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Giuseppe Da Prato

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Some results on elliptic and parabolic equations in Hilbert spaces

Memoria (*) di Giuseppe Da Prato

Abstract. — We consider elliptic and parabolic equations with infinitely many variables. We prove some results of existence, uniqueness and regularity of solutions.

Key words: Elliptic and parabolic equations in Hilbert spaces; Ornstein-Uhlenbeck semigroup; Schauder estimates.

Riassunto. — *Equazioni ellittiche e paraboliche negli spazi di Hilbert.* In questo lavoro si considerano equazioni ellittiche e paraboliche con un numero finito di variabili. Si provano risultati di esistenza, unicità e regolarità delle soluzioni.

1. Introduction

Let H be a separable Hilbert space (norm $|\cdot|$, inner product $\langle \cdot, \cdot \rangle$). We denote by $\mathcal{L}(H)$ the Banach algebra (norm $\|\cdot\|$) of all linear bounded operators from H into H, by $\mathcal{L}_1(H)$ (norm $\|\cdot\|_{\mathcal{L}_1(H)}$) the set of all trace-class operators and by $\mathcal{L}_2(H)$ (norm $\|\cdot\|_{\mathcal{L}_2(H)}$) the set of all Hilbert-Schmidt operators in H.

We are given a linear closed operator $A: D(A) \subset H \mapsto H$ and a symmetric bounded operator $Q \in \mathcal{L}(H)$. We assume

Hypothesis 1.1. (*i*) *A is the infinitesimal generator of an analytic semigroup e^{tA} in H, such that*

$$\|e^{tA}\| \leq 1, \qquad t \geq 0. \tag{1.1}$$

(*ii*) *There exists $\nu > 0$ such that*

$$(1/\nu) I \leq Q \leq \nu I. \tag{1.2}$$

(*iii*) *For any $t > 0$, $e^{tA} \in \mathcal{L}_2(H)$ and*

$$\int_0^t \mathrm{Tr}\,[e^{sA} Q e^{sA^*}]\, ds < +\infty. \tag{1.3}$$

If Hypothesis 1.1 holds then for arbitrary $t \geq 0$, the linear operator Q_t defined as

$$Q_t x = \int_0^t e^{sA} Q e^{sA^*} x\, dt, \qquad x \in H, \tag{1.4}$$

is well defined and trace-class.

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The following result is proved in [8].

PROPOSITION 1.1. *Under Hypothesis* 1.1 *one has*

$$e^{sA}(H) \subset Q_t^{1/2}(H), \qquad 0 < s \leqslant t. \tag{1.5}$$

Moreover setting

$$\Lambda_t = Q_t^{-1/2} e^{tA}, \qquad t > 0, \tag{1.6}$$

one has

$$\|\Lambda_t\| \leqslant \nu/\sqrt{t}, \qquad t > 0. \tag{1.7}$$

REMARK 1.2. Since

$$\Lambda_t = Q_t^{-1/2} e^{(t/2)A} e^{(t/2)A}, \qquad t > 0,$$

we have that $\Lambda_t \in \mathcal{L}_2(H)$ so that

$$\gamma(t) := \mathrm{Tr}[\Lambda_t \Lambda_t^*] < +\infty, \qquad \forall t > 0. \tag{1.8}$$

The main object of this paper is the *Ornstein-Uhlenbeck transition semigroup* P_t, $t \geqslant 0$ defined on $C_b(H)$, the Banach space of all uniformly continuous and bounded mappings from H into $\boldsymbol{R}$, endowed with the norm $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|$. We set for $t > 0$ (1)

$$P_t\varphi(x) = \int_H \varphi(x)\, \mathcal{N}(e^{tA}x, Q_t)(dy) = \int_H \varphi(e^{tA}x + y)\, \mathcal{N}(0, Q_t)(dy), \qquad \varphi \in C_b(H). \tag{1.9}$$

It is useful to note that, setting

$$G_t\varphi(x) = \int_H \varphi(x+y)\, \mathcal{N}(0, Q_t)(dy), \qquad \varphi \in C_b(H), \tag{1.10}$$

we have

$$P_t\varphi(x) = (G_t\varphi)(e^{tA}x), \quad \varphi \in C_b(H), \quad t \geqslant 0, \quad x \in H. \tag{1.11}$$

P_t, $t \geqslant 0$ is not a strongly continuous semigroup on $C_b(H)$, however it is *weakly continuous*, see [4]. In particular we have

$$\lim_{t \to 0} P_t\varphi(x) = \varphi(x), \qquad \forall \varphi \in C_b(H), \qquad \forall x \in H, \tag{1.12}$$

the convergence being uniform on the compact subsets of H.

In this paper we first study some regularity properties of the semigroup P_t, $t \geqslant 0$. Then we introduce its infinitesimal generator $\mathcal{M}$ and characterize the corresponding interpolation spaces. Finally we apply the obtained results to the study of the elliptic equation

$$\lambda\varphi - (1/2)\mathrm{Tr}[D^2\varphi] - \langle Ax, D\varphi\rangle = g, \qquad x \in H, \tag{1.13}$$

(1) For any $m \in H$ and any $S \in \mathcal{L}_1(H)$ symmetric nonnegative, we denote by $\mathcal{N}(m, S)$ the Gaussian measure with mean m and covariance operator S.

where $\lambda > 0$ and $g\colon H \mapsto \mathbf{R}$ is a suitable function, and to the initial value problem

$$(1.14)\qquad \begin{cases} du(t,x)/dt = (1/2)\operatorname{Tr}[D^2u(t,x)] + \langle Ax, Du(t,x)\rangle + F(t,x)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in]0,T]\,,\quad x \in H\,, \\ u(0,x) = \varphi(x)\,, \end{cases}$$

where $F\colon [0,T] \times H \mapsto \mathbf{R}$ and $\varphi\colon H \mapsto \mathbf{R}$ are given functions fulfilling suitable assumptions. We also study problems (1.13) and (1.14) in spaces $C_b^\theta(H)$ of Hölder continuos functions. In this case we will prove, following [3], Schauder estimates and we will characterize, under suitable hypotheses the domain of the infinitesimal generator $\mathcal{M}$ of P_t, $t \geq 0$.

Let us introduce our main notation. The following subspaces of $C_b(H)$ will be needed.

