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Representations of $sl_q(3)$ at the roots of unity

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Algebra. — *Representations of $sl_q(3)$ at the roots of unity.* Nota (*) di NICOLETTA CANTARINI, presentata dal Corrisp. C. De Concini.

ABSTRACT. — In this paper we study the irreducible finite dimensional representations of the quantized enveloping algebra $\mathcal{U}_q(g)$ associated to $g = sl(3)$, at the roots of unity. It is known that these representations are parametrized, up to isomorphisms, by the conjugacy classes of the group $G = SL(3)$. We get a complete classification of the representations corresponding to the submaximal unipotent conjugacy class and therefore a proof of the De Concini-Kac conjecture about the dimension of the $\mathcal{U}_q(g)$-modules at the roots of 1 in the case of $g = sl(3)$.

KEY WORDS: Enveloping algebra; Representation; Cartan matrix.

RIASSUNTO. — *Rappresentazioni di $sl_q(3)$ alle radici dell'unità*. Vengono studiate le rappresentazioni irriducibili, finito-dimensionali dell'algebra inviluppante quantizzata $\mathcal{U}_q(g)$ associata a $g = sl(3)$, alle radici dell'unità. È noto che tali rappresentazioni sono parametrizzate, a meno di isomorfismi, dalle classi di coniugio del gruppo $G = SL(3)$. Si ottiene una classificazione completa delle rappresentazioni corrispondenti alla classe di coniugio unipotente sottomassimale e quindi una prova, nel caso $g = sl(3)$, della congettura di De Concini, Kac sulla dimensione degli $\mathcal{U}_q(g)$-moduli alle radici dell'unità.

1. INTRODUCTION

In the papers [1, 3] the quantized enveloping algebra $\mathcal{U}_q(g)$ introduced by Drinfeld [5, 6] and Jimbo [8], has been studied in the case $q = \varepsilon$, ε being an odd, primitive root of unity.

In particular it has been shown that the irreducible finite dimensional representations of $\mathcal{U}_\varepsilon(g)$ are parametrized, up to equivalence, by the conjugacy classes of the corresponding complex Lie group G with trivial center (see Section 2 for the definitions and Section 3 for the main results).

In this paper we will study the subregular representations of the quantum group $sl_\varepsilon(3)$, *i.e.* the irreducible representations corresponding to the unipotent conjugacy class of $SL(3)$ of dimension 4.

The main result of this paper (see Theorem 4.8) consists in proving that any $sl_\varepsilon(3)$-subregular module can be induced by an irreducible $sl_\varepsilon(2)$-module in such a way that a suitable condition is satisfied (nice representation).

Hence we shall start from the construction of an induced module and study its irreducibility using a direct method (Propositions 4.4, 4.6, 4.7). In this way we shall be able to write a basis for any subregular module and to compute its dimension explicitly.

(*) Pervenuta all'Accademia l'11 luglio 1996.

2. Notations

2.1. Let (a_{ij}), $i, j = 1, \dots, n$, be a symmetric Cartan matrix and g the corresponding Lie algebra with Cartan subalgebra h and Chevalley generators e_i, f_i $(i = 1, \dots, n)$.

Let Q be the root system associated to (a_{ij}), R the root lattice $\mathcal{W}$ the Weyl group and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots. Then $Q = Q^+ \cup Q^-$ where Q^+ is the set of positive roots and Q^- is the set of negative roots.

Following Drinfeld [5, 6] and Jimbo [8] we consider the quantum group $\mathcal{U}_q(g)$ associated to the matrix (a_{ij}) *i.e.* the associative algebra over $\mathbf{C}(q)$ generated by E_i, F_i, K_i, K_i^{-1} $(i = 1, \dots, n)$ with the following relations:

$$K_i K_j = K_j K_i = K_{i+j}, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{2.1}$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \tag{2.2}$$

$$E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1})/(q - q^{-1}), \tag{2.3}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \quad \text{if } i \neq j, \tag{2.4}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix} F_i^{1-a_{ij}-s} F_j F_i^s = 0 \quad \text{if } i \neq j. \tag{2.5}$$

Here $\begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}$ is the Gaussian binomial coefficient $\begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_d$ with $d = 1$.

2.2. Recall that the Braid group $B_{\mathcal{W}}$ associated to (a_{ij}), with canonical generators T_i, acts on $\mathcal{U}_q(g)$ by automorphisms defined in [10] by:

$$T_i K_j = K_{s_i(\alpha_j)},$$

$$T_i E_i = -F_i K_i, \qquad T_i E_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)} \quad \text{if } i \neq j,$$

$$T_i F_i = -K_i^{-1} E_i, \qquad T_i F_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} \quad \text{if } i \neq j,$$

where for each $a \in \mathbf{N}$ we have $E_i^{(a)} = E_i^a / [a]!$, $F_i^{(a)} = F_i^a / [a]!$, $[a]! = [a] \dots [1]$ and $[a] = (q^a - q^{-a})/(q - q^{-1})$.

