GIUSEPPE DA PRATO

Some results on elliptic and parabolic equations in Hilbert spaces


Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1996_9_7_3_181_0>

L’utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l’utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma
*bdim* (Biblioteca Digitale Italiana di Matematica)

SIMAI & UMI

http://www.bdim.eu/
Some results on elliptic and parabolic equations in Hilbert spaces

Memoria (*) di Giuseppe Da Prato

ABSTRACT. — We consider elliptic and parabolic equations with infinitely many variables. We prove some results of existence, uniqueness and regularity of solutions.

KEY WORDS: Elliptic and parabolic equations in Hilbert spaces; Ornstein-Uhlenbeck semigroup; Schauder estimates.

1. INTRODUCTION

Let $H$ be a separable Hilbert space (norm $| \cdot |$, inner product $\langle \cdot, \cdot \rangle$). We denote by $\mathcal{L}(H)$ the Banach algebra (norm $\| \cdot \|$) of all linear bounded operators from $H$ into $H$, by $\mathcal{L}_1(H)$ (norm $\| \cdot \|_{\mathcal{L}_1(H)}$) the set of all trace-class operators and by $\mathcal{L}_2(H)$ (norm $\| \cdot \|_{\mathcal{L}_2(H)}$) the set of all Hilbert-Schmidt operators in $H$.

We are given a linear closed operator $A: D(A) \subset H \to H$ and a symmetric bounded operator $Q \in \mathcal{L}(H)$. We assume

HYPOTHESIS 1.1. (i) $A$ is the infinitesimal generator of an analytic semigroup $e^{tA}$ in $H$, such that

$$\|e^{tA}\| \leq 1, \quad t \geq 0.$$  

(ii) There exists $\nu > 0$ such that

$$\frac{1}{\nu} I \leq Q \leq \nu I.$$  

(iii) For any $t > 0$, $e^{tA} \in \mathcal{L}_2(H)$ and

$$\int_0^t \text{Tr}[e^{sA}Qe^{sA^*}]ds < +\infty.$$  

If Hypothesis 1.1 holds then for arbitrary $t \geq 0$, the linear operator $Q_t$ defined as

$$Q_t x = \int_0^t e^{sA}Qe^{sA^*} x \, dt, \quad x \in H,$$

is well defined and trace-class.

(*) Presentata nella seduta del 9 marzo 1996.
The following result is proved in [8].

**Proposition 1.1.** Under Hypothesis 1.1 one has

\[(1.5)\]  
\[e^{sA}(H) \subset Q_{t}^{1/2}(H), \quad 0 < s \leq t.\]

Moreover setting

\[(1.6)\]  
\[A_t = Q_t^{-1/2} e^{tA}, \quad t > 0,\]

one has

\[(1.7)\]  
\[\|A_t\| \leq \sqrt{\gamma t}, \quad t > 0.\]

**Remark 1.2.** Since

\[A_t = Q_t^{-1/2} e^{(t/2)A} e^{(t/2)A}, \quad t > 0,\]

we have that \(A_t \in \mathcal{L}_2(H)\) so that

\[(1.8)\]  
\[\gamma(t) := \text{Tr} [A_t A_t^\ast] < + \infty, \quad \forall t > 0.\]

The main object of this paper is the Ornstein-Uhlenbeck transition semigroup \(P_t, t \geq 0\) defined on \(C_b(H)\), the Banach space of all uniformly continuous and bounded mappings from \(H\) into \(\mathbb{R}\), endowed with the norm \(\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|\). We set for \(t > 0\)\(^{(1)}\)

\[(1.9)\]  
\[P_t \varphi(x) = \int_H \varphi(x + y) \mathcal{N}(0, Q_t)(dy), \quad \varphi \in C_b(H).\]

It is useful to note that, setting

\[(1.10)\]  
\[G_t \varphi(x) = \int_H \varphi(x + y) \mathcal{N}(0, Q_t)(dy), \quad \varphi \in C_b(H),\]

we have

\[(1.11)\]  
\[P_t \varphi(x) = (G_t \varphi)(e^{tA}x), \quad \varphi \in C_b(H), \quad t \geq 0, \quad x \in H.\]

\(P_t, t \geq 0\) is not a strongly continuous semigroup on \(C_b(H)\), however it is weakly continuous, see [4]. In particular we have

\[(1.12)\]  
\[\lim_{t \to 0} P_t \varphi(x) = \varphi(x), \quad \forall \varphi \in C_b(H), \quad \forall x \in H,\]

the convergence being uniform on the compact subsets of \(H\).

In this paper we first study some regularity properties of the semigroup \(P_t, t \geq 0\). Then we introduce its infinitesimal generator \(\mathcal{K}\) and characterize the corresponding interpolation spaces. Finally we apply the obtained results to the study of the elliptic equation

\[(1.13)\]  
\[\lambda \varphi - (1/2) \text{Tr} [D^2 \varphi] - \langle Ax, D\varphi \rangle = g, \quad x \in H,\]

\(^{(1)}\) For any \(m \in H\) and any \(S \in \mathcal{L}_1(H)\) symmetric nonnegative, we denote by \(\mathcal{N}(m, S)\) the Gaussian measure with mean \(m\) and covariance operator \(S\).
where $\lambda > 0$ and $g: H \to \mathbb{R}$ is a suitable function, and to the initial value problem
\[
\begin{aligned}
du(t, x)/dt &= (1/2) \text{Tr} [D^2 u(t, x)] + \langle Ax, Du(t, x) \rangle + F(t, x), \\
\lambda u(t, x) &= \varphi(x), \\
u(0, x) &= \varphi(x),
\end{aligned}
\]
(1.14) \hspace{1cm} t \in [0, T], \ x \in H,

where $F: [0, T] \times H \to \mathbb{R}$ and $\varphi: H \to \mathbb{R}$ are given functions fulfilling suitable assumptions. We also study problems (1.13) and (1.14) in spaces $C_c^0(H)$ of Hölder continuous functions. In this case we will prove, following [3], Schauder estimates and we will characterize, under suitable hypotheses the domain of the infinitesimal generator $\mathcal{M}$ of $P_t, t \geq 0$.

Let us introduce our main notation. The following subspaces of $C_b(H)$ will be needed.

