q' = \{ \alpha \in Q | \alpha \text{ vanishes on } b' \} \), then \( Q' = Z A' \cap Q \) where \( Z A' \) is a sublattice of \( R \) spanned by a proper subset \( A' \) of \( A \). We shall denote by \( g' \) the Lie algebra whose Chevalley generators are those of \( g \) corresponding to \( \alpha_i \in A' \) and by \( ' \mathcal{U} \) the subalgebra of \( \mathcal{U}_e \) generated by \( E_i, F_i \) with \( \alpha_i \in A' \) and \( K_j \) with \( j = 1, \ldots, n \). Put \( ' \mathcal{U} = ' \mathcal{U} \uplus ' \mathcal{U}^+ \) where \( ' \mathcal{U}^+ \) is the subalgebra of \( \mathcal{U}_e \) generated by \( E_i, K_i \) for \( i = 1, \ldots, n \). Then the following theorem holds:

**Theorem 3.7** [2]. If \( \sigma \in \text{Spec} \, \mathcal{U}_e \) is a non unipotent representation of \( \mathfrak{sl}_e(n) \) on a vector space \( V \) there exists a unique irreducible \( \mathcal{U}_e(g') \)-module \( V' \) such that:

1) \( V' \) is an irreducible \( ' \mathcal{U} \)-module;
2) \( V = \mathfrak{sl}_e(n) \otimes_{' \mathcal{U}} V' \); in particular \( \dim V = l' \dim V' \) where \( 2t = |Q/Q'| \).

The above theorem reduces the study of the irreducible representations of \( \mathfrak{sl}_e(n) \) to the study of its unipotent representations, since it states, in particular, that any \( \mathfrak{sl}_e(n) \)-module which is not unipotent is induced by a \( \mathfrak{sl}_e(r) \)-unipotent module, with \( r < n \).
We recall that the number of conjugacy classes of the unipotent elements in $SL(n)$ is finite and that each class is parametrized by the Jordan decomposition of its elements, i.e. by a partition of $n$. Moreover the following theorem holds:

**Theorem 3.8 [7].** Let $\mathcal{O}$ be a conjugacy class in $SL(n)$ parametrized by the partition $(h_i)$ of $n$. Then $\dim \mathcal{O} = n^2 - \sum h_i$, where $(h_i)$ is the dual partition.

**4. $U_e(sl(3))$: the subregular case**

In this section we will consider the case $g = sl(3)$ and study the subregular representations of the quantum group $U_e(sl(3))$ i.e. the irreducible representations which lie over the conjugacy class $\mathcal{O}$, parametrized by

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

through the correspondence (3.7). According to what stated in 3 this completes the proof of the recalled conjecture in the case of $sl_e(3)$. Indeed there are 3 conjugacy classes of unipotent elements in $SL(3)$:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
$$

In the first case ($\dim \mathcal{O} = 0$) the conjecture is empty and in the last case (the maximal case) it is proved by Theorem 3.5.

Let us fix a reduced expression for $w_0$, say $w_0 = s_2 s_1 s_2$. Then the following relations can be proved by induction on $r$:

$$
E_1 F_{12} = F_{12} E_1 - \left( \sum_{k=0}^{r-1} \varepsilon^{2k} \right) F_{12}^{-1} F_2 K_1^{-1} ;
$$

(4.8)

$$
E_2 F_{12} = F_{12} E_2 + \varepsilon \left( \sum_{k=0}^{r-1} \varepsilon^{-2k} \right) F_{12}^{-1} F_1 K_2.
$$

(4.9)

We recall that, with our choice of the reduced expression of $w_0$,

$$
F_1 F_{12} = \varepsilon^{-1} F_{12} F_1, \quad F_2 F_{12} = \varepsilon F_{12} F_2.
$$

Let us choose the representative element

$$
m = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
$$

of the class $\mathcal{O}$, then, using the definition of $q$, one sees that any representation in $q^{-1}(m)$ is such that $E_1^l = E_{12}^l = E_2^l = 0, K_1^l = K_2^l = 1, F_1^l = 0, F_2^l = 1 = F_{12}^l$, where the elements of $sl_e(3)$ are identified with their images through the representation.