Let w_0 be the longest element in $\mathcal{W}$ so that $w_0(Q^+) = Q^-$. Chosen a reduced expression for w_0: $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ with $N = |Q^+|$, we can define a convex total ordering of Q^+:

$$\beta_j = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j}) \qquad j = 1, \dots, N.$$

We introduce the corresponding root vectors [10]:

$$E_{\beta_j} = T_{i_1} \dots T_{i_{j-1}} E_{i_j}, \qquad F_{\beta_j} = T_{i_1} \dots T_{i_{j-1}} F_{i_j}, \qquad j = 1, \dots, N; \tag{2.6}$$

then we let

$$E^k = E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N}, \qquad F^k = \omega E^k$$

for $k = (k_1, \ldots, k_N) \in Z_+^N$, where ω is the conjugate-linear anti automorphism of $\mathcal{U}_q(g)$, as an algebra over C, defined by:

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_i) = K_i^{-1}, \quad \omega(q) = q^{-1}.$$

It is known that ω commutes with the action of the Braid group.

THEOREM 2.1 [9, 10]. (*a*) *The set* $\{F^k K_1^{m_1} \ldots K_n^{m_n} E^r : k, r \in Z_+^N, (m_1, \ldots, m_n) \in Z^n\}$ *is a basis of* $\mathcal{U}_q(g)$ *over* $C(q)$.

(*b*) *For* $i < j$ *one has*:

$$E_{\beta_i} E_{\beta_j} - q^{(\beta_i | \beta_j)} E_{\beta_j} E_{\beta_i} = \sum_{k \in Z_+^N} c_k E^k$$

where $c_k \in C[q, q^{-1}]$ *and* $c_k \neq 0$ *only when* $k = (k_1, \ldots, k_N)$ *is such that* $k_s = 0$ *for* $s \leqslant i$ *and* $s \geqslant j$.

Now, let l be an odd integer greater than 1 and ε a primitive l-th root of 1. We denote by $\mathcal{U}_\varepsilon \equiv \mathcal{U}_\varepsilon(g)$ the algebra over C obtained by specializing q to ε. More precisely, let $\mathcal{A} = C[q, q^{-1}]$ and denote by $\mathcal{U}_\mathcal{A}$ the $\mathcal{A}$ subalgebra of $\mathcal{U}_q(g)$ generated by E_i, F_i, K_i, K_i^{-1} and $(K_i - K_i^{-1})/(q - q^{-1})$ with $i = 1, \ldots, n$. Then $\mathcal{U}_\varepsilon = \mathcal{U}_\mathcal{A}/(q - \varepsilon)\mathcal{U}_\mathcal{A}$.

Denote by Z_ε the center of $\mathcal{U}_\varepsilon$. It is known [1] that E_α^l, F_α^l $(\alpha \in Q^+)$, K_i^l $(i = 1, \ldots, n)$ lie in Z_ε. Let Z_0 be the subalgebra of Z_ε generated by these elements and denote by Z_0^-, Z_0^0, Z_0^+ the subalgebras of Z_0 generated by F_α^l, K_j^l and E_α^l respectively, with $\alpha \in Q^+$, $j = 1, \ldots, n$. Then

$$Z_0 \simeq Z_0^- \otimes Z_0^0 \otimes Z_0^+ .$$

LEMMA 2.2 [1]. *The algebra* $\mathcal{U}_\varepsilon$ *is a free* Z_0*-module on the basis* $\{F^k K_1^{m_1} \ldots K_n^{m_n} E^r :$ $k = (k_1, \ldots, k_N)$, $r = (r_1, \ldots, r_N) \in Z_+^N$, $m_i \in Z$, $0 \leqslant k_i < l$, $0 \leqslant r_i < l$, $0 \leqslant m_i < l\}$.

3. BASIC CONSTRUCTION AND MAIN RESULTS

Let G be the connected complex Lie group with Lie algebra g and trivial center. Let T be the maximal torus of G corresponding to the Cartan subalgebra h of g, U_- and U_+ the maximal unipotent subgroups of G corresponding to Q^- and Q^+ respectively, $B_- = TU_-$ and $B_+ = TU_+$ Borel subgroups.

In this section we will recall the correspondence between the equivalence classes of the irreducible finite-dimensional representations of the quantized enveloping algebra $\mathcal{U}_\varepsilon(g)$ and the conjugacy classes of the group G, and we will collect the main results concerning this correspondence.

3.1. DEFINITION 3.1. *If* A *is an associative algebra by* Spec$A$ *we denote the set of the equivalence classes of the irreducible, finite dimensional representations of* A.

REMARK. Using Schur's lemma one can consider the canonical map

$$X\colon \operatorname{Spec} \mathcal{U}_\varepsilon \to \operatorname{Spec} Z_\varepsilon\,,$$
$$\sigma \mapsto \lambda_\sigma\,,$$

where σ is an irreducible representation of $\mathcal{U}_\varepsilon$ on a vector space V such that

$$\sigma(z)(v) = \lambda_\sigma(z)v \qquad \forall z \in Z_\varepsilon\,,\ \forall v \in V\,.$$

PROPOSITION 3.2 [4].

1) *The map* $X\colon \operatorname{Spec}\mathcal{U}_\varepsilon \to \operatorname{Spec} Z_\varepsilon$ *is surjective*;

2) *the points of* $\operatorname{Spec} Z_\varepsilon$ *parametrize the semisimple* l^N*-dimensional representations of* $\mathcal{U}_\varepsilon$;

3) *if* $\lambda \in \operatorname{Spec} Z_\varepsilon$, $X^{-1}(\lambda)$ *is the set of the irreducible components of the representation parametrized by* λ.