- $C_b^1(H)$ is the Banach space of all functions $\varphi \in C_b(H)$ which are Fréchet differentiable on H, with a bounded uniformly continuous derivative $D\varphi$, with the norm

$$\|\varphi\|_1 = \|\varphi\|_0 + [\varphi]_1\,,$$

where

$$[\varphi]_1 = \sup_{x \in H} |D\varphi(x)|\,.$$

If $\varphi \in C_b^1(H)$ and $x \in H$ we shall identify $D\varphi(x)$ with the element h of H such that

$$D\varphi(x)y = \langle h, y\rangle\,, \qquad \forall y \in H\,.$$

- $C_b^2(H)$ is the Banach space of all functions $\varphi \in C_b^1(H)$ which are twice Fréchet differentiable on H, with a bounded uniformly continuous second derivative $D^2\varphi$ with the norm

$$\|\varphi\|_2 = \|\varphi\|_1 + [\varphi]_2\,,$$

where

$$[\varphi]_2 = \sup_{x \in H} |D^2\varphi(x)|\,.$$

If $\varphi \in C_b^2(H)$ and $x \in H$ we shall identify $D^2\varphi(x)$ with the linear bounded operator $T \in \mathcal{L}(H)$ such that

$$D\varphi(x)(y,z) = \langle Ty, z\rangle\,, \qquad \forall y, z \in H\,.$$

- $C_b^n(H)$, $n \in \mathbf{N}$ is the Banach space of all functions $\varphi \in C_b(H)$ which are n times Fréchet differentiable on H, with bounded uniformly continuous derivatives of any order less or equal to n, with the norm

$$\|\varphi\|_n = \|\varphi\|_0 + \sum_{k=1}^{n} [\varphi]_k\,,$$

where

$$[\varphi]_k = \sup_{x \in H} |D^k\varphi(x)|\,, \qquad k = 1, \ldots, n\,.$$

We set

$$C_b^\infty(H) = \bigcap_{n=1}^{\infty} C_b^n(H).$$

- $C_b^\alpha(H)$, $\alpha \in]0, 1[$, is the Banach space of all α-Hölder continuous and bounded functions $\varphi \in C_b(H)$ with the norm

$$\|\varphi\|_\alpha = \|\varphi\|_0 + [\varphi]_\alpha,$$

where

$$[\varphi]_\alpha = \sup_{x, y \in H, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha} < +\infty.$$

- $C_b^{1+\alpha}(H)$, $\alpha \in]0, 1[$, is the set of all functions $\varphi \in C_b^1(H)$ such that

$$[D\varphi]_\alpha = \sup_{x, y \in H, x \neq y} \frac{|D\varphi(x) - D\varphi(y)|}{|x-y|^\alpha} < +\infty.$$

$C_b^{1+\alpha}(H)$ is a Banach space with the norm

$$\|\varphi\|_{1+\alpha} = \|\varphi\|_1 + [D\varphi]_\alpha.$$

- $C_b^{2+\alpha}(H)$, $\alpha \in]0, 1[$, is the set of all functions $\varphi \in C_b^2(H)$ such that

$$[D^2\varphi]_\alpha = \sup_{x, y \in H, x \neq y} \frac{\|D^2\varphi(x) - D^2\varphi(y)\|}{|x-y|^\alpha} < +\infty.$$

$C_b^{2+\alpha}(H)$ is a Banach space with the norm

$$\|\varphi\|_{2+\alpha} = \|\varphi\|_2 + [D^2\varphi]_\alpha.$$

We will also need some notations and results on Interpolation Theory.

Let first recall the definition of interpolation space, see [20]. Let X, $\|\cdot\|_X$ and Y, $\|\cdot\|_Y$ be Banach spaces such that $Y \subset X$ and

$$\|y\|_X \leqslant c\|y\|_Y, \qquad \forall y \in Y$$

for some constant $c > 0$.

Let $\alpha \in]0, 1]$. We denote by $(X, Y)_{\alpha, \infty}$ the real interpolation space consisting of all points $x \in X$ such that

$$\|x\|_{(X,Y)_{\alpha,\infty}} = \sup_{t>0} t^{-\alpha} K(t, x, X, Y) < +\infty,$$

where

$$K(t, x, X, Y) = \inf\{\|a\|_X + t\|b\|_Y : x = a + b, a \in X, b \in Y\}$$

$(X, Y)_{\alpha, \infty}$ is a Banach space with norm $\|\cdot\|_{(X,Y)_{\alpha,\infty}}$.

It is easy to see that x belongs to $(X, Y)_{\beta, \infty}$ if and only if for any $t \in [0, 1]$ there exists $a_t \in X$, $b_t \in Y$ and a constant $C > 0$ independent of t, such that $\|a_t\|_X \leqslant Ct^\beta$ and $\|b_t\|_Y \leqslant Ct^{\beta-1}$.

We also recall the following interpolation result, see [2]:

Proposition 1.3. *For all* $\theta \in]0, 1[$ *we have*

$$(C_b(H), C_b^1(H))_{\theta, \infty} = C_b^\theta(H).$$

2. Regularity properties of P_t, $t \geq 0$

We first recall a result proved in [8].

THEOREM 2.1. *For all $t > 0$ and for all $\varphi \in C_b(H)$, $P_t\varphi \in C_b^\infty(H)$. In particular, for any $h, k \in H$, we have*

$$(2.1)\quad \langle DP_t\varphi(x), h\rangle = \int_H \langle \Lambda_t h, Q_t^{-1/2}y\rangle \varphi(e^{tA}x + y)\,\mathcal{N}(0, Q_t)(dy)$$

and

$$(2.2)\quad \langle D^2P_t\varphi(x)h, k\rangle =$$

$$= \int_H \langle \Lambda_t h, Q_t^{-1/2}y\rangle\langle \Lambda_t k, Q_t^{-1/2}y\rangle \varphi(e^{tA}x + y)\,\mathcal{N}(0, Q_t)(dy) - \langle \Lambda_t h, \Lambda_t k\rangle P_t\varphi(x).$$

REMARK 2.2. By (2.1) and (2.2) the following estimates can be proved easily with the help of (1.7)

$$(2.3)\quad |DP_t\varphi(x)| \leq \nu t^{-1/2}\|\varphi\|_0, \quad x \in H,$$

$$(2.4)\quad \|D^2P_t\varphi(x)\| \leq (\sqrt{2}\nu^2/t)\|\varphi\|_0, \quad x \in H.$$

We will also need an estimate for the third derivative of $P_t\varphi$, that can be proved in a similar way

$$(2.5)\quad \|D^3P_t\varphi(x)\| \leq 2\sqrt{6}\nu^3 t^{-3/2}\|\varphi\|_0, \quad x \in H.$$

By Proposition 1.3 we easily obtain the following corollaries.

COROLLARY 2.3. *For all $t > 0$, $\alpha \in]0, 1[$, we have*

$$(2.6)\quad \|DP_t\varphi\|_\alpha \leq \nu^\alpha t^{-\alpha/2}\|\varphi\|_0, \quad \varphi \in C_b(H).$$

COROLLARY 2.4. *For all $t > 0$, $\theta \in]0, 1[$, $\alpha \in]\theta, 1[$ we have*

$$(2.7)\quad \|P_t\varphi\|_\alpha \leq \nu^{\alpha-\theta} t^{(\theta-\alpha)/2}\|\varphi\|_\theta, \quad \varphi \in C_b^\theta(H).$$

2.1. Existence of $\mathrm{Tr}\,[D^2P_t\varphi(x)]$

We show here that the linear operator $D^2P_t\varphi(x)$ is trace-class for all $t > 0$ and $x \in H$.