- $C^1_c(H)$ is the Banach space of all functions $\varphi \in C_b(H)$ which are Fréchet differentiable on $H$, with a bounded uniformly continuous derivative $D\varphi$, with the norm
  \[\|\varphi\|_1 = \|\varphi\|_0 + [\varphi],\]
  where
  \[[\varphi] = \sup_{x \in H} |D\varphi(x)|.\]

- $C^2_c(H)$ is the Banach space of all functions $\varphi \in C^1_c(H)$ which are twice Fréchet differentiable on $H$, with a bounded uniformly continuous second derivative $D^2\varphi$ with the norm
  \[\|\varphi\|_2 = \|\varphi\|_1 + [\varphi],\]
  where
  \[[\varphi]_2 = \sup_{x \in H} |D^2\varphi(x)|.\]

- $C^n_c(H), n \in \mathbb{N}$ is the Banach space of all functions $\varphi \in C^1_c(H)$ which are $n$ times Fréchet differentiable on $H$, with bounded uniformly continuous derivatives of any order less or equal to $n$, with the norm
  \[\|\varphi\|_n = \|\varphi\|_0 + \sum_{k=1}^{n} [\varphi],\]
  where
  \[[\varphi]_k = \sup_{x \in H} |D^k\varphi(x)|, \quad k = 1, \ldots, n.\]
We set

\[ C^a_b (H) = \bigcap_{n=1}^{\infty} C^n_b (H). \]

- \( C^a_b (H), \alpha \in ]0, 1[, \) is the Banach space of all \( \alpha \)-Hölder continuous and bounded functions \( \varphi \in C_b (H) \) with the norm

\[ \| \varphi \|_a = \| \varphi \|_0 + [\varphi]_a, \]

where

\[ [\varphi]_a = \sup_{x, y \in H, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} < + \infty. \]

- \( C^{1+a}_b (H), \alpha \in ]0, 1[, \) is the set of all functions \( \varphi \in C^1_b (H) \) such that

\[ [D\varphi]_a = \sup_{x, y \in H, x \neq y} \frac{|D\varphi(x) - D\varphi(y)|}{|x - y|^\alpha} < + \infty. \]

\( C^{1+a}_b (H) \) is a Banach space with the norm

\[ \| \varphi \|^{1+a} = \| \varphi \|_1 + [D\varphi]_a. \]

- \( C^{2+a}_b (H), \alpha \in ]0, 1[, \) is the set of all functions \( \varphi \in C^2_b (H) \) such that

\[ [D^2\varphi]_a = \sup_{x, y \in H, x \neq y} \frac{||D^2\varphi(x) - D^2\varphi(y)||}{|x - y|^\alpha} < + \infty. \]

\( C^{2+a}_b (H) \) is a Banach space with the norm

\[ \| \varphi \|^{2+a} = \| \varphi \|_2 + [D^2\varphi]_a. \]

We will also need some notations and results on Interpolation Theory.

Let first recall the definition of interpolation space, see \([20]\). Let \( X, \| \cdot \|_X \) and \( Y, \| \cdot \|_Y \) be Banach spaces such that \( Y \subset X \) and

\[ \| y \|_X \leq c \| y \|_Y, \quad \forall y \in Y \]

for some constant \( c > 0 \).

Let \( \alpha \in ]0, 1[ \). We denote by \((X, Y)_{\alpha}, \) the real interpolation space consisting of all points \( x \in X \) such that

\[ \| x \|_{(X, Y)_{\alpha},} = \sup_{t > 0} t^{-\alpha} K(t, x, X, Y) < + \infty, \]

where

\[ K(t, x, X, Y) = \inf\{ \| a \|_X + t \| b \|_Y : x = a + b, a \in X, b \in Y \}. \]

\((X, Y)_{\alpha}, \) is a Banach space with norm \( \| \cdot \|_{(X, Y)_{\alpha}}. \)

It is easy to see that \( x \) belongs to \((X, Y)_{\beta}, \) if and only if for any \( t \in ]0, 1[ \) there exists \( a_t \in X, b_t \in Y \) and a constant \( C > 0 \) independent of \( t \), such that \( \| a_t \|_X \leq Ct^\beta \) and \( \| b_t \|_Y \leq Ct^{\beta - 1} \).

We also recall the following interpolation result, see \([2]\):

**Proposition 1.3.** For all \( \theta \in ]0, 1[ \) we have

\[ (C^\theta_b (H), C^1_b (H))_{\theta,} = C^\theta_b (H). \]
2. Regularity properties of $P_t$, $t \geq 0$

We first recall a result proved in [8].

**Theorem 2.1.** For all $t > 0$ and for all $\varphi \in C_b^\infty(H)$, $P_t \varphi \in C_b^\infty(H)$. In particular, for any $h, k \in H$, we have

$$
\langle DP_t \varphi(x), h \rangle = \int_H \langle A_t b, Q_t^{-1/2} y \rangle \varphi(e^{itA}x + y) \mathcal{R}(0, Q_t)(dy)
$$

and

$$
\langle D^2 P_t \varphi(x) h, k \rangle = \int_H \langle A_t b, Q_t^{-1/2} y \rangle \langle A_t k, Q_t^{-1/2} y \rangle \varphi(e^{itA}x + y) \mathcal{R}(0, Q_t)(dy) - \langle A_t b, A_t k \rangle P_t \varphi.
$$

**Remark 2.2.** By (2.1) and (2.2) the following estimates can be proved easily with the help of (1.7)

$$
\|DP_t \varphi(x)\| \leq vt^{-1/2} \|\varphi\|_0, \quad x \in H,
$$

$$
\|D^2 P_t \varphi(x)\| \leq (\sqrt{2} v^2 / t) \|\varphi\|_0, \quad x \in H.
$$

We will also need an estimate for the third derivative of $P_t \varphi$, that can be proved in a similar way

$$
\|D^3 P_t \varphi(x)\| \leq 2 \sqrt{6} v^3 t^{-3/2} \|\varphi\|_0, \quad x \in H.
$$

By Proposition 1.3 we easily obtain the following corollaries.

**Corollary 2.3.** For all $t > 0$, $\alpha \in ]0, 1[$, we have

$$
\|DP_t \varphi\|_\alpha \leq v^\alpha t^{-\alpha/2} \|\varphi\|_0, \quad \varphi \in C_b(H).
$$

**Corollary 2.4.** For all $t > 0$, $\theta \in ]0, 1[$, $\alpha \in ]0, 1[$, we have

$$
\|P_t \varphi\|_\theta \leq v^\alpha - \theta t^{(\theta - \alpha)/2} \|\varphi\|_\theta, \quad \varphi \in C_b^\theta(H).
$$

2.1. Existence of $\text{Tr}[D^2 P_t \varphi(x)]$

We show here that the linear operator $D^2 P_t \varphi(x)$ is trace-class for all $t > 0$ and $x \in H$.