COROLLARY 3.3. *Any finite dimensional irreducible* $\mathcal{U}_\varepsilon$*-module has dimension less than or equal to* l^N.

Consider now the following sequence of canonical maps [3]:

$$\varphi\colon \operatorname{Spec}\mathcal{U}_\varepsilon \xrightarrow{X} \operatorname{Spec} Z_\varepsilon \xrightarrow{\tau} \operatorname{Spec} Z_0 \xrightarrow{\pi} G\,. \tag{3.7}$$

Here τ is induced by the inclusion $Z_0 \subset Z_\varepsilon$; it is finite with fibers of order less than or equal to l^n which are completely described in [1, 2]. The map π is constructed as follows: define

$$\pi^-\colon \operatorname{Spec} Z_0^- \to U_- \quad \text{and} \quad \pi^+\colon \operatorname{Spec} Z_0^+ \to U_+$$

respectively by the elements $\exp(y_{\beta_N} f_{\beta_N}) \ldots \exp(y_{\beta_1} f_{\beta_1})$ of $U_-(Z_0^-)$ and $\exp(T_0(y_{\beta_N})\,T_0(f_{\beta_N})) \ldots \exp(T_0(y_{\beta_1})\,T_0(f_{\beta_1}))$ of $U_+(Z_0^+)$, where $T_0 = T_{i_1} \ldots T_{i_N}$, $y_\alpha = (\varepsilon^{1/2(\alpha,\alpha)} - \varepsilon^{-1/2(\alpha,\alpha)})^l F_\alpha^l$ $(\alpha \in Q^+)$, and f_α are root vectors in g defined by formulas analogous to (2.6), through the action of $B_{\mathcal{W}}$ on g introduced by Tits [11]:

$$T_i = (\exp \operatorname{ad} f_i)(\exp \operatorname{ad} e_i)(\exp \operatorname{ad} f_i)\,.$$

We shall identify $\operatorname{Spec} Z_0^0$ with T through the isomorphism $R \to lR$ given by multiplication by l. Now consider the map

$$\pi\colon \operatorname{Spec} Z_0 = \operatorname{Spec} Z_0^- \times T \times \operatorname{Spec} Z_0^+ \to G\,,$$
$$\pi(a, t, b) = \pi^-(a)\,t^2\,\pi^+(b)\,;$$

the image of π is the big cell $(U_- T U_+)$ of the group G.

THEOREM 3.4 [3]. *There exists a canonical infinite dimensional group* $\widetilde{G}$ *of automorphisms of* $\mathcal{U}_\varepsilon$ *such that*:

a) $\widetilde{G}$ *stabilizes* Z_0 *and therefore acts on* $\operatorname{Spec} Z_0$:

$$(\tilde{g}\lambda)(z) = \lambda(\tilde{g}^{-1}z)\,, \qquad \lambda \in \operatorname{Spec} Z_0\,, \qquad z \in Z_0\,, \qquad \tilde{g} \in \widetilde{G}\,;$$

b) X *is an equivariant map with respect to the* $\widetilde{G}$*-action*;

c) *the set* F *of fixed points of* $\widetilde{G}$ *in* $\operatorname{Spec} Z_0$ *is* $(\pi)^{-1}(1)$;

d) if $\mathcal{O}$ is the conjugacy class of a non central element of G then $\pi^{-1}(\mathcal{O})$ is a single $\tilde{G}$-orbit and $(\mathrm{Spec}\, Z_0) - F$ *is a union of these $\tilde{G}$-orbits.*

The above theorem allows us to parametrize the equivalence classes of the irreducible $\mathcal{U}_q(g)$-modules by the conjugacy classes of the group G. The following conjecture states the existence of a linking between the geometry of these conjugacy classes and the structure of the corresponding representations in a more precise sense:

CONJECTURE [3]. *If $\sigma \in \mathrm{Spec}\, \mathcal{U}_\varepsilon$ is an irreducible representation of $\mathcal{U}_\varepsilon$ on a vector space V such that $\varphi(\sigma)$ belongs to a conjugacy class $\mathcal{O}$ in G then* $\dim V$ *is divisible by* $l^{\dim \mathcal{O}/2}$.

We recall that each conjugacy class in G has got even dimension less than or equal to $2N$. The above conjecture was proved in [4] in the maximal case:

THEOREM 3.5. *Any representation $\sigma \in \mathrm{Spec}\, \mathcal{U}_\varepsilon$ such that $\varphi(\sigma)$ lies in a regular conjugacy class of G has maximal dimension* $(= l^N)$.

From now on we consider the quantized enveloping algebra $\mathcal{U}_q(g)$ associated to $g = sl(n)$. Then $\mathcal{W} = S_n$ and $G = SL(n)$. We will denote the Borel subgroups of G of upper and lower triangular matrices by B_+ and B_- respectively, while U_+ and U_- will be the corresponding unipotent subgroups and T the maximal torus of diagonal matrices.

DEFINITION 3.6. *We say that $\sigma \in \mathcal{U}_\varepsilon$ is unipotent if $\varphi(\sigma)$ is a unipotent element in* $SL(n)$.