According to [1], we consider the irreducible $(j + 1)$-dimensional representation $V$ ($0 \leq j \leq l - 1$) of $sl_e(2)$ with a basis consisting of the vectors $v, F_2 v, \ldots, F_{12}^j v$, where $v$
is a non zero vector such that \( E_2 v = 0, K_2 v = e^i v, F_2^{i+1} v = 0 \). Let \( \tilde{U} \) be the subalgebra of \( \mathcal{U}_q \) with generators \( E_2, F_2, K_2, E_1, K_1, F_1^i, F_1^{12} \) and define an action of \( \tilde{U} \) on \( V \) by the relations:

\[
F_1^i = 1, \quad F_1^{12} = 1, \quad E_1 V = 0, \quad K_1 F_1^i v = e^i F_1^i v \quad \forall r = 0, \ldots, j
\]

where \( i \) is a fixed integer such that \( 0 \leq i \leq l - 1 \). \( V \) is then a left \( \tilde{U} \)-module, and we can consider the induced representation \( \text{Ind}(V) := \text{sl}_q(3) \otimes \tilde{U} \).

**Definition 4.1.** We say that the above defined representation \( \text{Ind}(V) \) is a representation of type \((i, j)\) of \( \text{sl}_q(3) \).

**Remark.** A representation of type \((i, j)\) has dimension \((j + 1)l^2\). Indeed, by definition, a basis of \( \text{Ind}(V) \) consists of the vectors

\[
\{ F_1^i F_1^{12} F_2^s v : 0 \leq r, t \leq l - 1, 0 \leq s \leq j \}.
\]

**Lemma 4.2.** Given \( x \in \text{Ind}(V) \), \( x = \sum a_k F_1^r F_1^{12} F_2^s v \), the following relations hold:

\[
E_1(x) = -\sum_{k=1}^n a_k e^{-sk-i} \frac{1-e^{2sk}}{1-e^2} F_1^r F_1^{12} F_2^s v + \sum_{k=1}^n a_k \frac{(1-e^{2sk})(e^{2-2sk-t_s+s_k+i}-e^{sk-s_k-i})}{(e-e^{-1})(1-e^2)} F_1^r F_1^{12} F_2^s v;
\]

\[
E_2(x) = \sum_{k=1}^n a_k \frac{(e^{j+2}-e^{-j+2sk})(1-e^{-2sk})}{(e^2-1)(e-e^{-1})} F_1^r F_1^{12} F_2^{s+1} v + \sum_{k=1}^n a_k \frac{1-e^{2sk}}{1-e^2} e^{2-2sk+j} F_1^r F_1^{12} F_2^{s+1} v.
\]

**Proof.** By using relation (4.8) we have:

\[
E_1(x) = E_1 \left( \sum_{k=1}^n a_k F_1^r F_1^{12} F_2^s v \right) = \sum_{k=1}^n a_k E_1 F_1^r F_1^{12} F_2^s v =
\]

\[
= \sum_{k=1}^n a_k \left( F_1^r E_1 + F_1^{r-1} \left( \sum_{s=0}^{r_k-1} e^{-2s} \right) K_1 - \sum_{s=0}^{r_k-1} e^{2s} \right) K_1^{-1} F_1^{12} F_2^s v =
\]

\[
= \sum_{k=1}^n a_k F_1^r E_1 F_1^{12} F_2^s v + \sum_{k=1}^n a_k F_1^{r-1} \left( 1-e^{-2sk} \right) \left( 1-e^{-2} \right) e^{-t_s+s_k+i} - \left( 1-e^{2sk} \right) \left( 1-e^{-2} \right) e^{t_s-s_k-i} \frac{e-e^{-1}}{e-e^{-1}}.
\]

\[
\cdot F_1^{12} F_2^s v = \sum_{k=1}^n a_k F_1^r \left( - \sum_{m=0}^{t_s-1} e^{2m} \right) F_1^{12-1} F_2 K_1^{-1} F_2^s v +
\]