**Proposition 2.5.** Let $\varphi \in C_b(H)$, $t > 0$ and $x \in H$. Then $D^2 P_t \varphi(x) \in \mathcal{L}_1(H)$ and

$$
\text{Tr}[D^2 P_t \varphi(x)] = \int_H |A_t^* Q_t^{-1/2} y|^2 \varphi(e^{itA}x + y) \mathcal{R}(0, Q_t)(dy) - \text{Tr}[A_t A_t^*] P_t \varphi(x).
$$

Moreover the following estimate holds

$$
\|D^2 P_t \varphi(x)\|_{\mathcal{L}_1(H)} \leq 2 \text{Tr}[A_t A_t^*] \|\varphi\|_0.
$$
PROOF. Since \( A_t \in \mathcal{L}_2(H) \) (see Remark 1.2) it is enough to show that the linear operator \( S_{t,x} \) defined as

\[
(S_{t,x}b, k) = \int_H \langle A_t b, Q_t^{-1/2}y \rangle \langle A_t k, Q_t^{-1/2}y \rangle \varphi(e^{tA}x + y) \mathcal{F}(0, Q_t)(dy), \quad b, k \in H,
\]

is trace-class for any \( t > 0 \) and \( x \in H \). For this it is enough to show, compare N. Dunford and J. T. Schwartz [10, Lemma 14(a), p. 1098], that there exists a constant \( C > 0 \) such that

\[
|\text{Tr}[NS_{t,x}]| \leq C\|N\|,
\]

for any symmetric positive operator \( N \in \mathcal{L}(H) \) of finite rank. To this purpose let \( \{e_j\} \) be a complete orthonormal system in \( H \). Then we have

\[
\text{Tr}[NS_{t,x}] = \sum_{j=1}^{\infty} \int_H \langle A_t e_j, Q_t^{-1/2}y \rangle \langle A_t N^* e_j, Q_t^{-1/2}y \rangle \varphi(e^{tA}x + y) \mathcal{F}(0, Q_t)(dy).
\]

It follows,

\[
|\text{Tr}[NS_{t,x}]| \leq \|\varphi\|_0 \text{Tr}[A_t^* A_t N] \leq \|\varphi\|_0 \|N\| \text{Tr}[A_t^* A_t].
\]

So (2.10) is fulfilled and we have proved that \( D^2 P, \varphi(x) \) is trace-class for any \( t > 0 \) and \( x \in H \). Moreover (2.8) and (2.9) follow setting \( N = I \) respectively in (2.11) and in (2.12).

REMARK 2.6. We want to describe in next example the behaviour of \( \gamma(t) = \text{Tr}[A_t A_t^*] \) near \( t = 0 \), in order to know whether it is integrable or not.

Assume that \( A \) is a negative self-adjoint operator, that \( Q = I \), and that there exists a complete orthonormal system \( \{e_k\} \) in \( H \) such that

\[
A e_k = -\lambda_k e_k, \quad \lambda_k \uparrow +\infty,
\]

with

\[
\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < +\infty.
\]

Then Hypothesis 1.1 is obviously fulfilled and we have

\[
Q_t = (e^{2tA} - 1)/(2A).
\]

It follows

\[
A_t A_t^* = 2A e^{2tA} / (e^{2tA} - 1), \quad t > 0,
\]

so that

\[
A_t = 2 \sum_{k=1}^{\infty} \lambda_k e^{-2t\lambda_k} / (1 - e^{-2t\lambda_k}) = (2/t) F(\lambda_k),
\]

where

\[
F(\xi) = \xi e^{-2\xi} / (1 - e^{-2\xi}), \quad \xi > 0.
\]
Let \( C_1 > 0, C_2 > 0 \) be such that
\[
C_1 e^{-3\xi} \leq F(\xi) \leq C_2 e^{-\xi}, \quad \xi > 0.
\]
So the behaviour of \( \gamma(t) \) near 0 is determined by
\[
\sum_{k=1}^{\infty} e^{-i\lambda_k t}.
\]
For instance if
\[
\lambda_k = k^{1+\alpha},
\]
where \( \alpha > 0 \), we have that \( \gamma(t) \) behaves at 0 as
\[
(1/t) \int_0^{+\infty} e^{-tx^{1+\alpha}} dx,
\]
and so as \( t^{-1-1/\alpha} \). In particular if \( \lambda_k = k^2 \) we have \( \gamma(t) \approx t^{-3/2} \).

2.2. ADDITIONAL REGULARITY RESULT WHEN \( \varphi \in C^1_b(H) \).

**Proposition 2.7.** Let \( \varphi \in C^1_b(H) \), \( t > 0 \) and \( x \in H \). Then we have

\[
(2.15) \quad \langle D^2 P_t \varphi(x) b, k \rangle = \int_{H} \langle \Lambda_i k, Q_t^{-1/2} y \rangle \langle D \varphi(e^{iA} x + y), e^{iA} b \rangle \mathcal{H}(0, Q_t)(dy).
\]

Moreover \( D^2 P_t \varphi(x) \in \mathcal{L}_1(H) \) and

\[
(2.16) \quad \text{Tr}[D^2 P_t \varphi(x)] = \int_{H} \langle A_i^* Q_t^{-1/2} y, e^{iA} D \varphi(e^{iA} x + y) \rangle \mathcal{H}(0, Q_t)(dy).
\]

Finally the following estimates hold

(2.17) \[ ||D^2 P_t \varphi(x)|| \leq (\nu/\sqrt{t}) ||\varphi||_1, \]

and

(2.18) \[ |\text{Tr}[D^2 P_t \varphi(x)]| \leq \{\text{Tr}[A_i A_i^*]\}^{1/2} ||\varphi||_1. \]

**Proof.** Let \( t > 0, x \in H \). Since
\[
\langle DP_t \varphi(x), b \rangle = \int_{H} \langle D \varphi(e^{iA} x + y), e^{iA} b \rangle \mathcal{H}(0, Q_t)(dy),
\]
(2.15) follows easily by differentiating (2.1). Moreover (2.16) is an immediate consequence of (2.15), recalling that, by Proposition 2.5, \( D^2 P_t \varphi(x) \) is trace-class.

We prove now (2.17). By (2.15), using Hölder's estimate, it follows
\[
|\langle D^2 P_t \varphi(x) b, k \rangle|^2 \leq ||\varphi||_2^2 ||b||^2 = \int_{H} |\langle A_i k, Q_t^{-1/2} y \rangle|^2 \mathcal{H}(0, Q_t)(dy) = ||\varphi||_2^2 ||b||^2 ||A_i k||^2 \leq (1/t) ||\varphi||_2^2 ||b||^2 ||k||^2,
\]
and (2.17) is proved. We prove finally (2.18). We have, using again Hölder’s estimate
\[
|\text{Tr} [D^2 P_t, \varphi(x)]|^2 \leq \|\varphi\|^2 \int_H |e^{tA} A_t^* Q_t^{-1/2} y|^2 \mathcal{F}(0, Q_t)(dy) = \\
\|\varphi\|^2 \text{Tr}[A_t e^{tA} e^{tA^*}] = \|\varphi\|^2 \text{Tr}[A_t A_t^*].
\]
The proof is complete. □

In a similar way we prove the following result.