Take a non unipotent element σ in $\mathrm{Spec}\, \mathcal{U}_\varepsilon$ and write $m = \varphi(\sigma) = m_s m_u$ where m_s and m_u are the semisimple and unipotent part of m respectively ($m_s \neq 1$). Define $T' =$ $=$ center (centralizer$_G(m_s)$) and put $h' := \mathrm{Lie}(T')$. Then h' will be a proper subalgebra of the Cartan subalgebra h of g. Let $Q' := \{\alpha \in Q \,|\, \alpha \text{ vanishes on } h'\}$, then $Q' = \mathbf{Z}\Delta' \cap Q$ where $\mathbf{Z}\Delta'$ is a sublattice of R spanned by a proper subset Δ' of Δ. We shall denote by g' the Lie algebra whose Chevalley generators are those of g corresponding to $\alpha_i \in \Delta'$ and by $\mathcal{U}'$ the subalgebra of $\mathcal{U}_\varepsilon$ generated by E_i, F_i with $\alpha_i \in \Delta'$ and K_j with $j = 1, \ldots, n$. Put $\tilde{\mathcal{U}} = \mathcal{U}'\mathcal{U}^+$ where $\mathcal{U}^+$ is the subalgebra of $\mathcal{U}_\varepsilon$ generated by E_i, K_i for $i = 1, \ldots, n$. Then the following theorem holds:

THEOREM 3.7 [2]. *If $\sigma \in \mathrm{Spec}\, \mathcal{U}_\varepsilon$ is a non unipotent representation of $sl_\varepsilon(n)$ on a vector space V there exists a unique irreducible $\mathcal{U}_\varepsilon(g')$-module V' such that:*

1) *V' is an irreducible $\tilde{\mathcal{U}}$-module;*
2) $V = sl_\varepsilon(n) \bigotimes_{\tilde{\mathcal{U}}} V'$; *in particular* $\dim V = l^t \dim V'$ *where* $2t = |Q/Q'|$.

The above theorem reduces the study of the irreducible representations of $sl_\varepsilon(n)$ to the study of its unipotent representations, since it states, in particular, that any $sl_\varepsilon(n)$-module which is not unipotent is induced by a $sl_\varepsilon(r)$-unipotent module, with $r < n$.

We recall that the number of conjugacy classes of the unipotent elements in $SL(n)$ is finite and that each class is parametrized by the Jordan decomposition of its elements, *i.e.* by a partition of n. Moreover the following theorem holds:

THEOREM 3.8 [7]. *Let $\mathcal{O}$ be a conjugacy class in $SL(n)$ parametrized by the partition (h_i) of n. Then* $\dim \mathcal{O} = n^2 - \sum \check{h}_i$, *where* $(\check{h}_i)$ *is the dual partition.*

4. $U_\varepsilon(sl(3))$: THE SUBREGULAR CASE

In this section we will consider the case $g = sl(3)$ and study the subregular representations of the quantum group $U_\varepsilon(sl(3))$ *i.e.* the irreducible representations which lie over the conjugacy class $\mathcal{O}$, parametrized by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

through the correspondence (3.7). According to what stated in 3 this completes the proof of the recalled conjecture in the case of $sl_\varepsilon(3)$. Indeed there are 3 conjugacy classes of unipotent elements in $SL(3)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the first case ($\dim \mathcal{O} = 0$) the conjecture is empty and in the last case (the maximal case) it is proved by Theorem 3.5.

Let us fix a reduced expression for w_0, say $w_0 = s_2 s_1 s_2$. Then the following relations can be proved by induction on r:

$$E_1 F_{12}^r = F_{12}^r E_1 - \left(\sum_{k=0}^{r-1} \varepsilon^{2k}\right) F_{12}^{r-1} F_2 K_1^{-1}\,; \tag{4.8}$$

$$E_2 F_{12}^r = F_{12}^r E_2 + \varepsilon\left(\sum_{k=0}^{r-1} \varepsilon^{-2k}\right) F_{12}^{r-1} F_1 K_2\,. \tag{4.9}$$

We recall that, with our choice of the reduced expression of w_0,

$$F_1 F_{12} = \varepsilon^{-1} F_{12} F_1\,, \qquad F_2 F_{12} = \varepsilon F_{12} F_2\,.$$

Let us choose the representative element

$$m = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

of the class $\mathcal{O}$, then, using the definition of φ, one sees that any representation in $\varphi^{-1}(m)$ is such that $E_1^l = E_{12}^l = E_2^l = 0$, $K_1^l = K_2^l = 1$, $F_2^l = 0$, $F_1^l = 1 = F_{12}^l$, where the elements of $sl_\varepsilon(3)$ are identified with their images through the representation.

According to [1], we consider the irreducible $(j+1)$-dimensional representation V $(0 \leqslant j \leqslant l-1)$ of $sl_\varepsilon(2)$ with a basis consisting of the vectors $v, F_2 v, \ldots, F_2^j v$, where v

is a non zero vector such that $E_2 v = 0, K_2 v = \varepsilon^j v, F_2^{j+1} v = 0$. Let $\tilde{\mathcal{U}}$ be the subalgebra of $\mathcal{U}_\varepsilon$ with generators $E_2, F_2, K_2, E_1, K_1, F_1^l, F_{12}^l$ and define an action of $\tilde{\mathcal{U}}$ on V by the relations:

$$F_1^l = 1, \quad F_{12}^l = 1, \quad E_1 V \equiv 0, \quad K_1 F_2^r v = \varepsilon^{i+r} F_2^r v \quad \forall r = 0, \dots, j$$

where i is a fixed integer such that $0 \leq i \leq l-1$. V is then a left $\tilde{\mathcal{U}}$-module, and we can consider the induced representation $\mathrm{Ind}(V) := sl_\varepsilon(3) \bigotimes_{\tilde{\mathcal{U}}} V$.