\[
+ \sum_{k=1}^n a_k \frac{1-e^{2sk}}{1-e^2} e^{2-2sk-t_s+s_k+i} - \left( 1-e^{2sk} \right) \left( 1-e^{-2} \right) e^{t_s-s_k-i} \frac{e-e^{-1}}{e-e^{-1}}.
\]
\[ F_{1}^{-1} F_{12} F_{2}^{-1} v = - \sum_{k=1}^{n} a_{k} \frac{1 - e^{2\lambda_{k}}}{1 - e^{2}} F_{1}^{-1} F_{12}^{k} F_{2}^{-1}^{k+1} v + \]
\[ + \sum_{k=1}^{n} a_{k} \frac{(1 - e^{2\lambda_{k}})(e^{2 - 2\lambda_{k} - i_{k} + s_{k} + i - e^{i_{k} - s_{k} - i})}{(e - e^{-1})(1 - e^{2})} F_{1}^{-1} F_{12}^{k} F_{2}^{k} v. \]

We compute \( E_{2}(x) \) in a similar way.  

Given a \( sl_{3}(\mathbb{C}) \)-module \( V \), we shall say that \( x \in V \) is a weight vector if it is a common eigenvector for the \( K_{i}'s \) for \( i = 1, 2 \).

**Lemma 4.3.** Each weight vector \( x \) in \( \text{Ind}(V) \) such that \( E_{2}(x) = 0 \) has the form

\[ x = \sum_{k=1}^{t+1} a_{k} F_{1}^{r} F_{12}^{k} F_{2}^{k} v. \]

with \( t, r \in \mathbb{N}, 0 \leq t \leq j, 0 \leq r \leq l - 1 \) and \( a_{k} \in \mathbb{C} - \{0\} \).

**Proof.** Let us take \( x \in \text{Ind}(V) \), then we can write \( x \) as a linear combination of the vectors in the basis (4.10): \( x = \sum_{k=1}^{n} a_{k} F_{1}^{r} F_{12}^{k} F_{2}^{k} v \).

If \( n = 1 \), relation (4.12) shows that \( E_{2}(x) = 0 \) if and only if \( s_{1} = t_{1} = 0 \). In this case \( x = F_{1}^{t_{1}} v \) spans the representation \( \text{Ind}(V) \) since \( F_{1} \) is invertible.

Suppose now \( n > 1 \). We rewrite (4.12) in the following way:

\[ E_{2}(x) = A + B = \sum_{k=1}^{n} \alpha_{k} A_{k} + \sum_{k=1}^{n} \beta_{k} B_{k} \]

with \( A_{k} = F_{1}^{r_{k}} F_{12}^{k} F_{2}^{k} v, B_{k} = F_{1}^{r_{k} + 1} F_{12}^{k} F_{2}^{k} v \); the vectors \( A_{k} \) are then linearly independent as well as the vectors \( B_{k} \), moreover \( A_{k} \neq B_{k} \) for the same \( k \). Now, if \( B_{k_{1}} = A_{k_{2}} \) for some \( k_{1} \neq k_{2} \), this means that

\[ \begin{cases} r_{k_{2}} = r_{k_{1}} + 1 \\ t_{k_{2}} = t_{k_{1}} - 1 \\ s_{k_{2}} = s_{k_{1}} + 1 \end{cases} \]

so that \( A_{k_{1}} \neq B_{k_{2}} \). In the same way, by induction, we get that if \( B_{k_{1}} = A_{k_{2}}, B_{k_{2}} = = A_{k_{1}}, \ldots, B_{k_{n-1}} = A_{k_{n}}, \) then \( k_{1}, \ldots, k_{n} \) must be different from each other and \( A_{k_{1}} \) is different from \( B_{k_{1}}, B_{k_{2}}, \ldots, B_{k_{n}} \). Therefore, \( E_{2}(x) = 0 \) if and only if there exists an ordering \( k_{1}, \ldots, k_{n} \) of the indeces such that

\[ \begin{cases} B_{k_{1}} = A_{k_{2}} \\ B_{k_{2}} = A_{k_{3}} \\ \vdots \\ B_{k_{n-1}} = A_{k_{n}} \end{cases} \]
and \( \alpha_{k_1} = 0, \beta_{k_n} = 0 \) i.e. \( s_{k_1} = 0, t_{k_n} = 0 \). Notice that system (4.15) is equivalent to the following:

\[
\begin{cases}
\alpha_{k_2} + \beta_{k_1} = 0 \\
\alpha_{k_1} + \beta_{k_2} = 0 \\
\vdots \\
\alpha_{k_n} + \beta_{k_{n-1}} = 0
\end{cases}
\]

and \( \alpha_{k_1} = 0, \beta_{k_n} = 0 \) i.e. \( s_{k_1} = 0, t_{k_n} = 0 \). Notice that system (4.15) is equivalent to the following:

\[
\begin{cases}
r_{k_2} = r_{k_1} + h - 1 \\
t_{k_n} = t_{k_1} - b + 1 \\
s_{k_2} = b - 1
\end{cases}
\]

with \( 2 \leq b \leq n \). Particularly \( t_{k_1} = t_{k_1} + n - 1 = n - 1 = s_{k_2} \), so that: \( 1 \leq t_{k_1} = n - 1 \leq \leq j \). Now we can write the relation \( \alpha_{k_2} + \beta_{k_1} = 0 \) explicitly:

\[
a_{k_2} \frac{(e^{j+2} - e^{-j+2t_{k_2}})(1 - e^{-2t_{k_2}})}{(e^2 - 1)(e - e^{-1})} + a_{k_2} \frac{1 - e^{2t_{k_2}}}{1 - e^2} e^{-t_{k_2} - 2t_{k_2} - j} = 0.
\]

We point out that, as in our hypothesis the coefficients of the previous equation are different from zero when \( 2 \leq b \leq n \), system (4.16) has got a solution \((a_{k_1}, \ldots, a_{k_n})\) with \( a_{k} \neq 0 \) for each \( j = 1, \ldots, n \), uniquely determined up to a scalar factor. Finally, if \( a_{k_j} = 0 \) for one \( j \) then \( x \equiv 0 \).

**Remark.** If \( t = 0 \) in (4.13) \( E_1(x) = 0 \) if and only if \( r = 0 \) or \( r = i + 1 \). These are the only cases in which a vector \( F_1^1 F_1^i F_2^j v \) is annihilated by both \( E_1 \) and \( E_2 \). Notice that, since \( F_1^1 = 1 \), the set \( \{ F_1^1 F_1^i F_2^j (F_1^{i+1} v): 0 \leq r, t \leq l - 1, 0 \leq s \leq j \} \) is a basis of \( \text{Ind}(V) \).

From now on we will suppose \( t > 0 \) in (4.13).

**Proposition 4.4.** Let \( x \) be of type (4.13), \( x \neq 0 \), such that \( E_1(x) = E_2(x) = 0 \). Then

\[
2 + i + j - t \equiv 0(\text{mod } l).
\]

**Proof.** Take \( x = \sum_{k=1}^{t+1} a_k F_1^{r+k-1} F_1^{i-k+1} F_2^{k-1} v \) as in Lemma 4.3. Then

\[
E_1(x) = - \sum_{k=1}^{t+1} a_k e^{-k+1-i} \frac{1 - e^{2(t-k-1)}}{1 - e^2} F_1^{r+k-1} F_1^{i-k+1} F_2^k v + \\
+ \sum_{k=1}^{t+1} a_k \frac{(1 - e^{2(r+k-1)}) (e^{2r-t+i} - e^{t-2k+1})}{(e - e^{-1})(1 - e^2)} F_1^{r+k-2} F_1^{i-k+1} F_2^{k-1} v.
\]

Since the first summand does not contain the vector \( F_1^{i-1} F_1^i v \), if \( E_1(x) = 0 \), we must have:

(A) \( r = 0 \)

or

(B) \( 1 - r - t + i \equiv 0(\text{mod } l) \).
Now, as
$$E_2(x) = \sum_{k=1}^{t+1} \frac{a_k}{(e^2 - 1)(e - e^{-1})} \left( e^{i+2} - e^{-j+2k-2} \right) (1 - e^{-2k+2}) F_{12}^{-k+1} F_{2}^{-2k+2} v +$$
$$+ \sum_{k=1}^{t+1} a_k \frac{1 - e^{2(t-k+1)}}{1 - e^2} \left( e^{3-j-k+j} F_{12}^{-k} F_{2}^{-k-1} v ,
$$
$$E_1(x) = E_2(x) = 0 \text{ if and only if the following system has got a non trivial solution for each } k = 2, \ldots, t+1:
$$
$$\begin{cases}
\frac{(e^{i+2} - e^{-j+2k-2})(1 - e^{-2k+2})}{(e^2 - 1)(e - e^{-1})} + a_{k-1} e^{4-t-k+j} \frac{1 - e^{2(t-k+2)}}{1 - e^2} = 0,
\frac{(1 - e^{2(r+k-1)})(e^{2-2r-t+i} - e^{t-2k+2-i})}{(e - e^{-1})(1 - e^2)} - a_{k-1} e^{-k+2-i} \frac{1 - e^{2(t-k+2)}}{1 - e^2} = 0.
\end{cases}
$$
Particularly, for $k = 2$ this is equivalent to require that
$$(e^{i+2} - e^{-j+2})(1 - e^{-2}) - e^{2-i+j-i}(1 - e^{2r+2})(e^{-2r+i-t+2} - e^{-i+t+2}) = 0.$$ We distinguish the following two different cases:

(A): $r = 0 \Rightarrow$
$$0 = (e^{i} - e^{-j})(e^2 - 1) - e^{2-t+i+j}(1 - e^2)(e^{i+2-t} - e^{-i+t+2}) =$$
$$= (e^2 - 1)(e^i - e^{-j} + e^{4-2t+2i+j} - e^i) \iff \iff e^{4-2t+2i+j} = e^{-j} \iff 2 - t + i + j \equiv 0 \text{ (mod } l),$$

(B): $1 - r + i - t \equiv 0 \text{ (mod } l) \Rightarrow$
$$0 = (e^{i} - e^{-j})(e^2 - 1) - e^{2-t+i+j}(1 - e^{2r+2})(e^{1-r} - e^{-r-1}) =$$
$$= (e^2 - 1)(e^i - e^{-j})(e^2 - 1) - e^{1+j}(1 - e^{2r+2})(e - e^{-1}) =$$
$$= (e^2 - 1)(e^i - e^{-j} - e^{2r+2+j}) \iff e^{2r+2+j} = e^{-j} \iff r + 1 + j \equiv 0.$$ The above relation, together with (B), is equivalent to (4.18). 

**Definition 4.5.** We say that a sl_3(3)-module is nice if it is of type $(i,j)$ with $2 + i + j \equiv l$ or $i = l - 1$.

**Proposition 4.6.** A nice representation is irreducible.

**Proof.** Let us consider a representation of type $(i,j)$ generated by a vector $v \neq 0$. Proposition 4.4 shows that if
$$2 + i + j \equiv t \text{ (mod } l)$$
for any $t$ such that $1 \leq t \leq j$, the representation $\text{Ind}(V)$ contains no weight vector $x \neq \alpha v, \beta F_{12}^{i+1} v$, with $\alpha, \beta \in \mathbb{C}$, such that $E_1(x) = 0 = E_2(x)$. Now, since $E_1^l = E_{12}^l = E_2^l = 0$, the algebra generated by $E_1, E_2$ is nilpotent, therefore if $W \subset \text{Ind}(V)$ is a subrepresentation of $\text{Ind}(V)$, there exists a weight vector $w \in W$ such that $E_1(w) = 0 = E_2(w)$. This forces $w$ to be a multiple scalar of $v$ or of $F_{12}^{i+1} v$ and therefore $W = \text{Ind}(V)$. 

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Finally it is easy to verify that $2 + i + j \neq t$ for any $t$ such that $1 \leq t \leq j$ if and only if $2 + i + j \leq l$ or $i = l - 1$.

**Proposition 4.7.** If $V$ is a $\mathfrak{sl}_q(3)$-module of type $(i, j)$ and is not nice there exists a nice submodule $W$ of $V$ such that the quotient $V/W$ is a nice representation.