**Proposition 2.8.** Let \( \varphi \in C^1_b(H) \), \( t > 0 \) and \( x \in H \). Then for all \( h, k, l \in H \) we have
\[
D^3 P_t \varphi(x)(h, k, l) = \int_H \langle A_t h, Q_t^{-1/2} y \rangle \langle A_t k, Q_t^{-1/2} y \rangle \\
\cdot \langle D\varphi(e^{tA} x + y), e^{tA} f \rangle \mathcal{F}(0, Q_t)(dy) - \langle A_t h, A_t k \rangle \langle D\varphi(x), l \rangle.
\]
Moreover the following estimate holds
\[
\|D^3 P_t \varphi(x)\| \leq (\sqrt{2} \sqrt{t}) \|\varphi\|_1.
\]

By interpolation we obtain the following results.

**Corollary 2.9.** Let \( \theta \in [0, 1[ \), \( \varphi \in C^1_b(H) \), \( t > 0 \) and \( x \in H \). Then we have
\[
\|D^3 P_t \varphi(x)\| \leq 2^{(1-\theta)/2} \sqrt{2-\theta} t^{\theta/2 - 1} \|\varphi\|_\theta,
\]
and
\[
\text{Tr} [D^2 P_t, \varphi(x)] \leq 2^{1-\theta} \left\{ \text{Tr} [A_t^* A_t] \right\}^{1-\theta/2} \|\varphi\|_\theta.
\]

**Corollary 2.10.** Let \( \theta \in [0, 1[ \), \( \varphi \in C^1_b(H) \), \( t > 0 \) and \( x \in H \). Then we have
\[
\|D^3 P_t \varphi\|_{\alpha} \leq 2^{(1-\alpha-\theta)/2} \sqrt{2+\alpha - \theta} t^{(\theta-\alpha)/2 - 1} \|\varphi\|_\theta.
\]

**Remark 2.11.** Assume that
\[
\gamma(t) \leq C t^{-3/2}.
\]
Then by (2.22) we have
\[
\text{Tr} [D^2 P_t, \varphi(x)] \leq 2^{1-\theta} C^{1-\theta/2} t^{-3/2 + 3\theta/4} \|\varphi\|_\theta.
\]
Thus \( \text{Tr}[D^2 P_t \varphi(x)] \) is integrable near 0 provided \( \theta > 2/3 \).

## 2.3. Kolmogorov Equation

We want to show here that if \( \varphi \in C_b(H) \) then for \( t > 0 \) the function \( u(t, x) = P_t \varphi(x) \) is a solution to the Kolmogorov equation
\[
u_t(t, x) = (1/2) \text{Tr} [D^2 u(t, x)] + \langle Ax, Du(t, x) \rangle,
\]
\( t > 0, x \in D(A). \)
If \(Du(t, x) \in D(A^*)\) we can write (2.24) as
\[
(2.25) \quad u_t(t, x) = \frac{1}{2} \text{Tr} [D^2 u(t, x)] + \langle x, A^* Du(t, x) \rangle, \quad t > 0, x \in H.
\]

**Proposition 2.12.** Let \(\varphi \in C_b(H), t > 0\) and \(x \in D(A)\). Then \(u(t, x) = P_t \varphi(x)\) is a solution to the Kolmogorov equation (2.25).

**Proof.** Let \(u(t, x) = P_t \varphi(x), t > 0, x \in H\). Then the term \(\text{Tr}[D^2 u(t, x)]\) is well defined by Proposition 2.5. Moreover also the term \(\langle x, A^* Du(t, x) \rangle\) is well defined, since, by (2.1) we have
\[
(2.26) \quad \langle x, A^* Du(t, x) \rangle = \int_{\mathcal{H}} \langle Q_t^{-1/2} e^{(t/2)A} A e^{(t/2)A} e^{-iA_y}, Q_t^{-1/2} y \rangle \varphi(e^{iA_y}x + y) \mathcal{R}(0, Q_t)(dy).
\]
By (2.26) we have
\[
(2.27) \quad |\langle x, A^* Du(t, x) \rangle| \leq K(t) \|\varphi\|_0 |x|,
\]
where
\[
(2.28) \quad K^2(t) = \|Q_t^{-1/2} e^{(t/2)A} A e^{(t/2)A} \|.
\]
It remains to show that \(u(t, x)\) is differentiable in \(t\) and that (2.25) holds. To this aim let us introduce the space of all exponential functions \(\mathcal{B}(H)\). We denote by \(\mathcal{B}(H)\) the linear subspace of \(C_b(H)\) spanned by all \(\zeta_b, b \in H\):
\[
\zeta_b(x) = e^{i(x, b)}, \quad x \in H.
\]
Since, as easily checked
\[
(2.29) \quad P_t \zeta_b(x) = e^{i(e^{it}x, b) - (1/2)\langle Q_t, b \rangle}, \quad x \in H,
\]
then the proposition holds when \(\varphi \in \mathcal{B}(H)\).

Let now \(\{\varphi_n\}\) be a sequence in \(\mathcal{B}(H)\) such that
\[
(i) \quad \lim_{n \to \infty} \varphi_n(x) = \varphi(x), \quad \forall x \in H,
\]
\[
(ii) \quad \|\varphi_n\|_0 \leq 2 \|\varphi\|_0,
\]
and set \(u_n(t, x) = P_t \varphi_n(x), t \geq 0, x \in H\). We fix now \(t > 0\). By (2.3)-(2.5) it follows that the sequence of functions \(\{u_n(t, \cdot)\}\) has all derivatives of order less than 3, bounded. This implies that
\[
\lim_{n \to \infty} u_n(t, \cdot) = u(t, \cdot), \quad \text{in} \ C_b^2(H),
\]
uniformly in \(t\) on compact subsets of \([0, + \infty[. \) Moreover by (2.9) it follows that the sequence in \(C_b(H)\) defined by \(\{\text{Tr}[D^2 u_n(t, \cdot)]\}\) is bounded, so that
\[
\lim_{n \to \infty} \text{Tr}[D^2 u_n(t, \cdot)] = \text{Tr}[D^2 u(t, \cdot)], \quad \text{in} \ C_b(H),
\]
uniformly in \(t\) on compact subsets of \([0, + \infty[. \) Finally from (2.27) it follows that
\[
\lim_{n \to \infty} \langle x, A^* Du_n(t, x) \rangle = \langle x, A^* Du(t, x) \rangle, \quad x \in H,
\]
uniformly in \(t\) on compact subsets of \([0, + \infty[\) and in \(x\) on bounded subsets of \(H\). This
implies that for any $x \in H$
\[
\lim_{n \to \infty} \frac{d}{dt} u_n(t, x) = \frac{d}{dt} u(t, x),
\]
for all $x \in H$ uniformly in $t$ on compact subsets of $]0, +\infty[$ and the conclusion follows. •