PROPOSITION 2.5. *Let $\varphi \in C_b(H)$, $t > 0$ and $x \in H$. Then $D^2P_t\varphi(x) \in \mathcal{L}_1(H)$ and*

$$(2.8)\quad \mathrm{Tr}\,[D^2P_t\varphi(x)] =$$

$$= \int_H |\Lambda_t^* Q_t^{-1/2}y|^2 \varphi(e^{tA}x + y)\,\mathcal{N}(0, Q_t)(dy) - \mathrm{Tr}\,[\Lambda_t\Lambda_t^*]P_t\varphi(x).$$

Moreover the following estimate holds

$$(2.9)\quad \|D^2P_t\varphi(x)\|_{\mathcal{L}_1(H)} \leq 2\,\mathrm{Tr}\,[\Lambda_t\Lambda_t^*]\|\varphi\|_0.$$

PROOF. Since $\Lambda_t \in \mathcal{L}_2(H)$ (see Remark 1.2) it is enough to show that the linear operator $S_{t,x}$ defined as

$$\langle S_{t,x}h, k\rangle = \int_H \langle \Lambda_t h, Q_t^{-1/2}y\rangle\langle \Lambda_t k, Q_t^{-1/2}y\rangle \varphi(e^{tA}x+y)\,\mathcal{N}(0,Q_t)(dy), \qquad h,k\in H,$$

is trace-class for any $t>0$ and $x \in H$. For this is enough to show, compare N. Dunford and J. T. Schwartz [10, Lemma 14 (*a*), p. 1098], that there exists a constant $C>0$ such that

$$|\mathrm{Tr}\,[NS_{t,x}]| \leq C\|N\|, \tag{2.10}$$

for any symmetric positive operator $N \in \mathcal{L}(H)$ of finite rank. To this purpose let $\{e_j\}$ be a complete orthonormal system in H. Then we have

$$\mathrm{Tr}\,[NS_{t,x}] = \sum_{j=1}^{\infty}\int_H \langle \Lambda_t e_j, Q_t^{-1/2}y\rangle\langle \Lambda_t N^* e_j, Q_t^{-1/2}y\rangle\cdot \tag{2.11}$$

$$\cdot\varphi(e^{tA}x+y)\,\mathcal{N}(0,Q_t)(dy) = \int_H |N^{1/2}\Lambda_t^* Q_t^{-1/2}y|^2\varphi(e^{tA}x+y)\,\mathcal{N}(0,Q_t)(dy).$$

It follows,

$$|\mathrm{Tr}\,[NS_{t,x}]| \leq \|\varphi\|_0\,\mathrm{Tr}\,[\Lambda_t^*\Lambda_t N] \leq \|\varphi\|_0\|N\|\,\mathrm{Tr}\,[\Lambda_t^*\Lambda_t]. \tag{2.12}$$

So (2.10) is fulfilled and we have proved that $D^2P_t\varphi(x)$ is trace-class for any $t>0$ and $x \in H$. Moreover (2.8) and (2.9) follow setting $N=I$ respectively in (2.11) and in (2.12). ■

REMARK 2.6. We want to describe in next example the behaviour of $\gamma(t) = \mathrm{Tr}[\Lambda_t\Lambda_t^*]$ near $t=0$, in order to know whether it is integrable or not.

Assume that A is a negative self-adjoint operator, that $Q=I$, and that there exists a complete orthonormal system $\{e_k\}$ in H such that

$$Ae_k = -\lambda_k e_k, \qquad \lambda_k \uparrow +\infty,$$

with

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k} < +\infty.$$

Then Hypothesis 1.1 is obviously fulfilled and we have

$$Q_t = (e^{2tA}-1)/(2A).$$

It follows

$$\Lambda_t\Lambda_t^* = 2Ae^{2tA}/(e^{2tA}-1), \qquad t>0,$$

so that

$$\Lambda_t = 2\sum_{k=1}^{\infty}\lambda_k e^{-2t\lambda_k}/(1-e^{-2t\lambda_k}) = (2/t)F(\lambda_k), \tag{2.13}$$

where

$$F(\xi) = \xi e^{-2\xi}/(1-e^{-2\xi}), \qquad \xi>0. \tag{2.14}$$

Let $C_1 > 0$, $C_2 > 0$ be such that

$$C_1 e^{-3\xi} \leq F(\xi) \leq C_2 e^{-\xi}, \qquad \xi > 0.$$

So the behaviour of $\gamma(t)$ near 0 is determined by

$$\sum_{k=1}^{\infty} e^{-t\lambda_k}.$$

For instance if

$$\lambda_k = k^{1+\alpha},$$

where $\alpha > 0$, we have that $\gamma(t)$ behaves at 0 as

$$(1/t) \int_0^{+\infty} e^{-tx^{1+\alpha}} dx,$$

and so as $t^{-1-1/\alpha}$. In particular if $\lambda_k = k^2$ we have $\gamma(t) \simeq t^{-3/2}$.

2.2. Additional regularity result when $\varphi \in C_b^1(H)$.

Proposition 2.7. *Let $\varphi \in C_b^1(H)$, $t > 0$ and $x \in H$. Then we have*

$$\langle D^2 P_t \varphi(x) h, k \rangle = \int_H \langle \Lambda_t k, Q_t^{-1/2} y \rangle \langle D\varphi(e^{tA} x + y), e^{tA} h \rangle \mathcal{N}(0, Q_t)(dy). \tag{2.15}$$

Moreover $D^2 P_t \varphi(x) \in \mathcal{L}_1(H)$ and

$$\mathrm{Tr}\,[D^2 P_t \varphi(x)] = \int_H \langle \Lambda_t^* Q_t^{-1/2} y, e^{tA^*} D\varphi(e^{tA} x + y) \rangle \mathcal{N}(0, Q_t)(dy). \tag{2.16}$$

Finally the following estimates hold

$$\|D^2 P_t \varphi(x)\| \leq (\nu/\sqrt{t}) \|\varphi\|_1, \tag{2.17}$$

and

$$|\mathrm{Tr}\,[D^2 P_t \varphi(x)]| \leq \{\mathrm{Tr}\,[\Lambda_t \Lambda_t^*]\}^{1/2} \|\varphi\|_1. \tag{2.18}$$

Proof. Let $t > 0$, $x \in H$. Since

$$\langle DP_t \varphi(x), h \rangle = \int_H \langle D\varphi(e^{tA} x + y), e^{tA} h \rangle \mathcal{N}(0, Q_t)(dy),$$

(2.15) follows easily by differentiating (2.1). Moreover (2.16) is an immediate consequence of (2.15), recalling that, by Proposition 2.5, $D^2 P_t \varphi(x)$ is trace-class.

We prove now (2.17). By (2.15), using Hölder's estimate, it follows

$$|\langle D^2 P_t \varphi(x) h, k \rangle|^2 \leq \|\varphi\|_1^2 |h|^2 = \int_H |\langle \Lambda_t k, Q_t^{-1/2} y \rangle|^2 \mathcal{N}(0, Q_t)(dy) =$$

$$= \|\varphi\|_1^2 |h|^2 |\Lambda_t k|^2 \leq (1/t) \|\varphi\|_1^2 |h|^2 |k|^2,$$

and (2.17) is proved. We prove finally (2.18). We have, using again Hölder's estimate

$$|\mathrm{Tr}\,[D^2 P_t\varphi(x)]|^2 \leq \|\varphi\|_1^2 \int_H |e^{tA}\Lambda_t^* Q_t^{-1/2}y|^2 \mathcal{N}(0, Q_t)(dy) =$$

$$= \|\varphi\|_1^2 \,\mathrm{Tr}[\Lambda_t e^{tA^*} e^{tA} \Lambda_t^*] = \|\varphi\|_1^2 \,\mathrm{Tr}\,[\Lambda_t \Lambda_t^*].$$

The proof is complete. ■

In a similar way we prove the following result.