DEFINITION 4.1. *We say that the above defined representation* $\mathrm{Ind}(V)$ *is a representation of type* (i,j) *of* $sl_\varepsilon(3)$.

REMARK. A representation of type (i,j) has dimension $(j+1)l^2$. Indeed, by definition, a basis of $\mathrm{Ind}(V)$ consists of the vectors

$$\{F_1^r F_{12}^t F_2^s v : 0 \leq r, t \leq l-1, 0 \leq s \leq j\}. \tag{4.10}$$

LEMMA 4.2. *Given* $x \in \mathrm{Ind}(V)$, $x = \sum_{k=1}^n a_k F_1^{r_k} F_{12}^{t_k} F_2^{s_k} v$, *the following relations hold*:

$$E_1(x) = -\sum_{k=1}^n a_k \varepsilon^{-s_k - i} \frac{1-\varepsilon^{2t_k}}{1-\varepsilon^2} F_1^{r_k} F_{12}^{t_k - 1} F_2^{s_k+1} v + \tag{4.11}$$
$$+ \sum_{k=1}^n a_k \frac{(1-\varepsilon^{2r_k})(\varepsilon^{2-2r_k-t_k+s_k+i} - \varepsilon^{t_k - s_k - i})}{(\varepsilon - \varepsilon^{-1})(1-\varepsilon^2)} F_1^{r_k - 1} F_{12}^{t_k} F_2^{s_k} v;$$

$$E_2(x) = \sum_{k=1}^n a_k \frac{(\varepsilon^{j+2} - \varepsilon^{-j+2s_k})(1-\varepsilon^{-2s_k})}{(\varepsilon^2 - 1)(\varepsilon - \varepsilon^{-1})} F_1^{r_k} F_{12}^{t_k} F_2^{s_k - 1} v + \tag{4.12}$$
$$+ \sum_{k=1}^n a_k \frac{1-\varepsilon^{2t_k}}{1-\varepsilon^2} \varepsilon^{2-t_k-2s_k+j} F_1^{r_k+1} F_{12}^{t_k - 1} F_2^{s_k} v.$$

PROOF. By using relation (4.8) we have:

$$E_1(x) = E_1\left(\sum_{k=1}^n a_k F_1^{r_k} F_{12}^{t_k} F_2^{s_k} v\right) = \sum_{k=1}^n a_k E_1 F_1^{r_k} F_{12}^{t_k} F_2^{s_k} v =$$

$$= \sum_{k=1}^n a_k \left(F_1^{r_k} E_1 + F_1^{r_k - 1} \frac{\left(\sum_{s=0}^{r_k - 1} \varepsilon^{-2s}\right) K_1 - \left(\sum_{s=0}^{r_k - 1} \varepsilon^{2s}\right) K_1^{-1}}{\varepsilon - \varepsilon^{-1}} \right) F_{12}^{t_k} F_2^{s_k} v =$$

$$= \sum_{k=1}^n a_k F_1^{r_k} E_1 F_{12}^{t_k} F_2^{s_k} v +$$

$$+ \sum_{k=1}^n a_k F_1^{r_k - 1} \frac{(1-\varepsilon^{-2r_k})/(1-\varepsilon^{-2})\,\varepsilon^{-t_k+s_k+i} - (1-\varepsilon^{2r_k})/(1-\varepsilon^2)\,\varepsilon^{t_k - s_k - i}}{\varepsilon - \varepsilon^{-1}} \cdot$$

$$\cdot F_{12}^{t_k} F_2^{s_k} v = \sum_{k=1}^n a_k F_1^{r_k} \left(-\sum_{m=0}^{t_k - 1} \varepsilon^{2m} \right) F_{12}^{t_k - 1} F_2 K_1^{-1} F_2^{s_k} v +$$

$$+ \sum_{k=1}^n a_k \frac{(1-\varepsilon^{2r_k})/(1-\varepsilon^2)\,\varepsilon^{2-2r_k-t_k+s_k+i} - (1-\varepsilon^{2r_k})/(1-\varepsilon^2)\,\varepsilon^{t_k-s_k-i}}{\varepsilon - \varepsilon^{-1}} \cdot$$

$$\cdot F_1^{r_k - 1} F_{12}^{t_k} F_2^{s_k} v = - \sum_{k=1}^{n} a_k \, \varepsilon^{-s_k - i} \frac{1 - \varepsilon^{2t_k}}{1 - \varepsilon^2} F_1^{r_k} F_{12}^{t_k - 1} F_2^{s_k + 1} v +$$

$$+ \sum_{k=1}^{n} a_k \frac{(1 - \varepsilon^{2r_k})(\varepsilon^{2 - 2r_k - t_k + s_k + i} - \varepsilon^{t_k - s_k - i})}{(\varepsilon - \varepsilon^{-1})(1 - \varepsilon^2)} F_1^{r_k - 1} F_{12}^{t_k} F_2^{s_k} v .$$

We compute $E_2(x)$ in a similar way. ■

Given a $sl_\varepsilon(3)$-module V, we shall say that $x \in V$ is a weight vector if it is a common eigenvector for the K_i's for $i = 1, 2$.