3. THE INFINITESIMAL GENERATOR

We proceed here as in [4], by introducing the Laplace transform of $P_t$, $t \geq 0$. For any $\lambda > 0$ we set
\[
F(\lambda) \varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t \varphi(x) dt, \quad x \in H, \varphi \in C_b(H).
\]
Note that the above integral is convergent for any fixed $x \in H$ and not in $C_b(H)$ in general. In [4] is shown that $F(\lambda)$ maps $C_b(H)$ into itself and that it is one-to-one. So there exists a unique closed operator $\mathfrak{M}$ in $C_b(H)$:
\[
\mathfrak{M}: D(\mathfrak{M}) \subset C_b(H) \mapsto C_b(H),
\]
such that the resolvent set $\rho(\mathfrak{M})$ of $\mathfrak{M}$ contains $]0, +\infty[$ and
\[
\mathfrak{M} \text{ is called the } \text{infinitesimal generator} \text{ of the semigroup } P_t, \quad t \geq 0.
\]
Let $\lambda > 0$, $g \in C_b(H)$ and set $\varphi = R(\lambda, \mathfrak{M}) g$. Then $\varphi$ is called a \textit{generalized solution} to the equation
\[
\lambda \varphi - (1/2) \text{Tr} [D^2 \varphi] - \langle Ax, D\varphi \rangle = g.
\]
It is also useful to introduce the concept of \textit{strict solution}. To this purpose we have to introduce a suitable restriction $\mathfrak{M}_0$ of $\mathfrak{M}$.

By definition the domain $D(\mathfrak{M}_0)$ of $\mathfrak{M}_0$ is the set of all functions $\varphi \in C_b(H)$ such that

(i) $\varphi \in C^2_b(H)$ and $D^2 \varphi(x) \in \mathcal{L}_1(H)$ for all $x \in H$.

(ii) $D\varphi(x) \in D(A^*)$ and the mapping
\[
H \mapsto R, \quad x \mapsto A^* D\varphi(x),
\]
belongs to $C_b(H)$.

Then we define the operator $\mathfrak{M}_0$ by setting
\[
\mathfrak{M}_0 \varphi = (1/2) \text{Tr} [D^2 \varphi] + \langle x, A^* D\varphi \rangle, \quad \forall \varphi \in D(\mathfrak{M}_0).
\]

Remark 3.1. In the paper [5], it is proved that the operator $\mathfrak{M}$ is the closure of $\mathfrak{M}_0$ with respect to the $\mathfrak{X}$-convergence. A sequence $\{\varphi_n\} \subset C_b(H)$ is said to be $\mathfrak{X}$-convergent to $\varphi \in C_b(H)$ if

(i) $\sup_{n \in N} \|\varphi_n\|_0 < +\infty$. 

For any compact subset $K$ in $H$, we have
\[ \limsup_{n \to \infty} \sup_{x \in K} |\varphi(x) - \varphi_n(x)| = 0. \]

From the regularity results of the semigroup $P_t$, $t \geq 0$, obtained in the previous section, one gets the following regularity results for the resolvent of $\mathcal{M}$.

**Proposition 3.2.** Let $\lambda > 0$, $g \in C_b(H)$, and set $\varphi = R(\lambda, \mathcal{M})g$. Then the following statements hold

(i) $\varphi \in C_b^1(H)$ and

\[ |D\varphi(x)| \leq \Gamma(1/2) \cdot \lambda^{-1/2} \|g\|_0, \quad x \in H, \]

where $\Gamma$ denotes the gamma Euler function.

(ii) For any $\alpha \in [0, 1[$, we have $\varphi \in C_b^{1+\alpha}(H)$ and

\[ |D\varphi|_{\alpha} \leq 2^{\alpha/2} \Gamma((1 - \alpha)/2) \cdot \lambda^{(1/2 - \alpha)/2} \|g\|_0, \]

(iii) If $g \in C_b^\theta(H)$ for some $\theta \in [0, 1[$, then $\varphi \in C_b^\theta(H)$, and

\[ \|D^2\varphi(x)\| \leq 2^{(1 - \theta)/2} \Gamma(\theta/2) \lambda^{-\theta/2} \|g\|_0, \quad x \in H. \]

(iv) If $g \in C_b^\theta(H)$ for some $\theta \in [0, 1[$, and if in addition $\gamma(t)^{1 - \theta/2}$ is integrable near 0, then $\varphi \in D(\mathcal{M}_0)$ and so it is a strict solution to equation (3.3).

**Remark 3.3.** If $\gamma(t) \leq C t^{-3/2}$, for some constant $C > 0$. Then condition (iv) is fulfilled provided $g \in C_b^\theta(H)$ with $\theta > 2/3$, see Remark 2.11.

## 3.1. Interpolation Spaces $D_{\mathcal{M}}(\theta, \infty)$

The semigroup $P_t$, $t \geq 0$ is not strongly continuous in $C_b(H)$, even when $H$ is finite-dimensional, see [4, 7]. The following proposition, proved in [6], gives a characterization of the maximal subspace $\mathcal{Y}$ of $C_b(H)$ where $P_t$, $t \geq 0$ is strongly continuous.

**Proposition 3.4.** Let $\varphi \in C_b(H)$. Then the following statements are equivalent

(i) $\lim_{t \to 0} P_t \varphi = \varphi$ in $C_b(H)$.

(ii) $\lim_{t \to 0} \varphi(e^{tA}x) = \varphi(x)$ in $C_b(H)$.

We shall set
\[ \mathcal{Y} = \{ \varphi \in C_b(H) : \lim_{t \to 0} \varphi(e^{tA}x) = \varphi(x) \text{ in } C_b(H) \}, \]

and for any $\theta \in [0, 1[$
\[ \mathcal{Y} = \{ \varphi \in C_b(H) : \exists C > 0, |\varphi(e^{tA}x) - \varphi(x)| \leq Ct^\theta, \forall x \in H \}. \]

We want now to characterize the interpolation spaces $(C_b(H), D(\mathcal{M}))_{\theta, \infty}$ that we shall denote by $D_{\mathcal{M}}(\theta, \infty)$. We need some preliminary result.
**Proposition 3.5.** Let \( \varphi \in C_b(H) \) and \( \theta \in ]0, 1[ \). Then the following statements hold.

(i) If \( \varphi \in D_{\mathfrak{M}}(\theta, \infty) \) then we have

\[
\sup_{\lambda > 0} \lambda^\theta \|\mathfrak{M} R(\lambda, \mathfrak{M}) \varphi\|_0 < +\infty .
\]

(ii) If \( \varphi \in C_b(H) \) and fulfills (3.8) then \( \varphi \in D_{\mathfrak{M}}(\theta, \infty) \).