PROPOSITION 2.8. *Let $\varphi \in C_b^1(H)$, $t > 0$ and $x \in H$. Then for all $h, k, l \in H$ we have*

$$D^3 P_t\varphi(x)(h, k, l) = \int_H \langle \Lambda_t h, Q_t^{-1/2}y\rangle\langle \Lambda_t k, Q_t^{-1/2}y\rangle \cdot \tag{2.19}$$

$$\cdot\langle D\varphi(e^{tA}x + y), e^{tA}j\rangle \mathcal{N}(0, Q_t)(dy) - \langle \Lambda_t h, \Lambda_t k\rangle\langle DP_t\varphi(x), l\rangle.$$

Moreover the following estimate holds

$$\|D^3 P_t\varphi(x)\| \leq (\sqrt{2}\,\nu^2/t)\,\|\varphi\|_1. \tag{2.20}$$

By interpolation we obtain the following results.

COROLLARY 2.9. *Let $\theta \in]0, 1[$, $\varphi \in C_b^\theta(H)$, $t > 0$ and $x \in H$. Then we have*

$$\|D^2 P_t\varphi(x)\| \leq 2^{(1-\theta)/2} \nu^{2-\theta} t^{\theta/2 - 1} \|\varphi\|_\theta, \tag{2.21}$$

and

$$|\mathrm{Tr}\,[D^2 P_t\varphi(x)]| \leq 2^{1-\theta} \{\mathrm{Tr}\,[\Lambda_t \Lambda_t^*]\}^{1-\theta/2} \|\varphi\|_\theta. \tag{2.22}$$

COROLLARY 2.10. *Let $\theta \in]0, 1[$, $\varphi \in C_b^\theta(H)$, $t > 0$ and $x \in H$. Then we have*

$$\|D^2 P_t\varphi\|_\alpha \leq 2^{(1-\alpha-\theta)/2} \nu^{2+\alpha-\theta} t^{(\theta-\alpha)/2 - 1} \|\varphi\|_\theta. \tag{2.23}$$

REMARK 2.11. Assume that

$$\gamma(t) \leq Ct^{-3/2}.$$

Then by (2.22) we have

$$|\mathrm{Tr}\,[D^2 P_t\varphi(x)]| \leq 2^{1-\theta} C^{1-\theta/2} t^{-3/2 + 3\theta/4} \|\varphi\|_\theta.$$

Thus $|\mathrm{Tr}[D^2 P_t\varphi(x)]|$ is integrable near 0 provided $\theta > 2/3$.

2.3. KOLMOGOROV EQUATION

We want to show here that if $\varphi \in C_b(H)$ then for $t > 0$ the function $u(t, x) = P_t\varphi(x)$ is a solution to the Kolmogorov equation

$$u_t(t, x) = (1/2)\,\mathrm{Tr}\,[D^2 u(t, x)] + \langle Ax, Du(t, x)\rangle, \quad t > 0,\ x \in D(A). \tag{2.24}$$

If $Du(t,x) \in D(A^*)$ we can write (2.24) as

$$u_t(t,x) = (1/2)\,\mathrm{Tr}\,[D^2 u(t,x)] + \langle x, A^* Du(t,x)\rangle, \qquad t > 0,\ x \in H. \tag{2.25}$$

PROPOSITION 2.12. *Let $\varphi \in C_b(H)$, $t > 0$ and $x \in D(A)$. Then $u(t,x) = P_t\varphi(x)$ is a solution to the Kolmogorov equation* (2.25).

PROOF. Let $u(t,x) = P_t\varphi(x)$, $t > 0$, $x \in H$. Then the term $\mathrm{Tr}\,[D^2 u(t,x)]$ is well defined by Proposition 2.5. Moreover also the term $\langle x, A^* Du(t,x)\rangle$ is well defined, since, by (2.1) we have

$$\langle x, A^* Du(t,x)\rangle = \int_H \langle Q_t^{-1/2} e^{(t/2)A} A e^{(t/2)A},\ Q_t^{-1/2} y\rangle \varphi(e^{tA}x + y)\,\mathcal{N}(0, Q_t)(dy). \tag{2.26}$$

By (2.26) we have

$$|\langle x, A^* Du(t,x)\rangle| \leq K(t)\|\varphi\|_0\,|x|, \tag{2.27}$$

where

$$K^2(t) = \|Q_t^{-1/2} e^{(t/2)A} A e^{(t/2)A}\|. \tag{2.28}$$

It remains to show that $u(t,x)$ is differentiable in t and that (2.25) holds. To this aim let us introduce the space of all *exponential functions* $\mathcal{E}(H)$. We denote by $\mathcal{E}(H)$ the linear subspace of $C_b(H)$ spanned by all ζ_h, $h \in H$:

$$\zeta_h(x) = e^{i\langle h, x\rangle}, \qquad x \in H.$$

Since, as easily checked

$$P_t\zeta_h(x) = e^{i\langle e^{tA}x, h\rangle - (1/2)\langle Q_t h, h\rangle}, \qquad x \in H, \tag{2.29}$$

then the proposition holds when $\varphi \in \mathcal{E}(H)$.

Let now $\{\varphi_n\}$ be a sequence in $\mathcal{E}(H)$ such that

(*i*) $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, $\forall x \in H$,

(*ii*) $\|\varphi_n\|_0 \leq 2\|\varphi\|_0$,

and set $u_n(t,x) = P_t\varphi_n(x)$, $t \geq 0$, $x \in H$. We fix now $t > 0$. By (2.3)-(2.5) it follows that the sequence of functions $\{u_n(t,\cdot)\}$ has all derivatives of order less than 3, bounded. This implies that

$$\lim_{n\to\infty} u_n(t,\cdot) = u(t,\cdot), \qquad \text{in } C_b^2(H),$$

uniformly in t on compact subsets of $]0, +\infty[$. Moreover by (2.9) it follows that the sequence in $C_b(H)$ defined by $\{\mathrm{Tr}\,[D^2 u_n(t,\cdot)]\}$ is bounded, so that

$$\lim_{n\to\infty} \mathrm{Tr}\,[D^2 u_n(t,\cdot)] = \mathrm{Tr}\,[D^2 u(t,\cdot)], \qquad \text{in } C_b(H),$$

uniformly in t on compact subsets of $]0, +\infty[$. Finally from (2.27) it follows that

$$\lim_{n\to\infty} \langle x, A^* Du_n(t,x)\rangle = \langle x, A^* Du(t,x)\rangle, \qquad x \in H,$$

uniformly in t on compact subsets of $]0, +\infty[$ and in x on bounded subsets of H. This implies that for any $x \in H$

$$\lim_{n \to \infty} \frac{d}{dt} u_n(t, x) = \frac{d}{dt} u(t, x),$$

for all $x \in H$ uniformly in t on compact subsets of $]0, +\infty[$ and the conclusion follows. ■