LEMMA 4.3. *Each weight vector x in* $\mathrm{Ind}(V)$ *such that* $E_2(x) = 0$ *has the form*

$$x = \sum_{k=1}^{t+1} a_k F_1^{r+k-1} F_{12}^{t-k+1} F_2^{k-1} v \tag{4.13}$$

with $t, r \in \mathbf{N}$, $0 \leqslant t \leqslant j$, $0 \leqslant r \leqslant l - 1$ *and* $a_k \in \mathbf{C} - \{0\}$.

PROOF. Let us take $x \in \mathrm{Ind}(V)$, then we can write x as a linear combination of the vectors in the basis (4.10): $x = \sum_{k=1}^{n} a_k F_1^{r_k} F_{12}^{t_k} F_2^{s_k} v$.

If $n = 1$, relation (4.12) shows that $E_2(x) = 0$ if and only if $s_1 = t_1 = 0$. In this case $x = F_1^{r_1} v$ spans the representation $\mathrm{Ind}(V)$ since F_1 is invertible.

Suppose now $n > 1$. We rewrite (4.12) in the following way:

$$E_2(x) = A + B = \sum_{k=1}^{n} \alpha_k A_k + \sum_{k=1}^{n} \beta_k B_k$$

with $A_k = F_1^{r_k} F_{12}^{t_k} F_2^{s_k} v$, $B_k = F_1^{r_k + 1} F_{12}^{t_k - 1} F_2^{s_k} v$; the vectors A_k are then linearly independent as well as the vectors B_k, moreover $A_k \neq B_k$ for the same k. Now, if $B_{k_1} = A_{k_2}$ for some $k_1 \neq k_2$, this means that

$$\begin{cases} r_{k_2} = r_{k_1} + 1 \\ t_{k_2} = t_{k_1} - 1 \\ s_{k_2} = s_{k_1} + 1 \end{cases} \tag{4.14}$$

so that $A_{k_1} \neq B_{k_2}$. In the same way, by induction, we get that if $B_{k_1} = A_{k_2}$, $B_{k_2} = A_{k_3}, \dots, B_{k_{n-1}} = A_{k_n}$, then $k_1, \dots, k_n$ must be different from each other and A_{k_1} is different from $B_{k_1}, B_{k_2}, \dots, B_{k_n}$. Therefore, $E_2(x) = 0$ if and only if there exists an ordering $k_1, \dots, k_n$ of the indeces such that

$$\begin{cases} B_{k_1} = A_{k_2} \\ B_{k_2} = A_{k_3} \\ \vdots \\ B_{k_{n-1}} = A_{k_n} \end{cases} \tag{4.15}$$

$$(4.16)\qquad \begin{cases} \alpha_{k_2} + \beta_{k_1} = 0 \\ \alpha_{k_3} + \beta_{k_2} = 0 \\ \vdots \\ \alpha_{k_n} + \beta_{k_{n-1}} = 0 \end{cases}$$

and $\alpha_{k_1} = 0$, $\beta_{k_n} = 0$ *i.e.* $s_{k_1} = 0$, $t_{k_n} = 0$. Notice that system (4.15) is equivalent to the following:

$$(4.17)\qquad \begin{cases} r_{k_h} = r_{k_1} + h - 1 \\ t_{k_h} = t_{k_1} - h + 1 \\ s_{k_h} = h - 1 \end{cases}$$

with $2 \leqslant h \leqslant n$. Particularly $t_{k_1} = t_{k_n} + n - 1 = n - 1 = s_{k_n}$, so that: $1 \leqslant t_{k_1} = n - 1 \leqslant \leqslant j$. Now we can write the relation $\alpha_{k_h} + \beta_{k_{h-1}} = 0$ explicitly:

$$a_{k_h} \frac{(\varepsilon^{j+2} - \varepsilon^{-j+2s_{k_h}})(1 - \varepsilon^{-2s_{k_h}})}{(\varepsilon^2 - 1)(\varepsilon - \varepsilon^{-1})} + a_{k_{h-1}} \frac{1 - \varepsilon^{2t_{k_{h-1}}}}{1 - \varepsilon^2} \varepsilon^{2 - t_{k_{h-1}} - 2s_{k_{h-1}} + j} = 0 .$$

We point out that, as in our hypothesis the coefficients of the previous equation are different from zero when $2 \leqslant h \leqslant n$, system (4.16) has got a solution $(a_{k_1}, \ldots, a_{k_n})$ with $a_{k_j} \neq 0$ for each $j = 1, \ldots, n$, uniquely determined up to a scalar factor. Finally, if $a_{k_j} = 0$ for one j then $x \equiv 0$. ■

REMARK. If $t = 0$ in (4.13) $E_1(x) = 0$ if and only if $r = 0$ or $r = i + 1$. These are the only cases in which a vector $F_1^r F_{12}^t F_2^s v$ is annihilated by both E_1 and E_2. Notice that, since $F_1^l = 1$, the set $\{F_1^r F_{12}^t F_2^s (F_1^{i+1} v) \colon 0 \leqslant r,\ t \leqslant l - 1,\ 0 \leqslant s \leqslant j\}$ is a basis of $\mathrm{Ind}(V)$.