**Proof.** (i) Let \( \varphi \in D_{\mathfrak{M}}(\theta, \infty) \). Then by the definition of interpolation space given in §1, for any \( t \in [0, 1] \) there exist \( \alpha_t \in C_b(H) \), \( \beta_t \in D(\mathfrak{M}) \), such that \( \varphi = \alpha_t + \beta_t \), and

\[
\|\alpha_t\|_0 \leq C t^\theta , \quad \|\beta_t\|_0 \leq C t^{\theta-1},
\]

for some \( C > 0 \). Now for any \( \lambda > 0 \) we have

\[
\mathfrak{M} R(\lambda, \mathfrak{M}) \varphi = \mathfrak{M} R(\lambda, \mathfrak{M})\alpha_{1/\lambda} + R(\lambda, \mathfrak{M}) \mathfrak{M} \beta_{1/\lambda}.
\]

It follows

\[
\|\mathfrak{M} R(\lambda, \mathfrak{M}) \varphi\|_0 \leq C \|\mathfrak{M} R(\lambda, \mathfrak{M})\|_0^{\lambda^{-\theta}} + C \|R(\lambda, \mathfrak{M})\|_0^{\lambda^{1-\theta}} \leq 3 \lambda^{-\theta},
\]

and the statement is proved.

(ii) Assume that \( \varphi \) fulfills (3.8). Define

\[
C_1 = \sup_{\lambda > 0} \lambda^\theta \|\mathfrak{M} R(\lambda, \mathfrak{M}) \varphi\|_0 ,
\]

and set

\[
\alpha_t = -\mathfrak{M} R((1/t), \mathfrak{M}) \varphi \cdot \beta_t = (1/t) R((1/t), \mathfrak{M}) \varphi .
\]

Then we have \( \alpha_t + \beta_t = \varphi \) and

\[
\|\alpha_t\|_0 \leq C_1 t^\theta , \quad \|\beta_t\|_0 \leq t^{1-\theta} , \quad t > 0 ,
\]

so that \( \varphi \in D_{\mathfrak{M}}(\theta, \infty) \). ■

**Lemma 3.6.** Let \( \theta \in ]0, 1/[ \), \( T > 0 \), \( \varphi \in C_b^{2\theta}(H) \). Then there exists \( C_T > 0 \) such that

\[
|G_t \varphi(x) - \varphi(x)| \leq C_T [\text{Tr}(Q_t)]^{\theta} [\varphi]_{2\theta}, \quad t \in [0, T].
\]

**Proof.** We have

\[
|G_t \varphi(x) - \varphi(x)| \leq \int_I |\varphi(x+y) - \varphi(x)| \mathcal{R}(0, Q_t)(dy) \leq
\]

\[
\leq [\varphi]_{2\theta} \int_I |y|^{2\theta} \mathcal{R}(0, Q_t)(dy) \leq D_\theta [\varphi]_{2\theta} [\text{Tr}(Q_t)]^{\theta},
\]

for some constant \( D_\theta \).

Now the conclusion follows. ■

**Lemma 3.7.** Let \( \theta \in [1/2, 1[ \), \( T > 0 \), \( \varphi \in C_b^{2\theta}(H) \). Then there exists \( C_{1,T} > 0 \) such that

\[
|G_t \varphi(x) - \varphi(x)| \leq C_{1,T} [\text{Tr}(Q_t)]^{\theta} [\varphi]_{2\theta}, \quad t \in [0, T].
\]
PROOF. We have
\[ G_t \varphi(x) - \varphi(x) = \int_{\mathcal{H}} [\varphi(x + y) - \varphi(x)] \mathcal{H}(0, Q_t)(dy) = \]
\[ = \int_0^1 \int_{\mathcal{H}} (D\varphi(x + \xi y) - D\varphi(x), y) \mathcal{H}(0, Q_t)(dy) d\xi. \]
It follows
\[ |G_t \varphi(x) - \varphi(x)| \leq \|\varphi\|_2^\theta \int_0^1 \|y\|^{2\theta - 1} \mathcal{H}(0, Q_t)(dy) d\xi, \]
and the conclusion follows as in the previous lemma.

**Proposition 3.8.** If \( \varphi \in D_{\mathcal{M}}(\theta, \infty), \theta \in \mathbb{R}, \) \( \theta \in \mathbb{R}, \) there exists \( C_T > 0 \) such that
\[
(3.11) \quad \|P_t \varphi - \varphi\| \leq C_T t^\theta, \quad t \in [0, T].
\]

**Proof.** Let \( \varphi \in D_{\mathcal{M}}(\theta, \infty). \) Then for any \( t > 0 \) there exists \( \alpha, \beta \in D(\mathcal{M}) \) such that \( \varphi = \alpha + \beta, \)
\[
(3.12) \quad \|\alpha\|_0 \leq Ct^\theta, \quad \|\beta\|_0 \leq Ct^{-1},
\]
for some constant \( C > 0. \) Since
\[
P_t \varphi - \varphi = (P_t \alpha_t - \alpha_t) + (P_t \beta_t - \beta_t) = (P_t \alpha_t - \alpha_t) + \int_0^t P_s \mathcal{M} b(t) ds,
\]
using (3.12), we find that (3.11) holds.

We can now prove the result

**Theorem 3.9.** For all \( \theta \in \mathbb{R}, \theta \in \mathbb{R}, \) \( \theta \in \mathbb{R}, \) we have
\[
(3.13) \quad D_{\mathcal{M}}(\theta, \infty) \subset C^2(\mathcal{H}) \cap \mathcal{F}_\theta.
\]

**Proof.** We only consider the case \( \theta \in \mathbb{R}, \theta \in \mathbb{R}, \) since the case \( \theta \in \mathbb{R}, \) can be treated in an analogous way.

**Step 1.** If \( \varphi \in D_{\mathcal{M}}(\theta, \infty) \) then there exists \( C_1 > 0 \) such that for all \( \lambda \geq 1 \) we have
\[
(3.14) \quad \|\lambda DR(\lambda, \mathcal{M}) \varphi\| \leq C_1 \lambda^{1/2 - \theta} \|\varphi\|_{D_{\mathcal{M}}(\theta, \infty)}.
\]

We first note that, since
\[
\frac{d}{d\lambda} [\lambda R(\lambda, \mathcal{M})] = R(\lambda, \mathcal{M}) - \lambda (R(\lambda, \mathcal{M}))^2,
\]
we have
\[
\lambda R(\lambda, \mathcal{M}) \varphi = R(1, \mathcal{M}) \varphi + \int_1^\lambda R(s, \mathcal{M})(1 - s R(s, \mathcal{M})) \varphi ds =
\]
\[
= R(1, \mathcal{M}) \varphi + \int_1^\lambda R(s, \mathcal{M}) \mathcal{M} R(s, \mathcal{M}) \varphi ds.
\]
By Proposition 3.2 (i) it follows
\[ D_x \lambda R(\lambda, \mathcal{M}) \varphi = D_x R(1, \mathcal{M}) \varphi + \int_1^{\lambda} D_x [R(s, \mathcal{M}) \mathcal{M} R(s, \mathcal{M}) \varphi] \, ds. \]