**Proof.** Let $V$ be a representation of $\mathfrak{sl}_q(3)$ of type $(i, j)$ with $2 + i + j \geq l + 1$, $i \neq l - 1$. Take $\bar{x} = F_2^{l-i-j-1}F_1^{i+1}v$, then $\bar{x}$ is a weight vector killed by both $E_1$ and $E_2$ which spans a proper subrepresentation $\Phi$ of $V$, with basis $\{F_iF_i'F_2^r\bar{x}: 0 \leq r, t \leq l - 1, 0 \leq s \leq l - i - 2\}$. $\Phi$ is irreducible, indeed it is the representation of type $([l - j - 2], l - i - 2)$, generated by $F_1^{i+1}\bar{x}$, which can be easily seen to be nice. (By $[l - j - 2]$ we mean the integer $k \in [0, l - 1]$ such that $k \equiv l - j - 2 \pmod{l}$). Finally the quotient $V/\Phi$ is the representation of type $(l - i - 2, i + j + 1 - l)$ generated by $F_1^{i+1}v$ and this is nice too.

**Theorem 4.8.** Every subregular representation of $U_q(\mathfrak{sl}(3))$ is a nice representation.

**Proof.** Let us take a subregular representation $W$ of $\mathfrak{u}_q(\mathfrak{sl}(3))$. As the algebra generated by $E_1$ and $E_2$ is nilpotent, the set

\[ B := \{ w \in W: E_1(w) = 0 = E_2(w) \} \]

is nontrivial; moreover $K_1$ and $K_2$ act diagonally on $B$. Take then $u \in B \setminus \{0\}$ such that $K_1u = e^u u$, $K_2u = e^u u$: $u$ spans $W$ since $W$ is irreducible. Consider the subspace $V$ of $W$ generated by the set $\{F_2^r u: 0 \leq r \leq l - 1\}$; $V$ is stable under the action of $F_2$, $E_2$, $K_1$, $K_2$. In particular $V$ defines a representation of the subalgebra $\tilde{\mathfrak{u}}$ of $\mathfrak{u}_q(\mathfrak{sl}(3))$ generated by $E_2$, $F_2$, $K_2$, $K_1$. Let $V'$ be an irreducible $\tilde{\mathfrak{u}}$-submodule of $V$. $V'$ is then an irreducible representation of $\mathfrak{sl}_q(2)$, since $K_2^2K_1$ is central in $\tilde{\mathfrak{u}}$. We then see that $V'$ is stable under $K_1$ as $K_1 = \lambda K_2^{j-1}/2$ with $\lambda \in \mathbb{C}$.

Define $\text{Ind}(V')$ as the representation induced by $V'$ on $\mathfrak{sl}_q(3)$ in the natural way. Then $W$ is a quotient of $\text{Ind}(V')$ since the set $\{F_iF_i'F_2^r\tilde{v}: \tilde{v} \in V', 0 \leq r, t \leq l - 1\}$ is stable under the action of $E_2$, $F_2$, $K_j$ for any $a \in \mathbb{Q}^+$, $j = 1, 2$. Now, if $\text{Ind}(V')$ is nice, $W = \text{Ind}(V')$. Otherwise, by Proposition 4.7, $\text{Ind}(V')$ contains a proper nice subrepresentation $\Phi$ such that $\text{Ind}(V')/\Phi$ is nice. Write $W = \text{Ind}(V')/T$ where $T$ is a subrepresentation of $\text{Ind}(V')$. Then, if $T \cap \Phi \neq \{0\}$, $T \supset \Phi$ so that $W = \text{Ind}(V')/T \subset \text{Ind}(V')/\Phi$, but since $\text{Ind}(V')/\Phi$ is irreducible, $W = \text{Ind}(V')/\Phi$.

On the contrary, if $T \cap \Phi = \{0\}$ then $W = \Phi$ so that $W = \Phi$.

**Corollary 4.9.** The dimension of any subregular representation of $U_q(\mathfrak{sl}(3))$ is divisible by $l^2$.

**Acknowledgements**

I would like to express my gratitude to Professor Corrado De Concini for the patient interest with which he followed this paper.
REFERENCES