3. The infinitesimal generator

We proceed here as in [4], by introducing the Laplace transform of P_t, $t \geq 0$. For any $\lambda > 0$ we set

$$F(\lambda)\,\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t \varphi(x)\,dt, \qquad x \in H, \varphi \in C_b(H). \tag{3.1}$$

Note that the above integral is convergent for any fixed $x \in H$ and not in $C_b(H)$ in general. In [4] is shown that $F(\lambda)$ maps $C_b(H)$ into itself and that it is one-to-one. So there exists a unique closed operator $\mathcal{M}$ in $C_b(H)$:

$$\mathcal{M}: D(\mathcal{M}) \subset C_b(H) \mapsto C_b(H),$$

such that the resolvent set $\varrho(\mathcal{M})$ of $\mathcal{M}$ contains $]0, +\infty[$ and

$$R(\lambda, \mathcal{M})\,\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t \varphi(x)\,dt, \qquad \forall \lambda > 0. \tag{3.2}$$

$\mathcal{M}$ is called the *infinitesimal generator* of the semigroup P_t, $t \geq 0$.

Let $\lambda > 0$, $g \in C_b(H)$ and set $\varphi = R(\lambda, \mathcal{M})g$. Then φ is called a *generalized solution* to the equation

$$\lambda\varphi - (1/2)\,\mathrm{Tr}\,[D^2\varphi] - \langle Ax, D\varphi\rangle = g. \tag{3.3}$$

It is also useful to introduce the concept of *strict solution*. To this purpose we have to introduce a suitable restriction $\mathcal{M}_0$ of $\mathcal{M}$.

By definition the domain $D(\mathcal{M}_0)$ of $\mathcal{M}_0$ is the set of all functions $\varphi \in C_b(H)$ such that

(*i*) $\varphi \in C_b^2(H)$ and $D^2\varphi(x) \in \mathcal{L}_1(H)$ for all $x \in H$.

(*ii*) $D\varphi(x) \in D(A^*)$ and the mapping

$$H \mapsto \mathbf{R}, \qquad x \mapsto A^* D\varphi(x),$$

belongs to $C_b(H)$.

Then we define the operator $\mathcal{M}_0$ by setting

$$\mathcal{M}_0\varphi = (1/2)\,\mathrm{Tr}\,[D^2\varphi] + \langle x, A^* D\varphi\rangle, \qquad \forall \varphi \in D(\mathcal{M}_0). \tag{3.4}$$

Remark 3.1. In the paper [5], it is proved that the operator $\mathcal{M}$ is the closure of $\mathcal{M}_0$ with respect to the $\mathcal{K}$-convergence. A sequence $\{\varphi_n\} \subset C_b(H)$ is said to be $\mathcal{K}$-convergent to $\varphi \in C_b(H)$ if

(*i*) $\sup_{n \in N} \|\varphi_n\|_0 < +\infty$.

(*ii*) For any compact subset K in H, we have

$$\lim_{n\to\infty} \sup_{x\in K} |\varphi(x) - \varphi_n(x)| = 0 .$$

From the regularity results of the semigroup P_t, $t \geq 0$, obtained in the previous section, one gets the following regularity results for the resolvent of $\mathcal{M}$.

PROPOSITION 3.2. *Let $\lambda > 0$, $g \in C_b(H)$, and set $\varphi = R(\lambda, \mathcal{M})g$. Then the following statements hold*

(*i*) *$\varphi \in C_b^1(H)$ and*

$$|D\varphi(x)| \leq \Gamma(1/2)\cdot\lambda^{-1/2}\|g\|_0 , \quad x \in H, \tag{3.5}$$

where Γ denotes the gamma Euler function.

(*ii*) *For any $\alpha \in]0, 1[$, we have $\varphi \in C_b^{1+\alpha}(H)$ and*

$$[D\varphi]_\alpha \leq 2^{\alpha/2}\Gamma((1-\alpha)/2)\cdot\lambda^{(\alpha-1)/2}\|g\|_0 , \tag{3.6}$$

(*iii*) *If $g \in C_b^\theta(H)$ for some $\theta \in]0, 1[$, then $\varphi \in C_b^2(H)$, and*

$$\|D^2\varphi(x)\| \leq 2^{(1-\theta)/2}\Gamma(\theta/2)\lambda^{-\theta/2}\|g\|_0 , \quad x \in H . \tag{3.7}$$

(*iv*) *If $g \in C_b^\theta(H)$ for some $\theta \in]0, 1[$, and if in addition $\gamma(t)^{1-\theta/2}$ is integrable near 0, then $\varphi \in D(\mathcal{M}_0)$ and so it is a strict solution to equation (3.3).*

REMARK 3.3. If

$$\gamma(t) \leq Ct^{-3/2} ,$$

for some constant $C > 0$. Then condition (*iv*) is fulfilled provided $g \in C_b^\theta(H)$ with $\theta > 2/3$, see Remark 2.11.

3.1. INTERPOLATION SPACES $D_{\mathcal{M}}(\theta, \infty)$

The semigroup P_t, $t \geq 0$ is not strongly continuous in $C_b(H)$, even when H is finite-dimensional, see [4, 7]. The following proposition, proved in [6], gives a characterization of the maximal subspace $\mathcal{Y}$ of $C_b(H)$ where P_t, $t \geq 0$ is strongly continuous.

PROPOSITION 3.4. *Let $\varphi \in C_b(H)$. Then the following statements are equivalent*

(*i*) $\lim_{t\to 0} P_t\varphi = \varphi$ *in* $C_b(H)$.

(*ii*) $\lim_{t\to 0} \varphi(e^{tA}x) = \varphi(x)$ *in* $C_b(H)$.

We shall set

$$\mathcal{Y} = \{\varphi \in C_b(H): \lim_{t\to 0} \varphi(e^{tA}x) = \varphi(x) \text{ in } C_b(H)\} ,$$

and for any $\theta \in]0, 1[$

$$\mathcal{Y} = \{\varphi \in C_b(H): \exists C > 0, |\varphi(e^{tA}x) - \varphi(x)| \leq Ct^\theta, \forall x \in H\} .$$

We want now to characterize the interpolation spaces $(C_b(H), D(\mathcal{M}))_{\theta,\infty}$ that we shall denote by $D_{\mathcal{M}}(\theta, \infty)$. We need some preliminary result.

PROPOSITION 3.5. *Let* $\varphi \in C_b(H)$ *and* $\theta \in]0, 1[$. *Then the following statements hold.*

(*i*) *If* $\varphi \in D_{\mathcal{M}}(\theta, \infty)$ *then we have*

$$\sup_{\lambda > 0} \lambda^\theta \|\mathcal{M} R(\lambda, \mathcal{M})\varphi\|_0 < +\infty . \tag{3.8}$$

(*ii*) *If* $\varphi \in C_b(H)$ *and fulfills* (3.8) *then* $\varphi \in D_{\mathcal{M}}(\theta, \infty)$.