Moreover, taking into account (3.5) and (3.8), we find
\[ \|R(\lambda, \mathcal{M}) \varphi\|_1 \leq C\lambda^{-1/2} \|\varphi\|_0, \quad \forall \lambda > 0, \]
we get
\[ \|R(\lambda, \mathcal{M}) \varphi\|_1 \leq C\|\varphi\|_0 + C \int_1^{\lambda} s^{-1/2 - \theta} [\varphi]_{2\mathcal{M}(\lambda, \mathcal{M})} \, ds = \]
\[ = C\|\varphi\|_0 + C/(1/2 - \theta)(\lambda^{1/2 - \theta} - 1)[\varphi]_{2\mathcal{M}(\lambda, \mathcal{M})}, \]
for some \( C > 0. \)

**Step 2.** \( D_{\mathcal{M}}(\theta, \infty) \subset C_b^{2\theta}(H). \)

Let \( x, y \in H \) such that \( |x - y| \leq 1, \) and let \( \lambda \geq 1. \) Then we have by (3.8) and (3.14),
\[ |\varphi(x) - \varphi(y)| \leq |\varphi(x) - \lambda R(\lambda, \mathcal{M}) \varphi(x)| + |\lambda R(\lambda, \mathcal{M}) \varphi(x) - \lambda R(\lambda, \mathcal{M}) \varphi(y)| + \]
\[ + |\lambda R(\lambda, \mathcal{M}) \varphi(y) - \varphi(y)| \leq 2[\varphi]_{2\mathcal{M}(\lambda, \mathcal{M})} \lambda^{-\theta} + ||D(\mathcal{M} R(\lambda, \mathcal{M}) \varphi)||_0 |x - y| \leq \]
\[ \leq 2[\varphi]_{2\mathcal{M}(\lambda, \mathcal{M})} \lambda^{-\theta} + C||\varphi||_{2\mathcal{M}(\lambda, \mathcal{M})} (\lambda^{1/2 - \theta} + 1)|x - y|. \]
Choosing \( \lambda = |x - y|^{-2} \) we have
\[ |\varphi(x) - \varphi(y)| \leq 2[\varphi]_{2\mathcal{M}(\lambda, \mathcal{M})} |x - y|^{2\theta} + C||\varphi||_{2\mathcal{M}(\lambda, \mathcal{M})} (|x - y|^{2\theta} + |x - y|), \]
and the conclusion follows easily.

**Step 3.** \( D_{\mathcal{M}}(\theta, \infty) \subset Y_\theta. \)

Let \( \varphi \in D_{\mathcal{M}}(\theta, \infty). \) Then we have
\[ \|\varphi(e^{iB} x) - \varphi(x)\| \leq \|\varphi(e^{iB} x) - G_t \varphi(e^{iB} x)\| + \|P_t \varphi(x) - \varphi(x)\|. \]
Since \( \varphi \in C_b^{2\theta}(H) \) by (3.9) we find
\[ |\varphi(e^{iB} x) - G_t \varphi(e^{iB} x)| \leq C t^{\theta} [\varphi]_{2\theta}, \quad t \in [0, T]. \]
Moreover from (3.11) it follows
\[ \|P_t \varphi - \varphi\|_0 \leq C_T t^{\theta}, \quad t \in [0, T]. \]
Substituting (3.16) and (3.17) into (3.15) we get finally
\[ |\varphi(e^{iB} x) - \varphi(x)| \leq (C + C_T) t^{\theta} [\varphi]_{2\theta}, \]
and the proof of the theorem is complete. ■

4. Maximal regularity results for elliptic equations

The following result is proved in [3]. We give a sketch of the proof for the reader convenience.
PROPOSITION 4.1. Assume that $\theta \in ]0, 1[,$ $g \in C^0_b(H),$ and $\lambda > 0.$ Then the function $\phi = R(\lambda, \mathcal{M})g$ belongs to $C^{2+\theta}_b(H).$

PROOF. The proof is based on a general interpolation argument due to A. Lunardi see [16], in particular on the following inclusion result

\[(C^a_b(H), C^{2+a}_b(H))_{1-\frac{(a-\theta)}{2}} \subset C^{2+\theta}_b(H),\]

for any $a \in ]0, 1[.$ Consequently, in order to prove the theorem it will be enough to show that for some $a \in ]0, 1[,$ we have

\[(4.2) \phi \in (C^a_b(H), C^{a+\theta}_b(H))_{1-\frac{(a-\theta)}{2}}.\]

To prove (4.2) we set

$$\phi(x) = a(t,x) + b(t,x),$$

where

$$a(t,x) = \int_0^t e^{-\lambda s} P_s g(x) ds,$$

and

$$b(t,x) = \int_t^\infty e^{-\lambda s} P_s g(x) ds.$$

Then from (2.7) it follows that

$$\|a(\cdot, t)\|_a \leq C(\alpha, \theta) \int_0^t e^{-\lambda s} \sigma^{-(a-\theta)/2} ds \|g\|_\theta \leq C(\alpha, \theta) t^{1-(a-\theta)/2} \|g\|_\theta,$$

and from (2.8) that

$$\|b(\cdot, t)\|_{2+a} \leq C(\alpha, \theta) \int_1^\infty e^{-\lambda s} \sigma^{-(a-\theta)/2} - 1 ds \|g\|_\theta \leq C(\alpha, \theta) \frac{t^{\theta-a}}{\alpha-\theta} \|g\|_\theta.$$}

This implies (4.2). \(\blacksquare\)

By Proposition 4.1 and 3.2 (iv) we find the result.

THEOREM 4.2. Assume that $\theta \in ]0, 1[,$ $g \in C^0_b(H),$ $\lambda > 0,$ and in addition that

\[(4.3) \int_0^\infty [\text{Tr}(A_t A_t^*)]^{1-\theta/2} dt < +\infty.\]

Then, setting $\phi = R(\lambda, \mathcal{M})g,$ the following statements hold.
(i) $\varphi \in C_b^{2+\theta}(H)$ and $D^2 \varphi(x) \in \mathcal{L}_1(H)$ for any $x \in H$.

(ii) $\text{Tr}[D^2 \varphi(\cdot)] \in C_b(H)$.

(iii) $x \mapsto \langle x, A^* D \varphi \rangle \in C_b(H)$.

Moreover

\begin{equation}
\lambda \varphi(x) - \left(1/2\right) \text{Tr}[D^2 \varphi(x)] - \langle x, A^* D \varphi \rangle = g(x),
\end{equation}

for all $x \in H$.