From now on we will suppose $t > 0$ in (4.13).

PROPOSITION 4.4. *Let x be of type* (4.13), $x \neq 0$, *such that* $E_1(x) = E_2(x) = 0$. *Then*

$$(4.18)\qquad 2 + i + j - t \equiv 0 (\mathrm{mod}\, l) .$$

PROOF. Take $x = \sum_{k=1}^{t+1} a_k F_1^{r+k-1} F_{12}^{t-k+1} F_2^{k-1} v$ as in Lemma 4.3. Then

$$E_1(x) = - \sum_{k=1}^{t+1} a_k \varepsilon^{-k+1-i} \frac{1 - \varepsilon^{2(t-k+1)}}{1 - \varepsilon^2} F_1^{r+k-1} F_{12}^{t-k} F_2^k v + $$
$$+ \sum_{k=1}^{t+1} a_k \frac{(1 - \varepsilon^{2(r+k-1)})(\varepsilon^{2-2r-t+i} - \varepsilon^{t-2k+2-i})}{(\varepsilon - \varepsilon^{-1})(1 - \varepsilon^2)} F_1^{r+k-2} F_{12}^{t-k+1} F_2^{k-1} v .$$

Since the first summand does not contain the vector $F_1^{r-1} F_{12}^t v$, if $E_1(x) = 0$, we must have:

$$(A)\qquad r = 0$$

or

$$(B)\qquad 1 - r - t + i \equiv 0 (\mathrm{mod}\, l) .$$

Now, as

$$E_2(x) = \sum_{k=1}^{t+1} a_k \frac{(\varepsilon^{j+2} - \varepsilon^{-j+2k-2})(1-\varepsilon^{-2k+2})}{(\varepsilon^2-1)(\varepsilon-\varepsilon^{-1})} F_1^{r+k-1} F_{12}^{t-k+1} F_2^{k-2} v +$$

$$+ \sum_{k=1}^{t+1} a_k \frac{1-\varepsilon^{2(t-k+1)}}{1-\varepsilon^2} \varepsilon^{3-t-k+j} F_1^{r+k} F_{12}^{t-k} F_2^{k-1} v ,$$

$E_1(x) = E_2(x) = 0$ if and only if the following system has got a non trivial solution for each $k = 2, \ldots, t+1$:

$$\begin{cases} a_k \dfrac{(\varepsilon^{j+2} - \varepsilon^{-j+2k-2})(1-\varepsilon^{-2k+2})}{(\varepsilon^2-1)(\varepsilon-\varepsilon^{-1})} + a_{k-1}\varepsilon^{4-t-k+j} \dfrac{1-\varepsilon^{2(t-k+2)}}{1-\varepsilon^2} = 0 , \\ a_k \dfrac{(1-\varepsilon^{2(r+k-1)})(\varepsilon^{2-2r-t+i} - \varepsilon^{t-2k+2-i})}{(\varepsilon-\varepsilon^{-1})(1-\varepsilon^2)} - a_{k-1}\varepsilon^{-k+2-i} \dfrac{1-\varepsilon^{2(t-k+2)}}{1-\varepsilon^2} = 0 . \end{cases}$$

Particularly, for $k = 2$ this is equivalent to require that

$$(\varepsilon^{j+2} - \varepsilon^{-j+2})(1-\varepsilon^{-2}) - \varepsilon^{2-t+j+i}(1-\varepsilon^{2r+2})(\varepsilon^{-2r+i-t+2} - \varepsilon^{-i+t-2}) = 0 .$$

We distinguish the following two different cases:

(A): $r = 0 \Rightarrow$

$$0 = (\varepsilon^j - \varepsilon^{-j})(\varepsilon^2-1) - \varepsilon^{2-t+i+j}(1-\varepsilon^2)(\varepsilon^{i+2-t} - \varepsilon^{-i+t-2)} =$$

$$= (\varepsilon^2-1)(\varepsilon^j - \varepsilon^{-j} + \varepsilon^{4-2t+2i+j} - \varepsilon^j) \Leftrightarrow$$

$$\Leftrightarrow \varepsilon^{4-2t+2i+j} = \varepsilon^{-j} \Leftrightarrow 2 - t + i + j \equiv 0 \pmod{l} ,$$

(B): $1 - r + i - t \equiv 0 \pmod{l} \Rightarrow$

$$0 = (\varepsilon^j - \varepsilon^{-j})(\varepsilon^2-1) - \varepsilon^{2-t+i+j}(1-\varepsilon^{2r+2})(\varepsilon^{1-r} - \varepsilon^{-r-1}) =$$

$$= (\varepsilon^j - \varepsilon^{-j})(\varepsilon^2-1) - \varepsilon^{1+j}(1-\varepsilon^{2r+2})(\varepsilon - \varepsilon^{-1}) =$$

$$= (\varepsilon^2-1)(\varepsilon^j - \varepsilon^{-j} - \varepsilon^j + \varepsilon^{2r+2+j}) \Leftrightarrow \varepsilon^{2r+2+j} = \varepsilon^{-j} \Leftrightarrow r + 1 + j \equiv 0 .$$

The above relation, together with (B), is equivalent to (4.18). ■

DEFINITION 4.5. *We say that a $sl_\varepsilon(3)$-module is nice if it is of type (i,j) with $2 + i + j \leq l$ or $i = l - 1$.*