PROOF. (*i*) Let $\varphi \in D_{\mathcal{M}}(\theta, \infty)$. Then by the definition of interpolation space given in § 1, for any $t \in [0, 1]$ there exist $\alpha_t \in C_b(H)$, $\beta_t \in D(\mathcal{M})$, such that $\varphi = \alpha_t + \beta_t$ and

$$\|\alpha_t\|_0 \leq Ct^\theta, \qquad \|\mathcal{M}\beta_t\|_0 \leq Ct^{\theta-1},$$

for some $C > 0$. Now for any $\lambda > 0$ we have

$$\mathcal{M}R(\lambda, \mathcal{M})\varphi = \mathcal{M}R(\lambda, \mathcal{M})\alpha_{1/\lambda} + R(\lambda, \mathcal{M})\mathcal{M}\beta_{1/\lambda} .$$

It follows

$$\|\mathcal{M}R(\lambda, \mathcal{M})\varphi\|_0 \leq C\|\mathcal{M}R(\lambda, \mathcal{M})\|\lambda^{-\theta} + C\|R(\lambda, \mathcal{M})\|\lambda^{1-\theta} \leq 3\lambda^{-\theta},$$

and the statement is proved.

(*ii*) Assume that φ fulfills (3.8). Define

$$C_1 = \sup_{\lambda > 0} \lambda^\theta \|\mathcal{M}R(\lambda, \mathcal{M})\varphi\|_0 ,$$

and set

$$\alpha_t = -\mathcal{M}R((1/t), \mathcal{M})\varphi \cdot \beta_t = (1/t)R((1/t), \mathcal{M})\varphi .$$

Then we have $\alpha_t + \beta_t = \varphi$ and

$$\|\alpha_t\|_0 \leq C_1 t^\theta, \qquad \|\beta_t\|_0 \leq t^{1-\theta}, \qquad t > 0,$$

so that $\varphi \in D_{\mathcal{M}}(\theta, \infty)$. ■

LEMMA 3.6. *Let* $\theta \in]0, 1/2[$, $T > 0$, $\varphi \in C_b^{2\theta}(H)$. *Then there exists* $C_T > 0$ *such that*

$$|G_t\varphi(x) - \varphi(x)| \leq C_T[\mathrm{Tr}(Q_t)]^\theta [\varphi]_{2\theta}, \qquad t \in [0, T]. \tag{3.9}$$

PROOF. We have

$$|G_t\varphi(x) - \varphi(x)| \leq \int_H |\varphi(x+y) - \varphi(x)| \mathcal{N}(0, Q_t)(dy) \leq$$

$$\leq [\varphi]_{2\theta} \int_H |y|^{2\theta} \mathcal{N}(0, Q_t)(dy) \leq D_\theta [\varphi]_{2\theta} [\mathrm{Tr}(Q_t)]^\theta ,$$

for some constant D_θ.

Now the conclusion follows. ■

LEMMA 3.7. *Let* $\theta \in]1/2, 1[$, $T > 0$, $\varphi \in C_b^{2\theta}(H)$. *Then there exists* $C_{1,T} > 0$ *such that*

$$|G_t\varphi(x) - \varphi(x)| \leq C_{1,T}[\mathrm{Tr}(Q_t)]^\theta [\varphi]_{2\theta}, \qquad t \in [0, T]. \tag{3.10}$$

PROOF. We have

$$G_t\varphi(x)-\varphi(x)=\int_H[\varphi(x+y)-\varphi(x)]\,\mathcal{N}(0,Q_t)(dy)=$$

$$=\int_0^1\int_H\langle D\varphi(x+\xi y)-D\varphi(x),y\rangle\,\mathcal{N}(0,Q_t)(dy)\,d\xi\,.$$

It follows

$$|G_t\varphi(x)-\varphi(x)|\leqslant[\varphi]_{2\theta}\int_0^1\int_H|y|^{2\theta}\xi^{2\theta-1}\,\mathcal{N}(0,Q_t)(dy)\,d\xi\,,$$

and the conclusion follows as in the previous lemma. ■

PROPOSITION 3.8. *If* $\varphi\in D_{\mathcal{M}}(\theta,\infty)$, $\theta\in]0,1[$, *there exists* $C_T>0$ *such that*

$$\|P_t\varphi-\varphi\|_0\leqslant C_Tt^\theta\,,\qquad t\in[0,T]\,. \tag{3.11}$$

PROOF. Let $\varphi\in D_{\mathcal{M}}(\theta,\infty)$. Then for any $t>0$ there exists $\alpha_t\in C_b(H)$, $\beta_t\in D(\mathcal{M})$ such that $\varphi=\alpha_t+\beta_t$,

$$\|\alpha_t\|_0\leqslant Ct^\theta\,,\qquad\|\mathcal{M}\beta_t\|_0\leqslant Ct^{\theta-1}\,, \tag{3.12}$$

for some constant $C>0$. Since

$$P_t\varphi-\varphi=(P_ta_t-a_t)+(P_tb_t-b_t)=(P_ta_t-a_t)+\int_0^tP_s\mathcal{M}b(t)\,ds\,,$$

using (3.12), we find that (3.11) holds. ■

We can now prove the result

THEOREM 3.9. *For all* $\theta\in]0,1/2[\cup]1/2,1[$ *we have*

$$D_{\mathcal{M}}(\theta,\infty)\subset C_b^{2\theta}(H)\cap\mathcal{Y}_\theta\,. \tag{3.13}$$

PROOF. We only consider the case $\theta\in]0,1/2[$, since the case $\theta\in]1/2,1[$ can be treated in an analogous way.

STEP 1. If $\varphi\in D_{\mathcal{M}}(\theta,\infty)$ then there exists $C_1>0$ such that for all $\lambda\geqslant1$ we have

$$\|\lambda DR(\lambda,\mathcal{M})\varphi\|_0\leqslant C_1\lambda^{1/2-\theta}\|\varphi\|_{D_{\mathcal{M}}(\theta,\infty)}\,. \tag{3.14}$$

We first note that, since

$$\frac{d}{d\lambda}[\lambda R(\lambda,\mathcal{M})]=R(\lambda,\mathcal{M})-\lambda(R(\lambda,\mathcal{M}))^2\,,$$

we have

$$\lambda R(\lambda,\mathcal{M})\varphi=R(1,\mathcal{M})\varphi+\int_1^\lambda R(s,\mathcal{M})(1-sR(s,\mathcal{M}))\varphi\,ds=$$

$$=R(1,\mathcal{M})\varphi+\int_1^\lambda R(s,\mathcal{M})\mathcal{M}R(s,\mathcal{M})\varphi\,ds\,.$$

By Proposition 3.2 (*i*) it follows

$$D_x \lambda R(\lambda, \mathcal{M}) \varphi = D_x R(1, \mathcal{M}) \varphi + \int_1^\lambda D_x [R(s, \mathcal{M}) \mathcal{M} R(s, \mathcal{M}) \varphi] \, ds .$$

Moreover, taking into account (3.5) and (3.8), we find

$$\|R(\lambda, \mathcal{M}) \varphi\|_1 \leqslant C\lambda^{-1/2} \|\varphi\|_0 , \qquad \forall \lambda > 0 ,$$

we get

$$\|R(\lambda, \mathcal{M}) \varphi\|_1 \leqslant C\|\varphi\|_0 + C \int_1^\lambda s^{-1/2-\theta} [\varphi]_{D_{\mathcal{M}}(\theta, \infty)} \, ds =$$

$$= C\|\varphi\|_0 + C/(1/2 - \theta)(\lambda^{1/2-\theta} - 1)[\varphi]_{D_{\mathcal{M}}(\theta, \infty)} ,$$

for some $C > 0$.