**Remark 4.3.** Let us consider the restriction $P_t^\theta$, $t \geq 0$ of the semigroup $P_t$, $t \geq 0$ to $C_b^\theta(H)$, $\theta \in ]0, 1[$. We can still define the infinitesimal generator $\mathfrak{m}^\theta$ of $P_t$, $t \geq 0$ to $C_b^\theta(H)$ by the Laplace, transform setting

\begin{equation}
R(\lambda, \mathfrak{m}^\theta) \varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t^\theta \varphi(x) dt.
\end{equation}

It is easy to check that $\mathfrak{m}^\theta$ is the part of $\mathfrak{m}$ in $C_b^\theta(H)$:

$$
D(\mathfrak{m}^\theta) = \{ \varphi \in D(\mathfrak{m}) \cap C_b^\theta(H) : \mathfrak{m} \varphi \in C_b^\theta(H) \}.
$$

Theorem 4.2 enable us to characterize, under suitable assumptions, the domain of $M^\theta$.

We have

$$
D(\mathfrak{m}^\theta) = \{ \varphi \in C_b^{2+\theta}(H) : \langle A^*, D \varphi \rangle \in C_b(H) \}.
$$

If $H$ is finite-dimensional this characterization of $D(\mathfrak{m}^\theta)$ was obtained in [7].

Under the hypotheses of Theorem 4.2 we can give the following definition of $D(\mathfrak{m}^\theta)$

\begin{equation}
D(\mathfrak{m}^\theta) = \{ \varphi \in C_b^{2+\theta}(H) : D^2 \varphi(x) \in \mathcal{L}_1(H), \forall x \in H, \text{Tr}[D^2 \varphi(x)] \in C_b(H), \langle A^*, D \varphi \rangle \in C_b(H) \}.
\end{equation}

### 5. Maximal regularity results for parabolic equations

We are here concerned with the initial value problem

\begin{equation}
\begin{cases}
\frac{du(t, x)}{dt} = (1/2) \text{Tr}[D^2 u(t, x)] + \langle Ax, Du(t, x) \rangle + F(t, x), & t \in ]0, T], \ x \in H, \\
u(0, x) = \varphi(x),
\end{cases}
\end{equation}

where $F \in C([0, T]; C_b(H))$ and $\varphi \in C_b(H)$.

Following S. Cerrai and F. Gozzi [5], we call the function $u : [0, T] \times H \to \mathbb{R}$ defined as

\begin{equation}
u(t, x) = P_t \varphi(x) + \int_0^t P_{t-s} F(s, \cdot)(x) ds = u_1(t, x) + u_2(t, x),
\end{equation}

the mild solution to (5.1). Several properties of the mild solution $u$ are described in the quoted paper [5]. Here we will discuss only some new maximal regularity results for $u_1$ and $u_2$. Concerning $u_1$ we have the following proposition.
Proposition 5.1. The following statements are equivalent

(i) $u_1 \in C^\theta ([0, T]; C_b (H))$.

(ii) $\varphi \in D_\mathcal{\mathcal{M}} (\theta, \infty)$.

Proof. $(i) \Rightarrow (ii)$. It is enough to show that

$$
\sup_{\lambda > 0} \lambda^\theta \| \mathcal{M} R(\mathcal{M}, \lambda) \varphi \|_0 < +\infty .
$$

In fact, by Proposition 3.5, if (5.3) holds, we have $\varphi \in D_\mathcal{\mathcal{M}} (\theta, \infty)$, and by Theorem 3.9 this implies $(ii)$. By hypothesis $(i)$ there exists $K > 0$ such that

$$
|P_t \varphi (x) - \varphi (x)| \leq K t^\theta, \quad t \in [0, T].
$$

It follows

$$
|\mathcal{M} R(\mathcal{M}) \varphi (x)| \leq K \lambda \int_0^\infty e^{-\lambda t} t^\theta \mathrm{d} t \leq \frac{K (\theta + 1)}{\lambda^\theta},
$$

and (5.3) holds.

$(ii) \Rightarrow (i)$. Let $t > s \geq 0$. Then by Proposition 3.8, we have

$$
|P_t \varphi (x) - P_s \varphi (x)| \leq |P_{t-s} \varphi (x) - \varphi (x)| \leq C T |t-s|^\theta,
$$

and $(i)$ is proved. $lacksquare$

We conclude this section, by studying the regularity of $u_2$.

Theorem 5.2. Let $F \in C([0, T]; C_b (H))$, and assume that, for some $\theta \in ]0, 1[$, we have $F(t, \cdot) \in C_b^\theta (H)$ and that

$$
\sup_{t \in [0, T]} \| F(t, \cdot) \|_\theta < +\infty.
$$

Then $u \in C([0, T]; C_b (H))$, $u(t, \cdot) \in C_b^{2+\theta} (H)$ and

$$
\sup_{t \in [0, T]} \| u(t, \cdot) \|_{2+\theta} < +\infty.
$$

Proof. We fix $t > 0$. Arguing as in the proof of Proposition 4.1 it is enough to prove that

$$
u(t, \cdot) \in \left( C_b^a (H), C_b^{2+\alpha} (H) \right)_{1-(\alpha-\theta)/2, \infty},
$$

for some $\alpha \in ]\theta, 1[$. To this purpose we set

$$
a(t, \tau, x) = \int_0^\tau P_s u(t-s, \cdot) (x) \mathrm{d} s, \quad \tau \in [0, t],
$$

$$
b(t, \tau, x) = \int_\tau^t P_s u(t-s, \cdot) (x) \mathrm{d} s, \quad \tau \in [0, t].
$$
Then by (2.7) we have
\[
\|\mu(t, \tau, \cdot)\|_a \leq C(\alpha, \theta) \sup_{s \in [0, T]} \|\mu(s, \cdot)\|_\theta \int_0^t \frac{ds}{s^{(\alpha - \theta)/2}} \leq \frac{C(\alpha, \theta)}{1 - (\alpha - \theta)/2} \sup_{s \in [0, T]} \|\mu(s, \cdot)\|_\theta \tau^{1 - (\alpha - \theta)/2}.
\]
Moreover by (2.8) we have
\[
\|\rho(t, \tau, \cdot)\|_2 + \alpha \leq C(\alpha, \theta) \sup_{s \in [0, T]} \|\mu(s, \cdot)\|_\theta \int_0^t \frac{ds}{s^{(\alpha - \theta)/2 + 1}} \leq \frac{2C(\alpha, \theta)}{(\alpha - \theta)/2} \sup_{s \in [0, T]} \|\mu(s, \cdot)\|_\theta \tau^{-(\alpha - \theta)/2}.
\]
This implies (5.7). 

**References**


Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 Pisa