PROPOSITION 4.6. *A nice representation is irreducible.*

PROOF. Let us consider a representation of type (i,j) generated by a vector $v \neq 0$. Proposition 4.4 shows that if

$$2 + i + j \not\equiv t \pmod{l}$$

for any t such that $1 \leq t \leq j$, the representation $\mathrm{Ind}(V)$ contains no weight vector $x \neq \alpha v, \beta F_1^{i+1} v$, with $\alpha, \beta \in C$, such that $E_1(x) = 0 = E_2(x)$. Now, since $E_1^l = E_{12}^l = E_2^l = 0$, the algebra generated by E_1, E_2 is nilpotent, therefore if $W \subset \mathrm{Ind}(V)$ is a subrepresentation of $\mathrm{Ind}(V)$, there exists a weight vector $w \in W$ such that $E_1(w) = 0 = E_2(w)$. This forces w to be a multiple scalar of v or of $F_1^{i+1} v$ and therefore $W = \mathrm{Ind}(V)$.

Finally it is easy to verify that $2+i+j \not\equiv t$ for any t such that $1 \leqslant t \leqslant j$ if and only if $2+i+j \leqslant l$ or $i=l-1$. ■

PROPOSITION 4.7. *If V is a $sl_\varepsilon(3)$-module of type (i,j) and is not nice there exists a nice submodule W of V such that the quotient V/W is a nice representation.*

PROOF. Let V be a representation of $sl_\varepsilon(3)$ of type (i,j) with $2+i+j \geqslant l+1$, $i \neq l-1$. Take $\bar{x} = F_2^{2+i+j-l} F_1^{i+1} v$, then $\bar{x}$ is a weight vector killed by both E_1 and E_2 which spans a proper subrepresentation Φ of V, with basis $\{F_1^r F_{12}^t F_2^s \bar{x}\colon 0 \leqslant r, t \leqslant l-1, 0 \leqslant s \leqslant l-i-2\}$. Φ is irreducible, indeed it is the representation of type $([l-j-2], l-i-2)$, generated by $F_1^{i+1}\bar{x}$, which can be easily seen to be nice. (By $[l-j-2]$ we mean the integer $k \in [0, l-1]$ such that $k \equiv l-j-2 \pmod{l}$). Finally the quotient V/Φ is the representation of type $(l-i-2, i+j+1-l)$ generated by $F_1^{i+1} v$ and this is nice too. ■

THEOREM 4.8. *Every subregular representation of $U_\varepsilon(sl(3))$ is a nice representation.*

PROOF. Let us take a subregular representation W of $\mathcal{U}_\varepsilon(sl(3))$. As the algebra generated by E_1 and E_2 is nilpotent, the set

$$B := \{w \in W\colon E_1(w) = 0 = E_2(w)\}$$

is nontrivial; moreover K_1 and K_2 act diagonally on B. Take then $u \in B \backslash \{0\}$ such that $K_1 u = \varepsilon^x u$, $K_2 u = \varepsilon^y u$: u spans W since W is irreducible. Consider the subspace V of W generated by the set $\{F_2^r u\colon 0 \leqslant r \leqslant l-1\}$; V is stable under the action of F_2, E_2, K_2, K_1. In particular V defines a representation of the subalgebra $\tilde{\mathcal{U}}$ of $\mathcal{U}_\varepsilon(sl(3))$ generated by $E_2, F_2, K_2, K_1^2 K_2$. Let V' be an irreducible $\tilde{\mathcal{U}}$-submodule of V. V' is then an irreducible representation of $sl_\varepsilon(2)$, since $K_1^2 K_2$ is central in $\tilde{\mathcal{U}}$. We then see that V' is stable under K_1 as $K_1 = \lambda K_2^{(l-1)/2}$ with $\lambda \in \mathbf{C}$.

Define $\operatorname{Ind}(V')$ as the representation induced by V' on $sl_\varepsilon(3)$ in the natural way. Then W is a quotient of $\operatorname{Ind}(V')$ since the set $\{F_1^r F_{12}^t \tilde{v}\colon \tilde{v} \in V', 0 \leqslant r, t \leqslant l-1\}$ is stable under the action of E_α, F_α, K_j for any $\alpha \in Q^+, j = 1, 2$. Now, if $\operatorname{Ind}(V')$ is nice, $W = \operatorname{Ind}(V')$. Otherwise, by Proposition 4.7, $\operatorname{Ind}(V')$ contains a proper nice subrepresentation Φ such that $\operatorname{Ind}(V')/\Phi$ is nice. Write $W = \operatorname{Ind}(V')/T$ where T is a subrepresentation of $\operatorname{Ind}(V')$. Then, if $T \cap \Phi \neq \{0\}$, $T \supset \Phi$ so that $W = \operatorname{Ind}(V')/T \subset \operatorname{Ind}(V')/\Phi$, but since $\operatorname{Ind}(V')/\Phi$ is irreducible, $W = \operatorname{Ind}(V')/\Phi$.

On the contrary, if $T \cap \Phi = \{0\}$ then $W \supset \Phi/T \simeq \Phi$ so that $W = \Phi$. ■

COROLLARY 4.9. *The dimension of any subregular representation of $U_\varepsilon(sl(3))$ is divisible by l^2.*

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