STEP 2. $D_{\mathcal{M}}(\theta, \infty) \subset C_b^{2\theta}(H)$.

Let $x, y \in H$ such that $|x - y| \leqslant 1$, and let $\lambda \geqslant 1$. Then we have by (3.8) and (3.14),

$$|\varphi(x) - \varphi(y)| \leqslant |\varphi(x) - \lambda R(\lambda, \mathcal{M}) \varphi(x)| + |\lambda R(\lambda, \mathcal{M}) \varphi(x) - \lambda R(\lambda, \mathcal{M}) \varphi(y)| +$$
$$+ |\lambda R(\lambda, \mathcal{M}) \varphi(y) - \varphi(y)| \leqslant 2[\varphi]_{D_{\mathcal{M}}(\theta, \infty)} \lambda^{-\theta} + \|D(\mathcal{M} R(\lambda, \mathcal{M}) \varphi)\|_0 \, |x - y| \leqslant$$
$$\leqslant 2[\varphi]_{D_{\mathcal{M}}(\theta, \infty)} \lambda^{-\theta} + C\|\varphi\|_{D_{\mathcal{M}}(\theta, \infty)} (\lambda^{1/2-\theta} + 1)|x - y| .$$

Choosing $\lambda = |x - y|^{-2}$ we have

$$|\varphi(x) - \varphi(y)| \leqslant 2[\varphi]_{D_{\mathcal{M}}(\theta, \infty)} |x - y|^{2\theta} + C\|\varphi\|_{D_{\mathcal{M}}(\theta, \infty)} (|x - y|^{2\theta} + |x - y|) ,$$

and the conclusion follows easily.

STEP 3. $D_{\mathcal{M}}(\theta, \infty) \subset Y_\theta$.

Let $\varphi \in D_{\mathcal{M}}(\theta, \infty)$. Then we have

$$|\varphi(e^{tB}x) - \varphi(x)| \leqslant |\varphi(e^{tB}x) - G_t \varphi(e^{tB}x)| + |P_t \varphi(x) - \varphi(x)| . \tag{3.15}$$

Since $\varphi \in C_b^{2\theta}(H)$ by (3.9) we find

$$|\varphi(e^{tB}x) - G_t \varphi(e^{tB}x)| \leqslant Ct^\theta [\varphi]_{2\theta} , \qquad t \in [0, T] . \tag{3.16}$$

Moreover from (3.11) it follows

$$\|P_t \varphi - \varphi\|_0 \leqslant C_T t^\theta , \qquad t \in [0, T] . \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15) we get finally

$$|\varphi(e^{tB}x) - \varphi(x)| \leqslant (C + C_T) t^\theta [\varphi]_{2\theta} ,$$

and the proof of the theorem is complete. ■

4. MAXIMAL REGULARITY RESULTS FOR ELLIPTIC EQUATIONS

The following result is proved in [3]. We give a sketch of the proof for the reader convenience.

PROPOSITION 4.1. *Assume that* $\theta \in]0, 1[$, $g \in C_b^\theta(H)$, *and* $\lambda > 0$. *Then the function* $\varphi = R(\lambda, \mathcal{M})g$ *belongs to* $C_b^{2+\theta}(H)$.

PROOF. The proof is based on a general interpolation argument due to A. Lunardi see [16], in particular on the following inclusion result

$$(C_b^\alpha(H), C_b^{2+\alpha}(H))_{1-(\alpha-\theta)/2,\infty} \subset C_b^{2+\theta}(H), \tag{4.1}$$

for any $\alpha \in]\theta, 1[$. Consequently, in order to prove the theorem it will be enough to show that for some $\alpha \in]\theta, 1[$, we have

$$\varphi \in (C_b^\alpha(H), C_b^{2+\alpha}(H))_{1-(\alpha-\theta)/2,\infty}. \tag{4.2}$$

To prove (4.2) we set

$$\varphi(x) = a(t,x) + b(t,x),$$

where

$$a(t,x) = \int_0^t e^{-\lambda s} P_s g(x)\, ds,$$

and

$$b(t,x) = \int_t^{+\infty} e^{-\lambda s} P_s g(x)\, ds.$$

Then from (2.7) it follows that

$$\|a(\cdot,t)\|_\alpha \leq C(\alpha,\theta) \int_0^t e^{-\lambda s} s^{-(\alpha-\theta)/2}\, ds\, \|g\|_\theta =$$

$$= C(\alpha,\theta)\, t^{1-(\alpha-\theta)/2} \int_0^1 e^{-\lambda t\sigma} \sigma^{-(\alpha-\theta)/2}\, d\sigma\, \|g\|_\theta \leq \frac{C(\alpha,\theta)}{1-(\alpha-\theta)/2}\, t^{1-(\alpha-\theta)/2} \|g\|_\theta,$$

and from (2.8) that

$$\|b(\cdot,t)\|_{2+\alpha} \leq C(\alpha,\theta) \int_t^{+\infty} e^{-\lambda s} s^{-((\alpha-\theta)/2)-1}\, ds\, \|g\|_\theta =$$

$$= C(\alpha,\theta)\, t^{-(\alpha-\theta)/2} \int_1^{+\infty} e^{-\lambda t\sigma} \sigma^{-((\alpha-\theta)/2)-1}\, d\sigma\, \|g\|_\theta \leq \frac{C(\alpha,\theta)}{\alpha-\theta}\, t^{(\theta-\alpha)/2} \|g\|_\theta.$$

This implies (4.2). ■

By Proposition 4.1 and 3.2 (*iv*) we find the result.

THEOREM 4.2. *Assume that* $\theta \in]0, 1[$, $g \in C_b^\theta(H)$, $\lambda > 0$, *and in addition that*

$$\int_0^1 [\mathrm{Tr}(\Lambda_t \Lambda_t^*)]^{1-\theta/2}\, dt < +\infty. \tag{4.3}$$

Then, setting $\varphi = R(\lambda, \mathcal{M})g$, *the following statements hold.*

(*i*) $\varphi \in C_b^{2+\theta}(H)$ *and* $D^2\varphi(x) \in \mathcal{L}_1(H)$ *for any* $x \in H$.

(*ii*) $\mathrm{Tr}\,[D^2\varphi(\cdot)] \in C_b(H)$.

(*iii*) $x \to \langle x, A^* D\varphi\rangle \in C_b(H)$.

Moreover

$$\lambda\varphi(x) - (1/2)\,\mathrm{Tr}\,[D^2\varphi(x)] - \langle x, A^* D\varphi\rangle = g(x)\,, \tag{4.4}$$

for all $x \in H$.

REMARK 4.3. Let us consider the restriction P_t^θ, $t \geq 0$ of the semigroup P_t, $t \geq 0$ to $C_b^\theta(H)$, $\theta \in]0, 1[$. We can still define the infinitesimal generator $\mathcal{M}^\theta$ of P_t, $t \geq 0$ to $C_b^\theta(H)$ by the Laplace, transform setting

$$R(\lambda, \mathcal{M}^\theta)\,\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t^\theta \varphi(x)\,dt\,. \tag{4.5}$$

It is easy to check that $\mathcal{M}^\theta$ is the part of $\mathcal{M}$ in $C_b^\theta(H)$:

$$D(\mathcal{M}^\theta) = \{\varphi \in D(\mathcal{M}) \cap C_b^\theta(H) \colon \mathcal{M}\varphi \in C_b^\theta(H)\}\,.$$

Theorem 4.2 enable us to characterize, under suitable assumptions, the domain of M^θ.
We have

$$D(\mathcal{M}^\theta) = \{\varphi \in C_b^{2+\theta}(H) \colon \langle A\,\cdot, D\varphi\rangle \in C_b(H)\}\,.$$

If H is finite-dimensional this characterization of $D(\mathcal{M}^\theta)$ was obtained in [7].

Under the hypotheses of Theorem 4.2 we can give the following definition of $D(\mathcal{M}^\theta)$

$$D(\mathcal{M}^\theta) = \{\varphi \in C_b^{2+\theta}(H) \colon D^2\varphi(x) \in \mathcal{L}_1(H),\ \forall x \in H,\ \mathrm{Tr}\,[D^2\varphi(x)] \in C_b(H), \langle A\,\cdot, D\varphi\rangle \in C_b(H)\}\,. \tag{4.6}$$

5. MAXIMAL REGULARITY RESULTS FOR PARABOLIC EQUATIONS

We are here concerned with the initial value problem

$$\begin{cases} du(t,x)/dt = (1/2)\,\mathrm{Tr}\,[D^2u(t,x)] + \langle Ax, Du(t,x)\rangle + F(t,x)\,, \\ \qquad\qquad t \in]0, T],\quad x \in H\,, \\ u(0,x) = \varphi(x)\,, \end{cases} \tag{5.1}$$

where $F \in C([0, T]; C_b(H))$ and $\varphi \in C_b(H)$.

Following S. Cerrai and F. Gozzi [5], we call the function $u\colon [0, T] \times H \mapsto \mathbf{R}$ defined as

$$u(t,x) = P_t\varphi(x) + \int_0^t P_{t-s}F(s,\cdot)(x)\,ds = u_1(t,x) + u_2(t,x)\,, \tag{5.2}$$

the *mild* solution to (5.1). Several properties of the mild solution u are described in the quoted paper [5]. Here we will discuss only some new maximal regularity results for u_1 and u_2. Concerning u_1 we have the following proposition.

PROPOSITION 5.1. *The following statements are equivalent*

(*i*) $u_1 \in C^\theta([0, T]; C_b(H))$.

(*ii*) $\varphi \in D_{\mathcal{M}}(\theta, \infty)$.

PROOF. (*i*) $\Rightarrow$ (*ii*). It is enough to show that

$$\sup_{\lambda > 0} \lambda^\theta \|\mathcal{M} R(\mathcal{M}, \lambda)\varphi\|_0 < +\infty . \tag{5.3}$$

In fact, by Proposition 3.5, if (5.3) holds, we have $\varphi \in D_{\mathcal{M}}(\theta, \infty)$, and by Theorem 3.9 this implies (*ii*). By hypothesis (*i*) there exists $K > 0$ such that

$$|P_t \varphi(x) - \varphi(x)| \leq K t^\theta , \qquad t \in [0, T]. \tag{5.4}$$

It follows

$$|\mathcal{M} R(\lambda, \mathcal{M})\varphi(x)| \leq K\lambda \int_0^{+\infty} e^{-\lambda t} t^\theta \, dt \leq \frac{K\Gamma(\theta + 1)}{\lambda^\theta} ,$$

and (5.3) holds.

(*ii*) $\Rightarrow$ (*i*). Let $t > s \geq 0$. Then by Proposition 3.8, we have

$$|P_t \varphi(x) - P_s \varphi(x)| \leq |P_{t-s}\varphi(x) - \varphi(x)| \leq C_T |t - s|^\theta ,$$

and (*i*) is proved. ■

We conclude this section, by studying the regularity of u_2.

THEOREM 5.2. *Let* $F \in C([0, T]; C_b(H))$, *and assume that, for some* $\theta \in]0, 1[$, *we have* $F(t, \cdot) \in C_b^\theta(H)$ *and that*

$$\sup_{t \in [0, T]} \|F(t, \cdot)\|_\theta < +\infty . \tag{5.5}$$

Then $u \in C([0, T]; C_b(H))$, $u(t, \cdot) \in C_b^{2+\theta}(H)$ *and*

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{2+\theta} < +\infty . \tag{5.6}$$

PROOF. We fix $t > 0$. Arguing as in the proof of Proposition 4.1 it is enough to prove that

$$u(t, \cdot) \in (C_b^\alpha(H), C_b^{2+\alpha}(H))_{1 - (\alpha - \theta)/2, \infty} , \tag{5.7}$$

for some $\alpha \in]\theta, 1[$. To this purpose we set

$$a(t, \tau, x) = \int_0^\tau P_s u(t - s, \cdot)(x) \, ds , \qquad \tau \in [0, t],$$

$$b(t, \tau, x) = \int_\tau^t P_s u(t - s, \cdot)(x) \, ds , \qquad \tau \in [0, t].$$

Then by (2.7) we have

$$\|a(t,\tau,\cdot)\|_\alpha \leqslant C(\alpha,\theta) \sup_{s\in[0,T]} \|u(s,\cdot)\|_\theta \int_0^\tau \frac{ds}{s^{(\alpha-\theta)/2}} \leqslant$$

$$\leqslant \frac{C(\alpha,\theta)}{1-(\alpha-\theta)/2} \sup_{s\in[0,T]} \|u(s,\cdot)\|_\theta \tau^{1-(\alpha-\theta)/2}.$$

Moreover by (2.8) we have

$$\|b(t,\tau,\cdot)\|_{2+\alpha} \leqslant C(\alpha,\theta) \sup_{s\in[0,T]} \|u(s,\cdot)\|_\theta \int_\tau^t \frac{ds}{s^{(\alpha-\theta)/2+1}} \leqslant$$

$$\leqslant \frac{2C(\alpha,\theta)}{(\alpha-\theta)/2} \sup_{s\in[0,T]} \|u(s,\cdot)\|_\theta \tau^{-(\alpha-\theta)/2}.$$

This implies (5.7). ■

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Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 Pisa