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The parabolic mixed Cauchy-Dirichlet problem in spaces of functions which are hölder continuous with respect to space variables

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Equazioni a derivate parziali. — *The parabolic mixed Cauchy-Dirichlet problem in spaces of functions which are hölder continuous with respect to space variables.* Nota di DAVIDE GUIDETTI, presentata (*) dal Corrisp. G. Da Prato.

ABSTRACT. — We give a new proof, based on analytic semigroup methods, of a maximal regularity result concerning the classical Cauchy-Dirichlet's boundary value problem for second order parabolic equations. More specifically, we find necessary and sufficient conditions on the data in order to have a strict solution u which is bounded with values in $C^{2+\theta}(\overline{\Omega})$ $(0 < \theta < 1)$, with $\partial_t u$ bounded with values in $C^{\theta}(\overline{\Omega})$.

KEY WORDS: Parabolic equations; Cauchy-Dirichlet problem; Maximal regularity; Analytic semigroups.

RIASSUNTO. — *Il problema misto di Cauchy-Dirichlet per equazioni paraboliche in spazi di funzioni hölderiane.* Si dà una nuova dimostrazione, basata su metodi di semigruppi analitici, di un risultato di regolarità massimale per il classico problema al contorno di Cauchy-Dirichlet per equazioni paraboliche del secondo ordine. Più specificamente, si trovano condizioni necessarie e sufficienti sui dati per avere una soluzione stretta u che sia limitata a valori in $C^{2+\theta}(\overline{\Omega})$ con $\partial_t u$ limitata a valori in $C^{\theta}(\overline{\Omega})$.

INTRODUCTION

Let $\mathcal{A} = \mathcal{A}(x, \partial_x)$ be a second order strongly elliptic operator in a domain Ω of $\boldsymbol{R}^n$ with conveniently smooth boundary; consider the linear parabolic operator $L := \partial_t - \mathcal{A}$ and the corresponding mixed Cauchy-Dirichlet problem in the cylinder $Q := [0, T] \times \times \overline{\Omega}$

$$(1) \qquad \begin{cases} Lu(t,x) = f(t,x), (t,x) \in Q\,, \\ u(t,x') = g(t,x'), (t,x') \in \Gamma\,, \\ u(0,x) = u_0(x), x \in \overline{\Omega}\,, \end{cases}$$

where we have indicated with $\partial\Omega$ the topological boundary of Ω and with Γ the product $[0, T] \times \partial\Omega$. We are interested in the existence and uniqueness of strict solutions of (1), that is, of solutions which are continuous in Q together with their first derivate with respect to t and their first and second order derivatives with respect to x. Connected with this, there are well known theorems of optimal regularity, giving necessary and sufficient conditions (under suitable assumptions on Ω and the regularity of the coefficients of $\mathcal{A}$) on the data f, g and u_0 in order to have a solution u whose first derivative with respect to t and first and second derivatives with respect to x are hölder-continuous with respect to the parabolic distance in Q (see [10, 8]). But also the problem with a datum f with is hölder continuous with respect to the space variables only has been considered. In this framework results of interior optimal regularity have been for example given in [4, 5] (in [5] a problem in $\boldsymbol{R}^n$ without boundary conditions is considered); the Cauchy-Dirichlet problem was treated by Sinestrari and von Wahl [9], who

(*) Nella seduta dell'11 maggio 1996.

considered the case $g \equiv 0$ and assumed the boundary of Ω of class $C^{2+\theta}$ for a certain $\theta > 0$, $f \in C(Q)$ such that for every $t \in [0, T]\, f(t, \cdot) \in C^{\theta}(\overline{\Omega})$ uniformly in t (that is, $f \in B([0, T]; C^{\theta}(\overline{\Omega}))$, $u_0 \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$, with $\gamma_0 u_0 = 0$, $\mathcal{A}u \in C(\overline{\Omega})$ and $\gamma_0(\mathcal{A}u_0 + f(0, \cdot)) = 0$, where we have indicated with γ_0 the trace operator on $\partial\Omega$; they showed the existence of a solution u with many properties of regularity (among them the interior optimal regularity) but did not obtain (of course even assuming $u_0 \in C^{2+\theta}(\overline{\Omega})$) the expected results that the first derivative with respect to t and the derivatives of order less or equal to two with respect to x belong to $B([0, T]; C^{\theta}(\overline{\Omega}))$; in fact [9] contains a counterexample due to Wiegner showing that, for example, the assumptions $f \in C(Q) \cap B([0, T]; C^{\theta}(\overline{\Omega}))$, $\gamma_0 f(0, \cdot) = 0$, $u_0 = 0$ and $g \equiv 0$ are not sufficient to guarantee that the solution has the desired regularity. There is in fact something lacking; such lacking condition was given for the first time by M. Lopéz Morales in [6] and, in case $g \equiv 0$, is the $\theta/2$-hölder regularity with respect to t of the trace $\gamma_0 f$.

The aim of this *Note* is to give an alternative proof of the main result of [6], which was obtained through potential theory, using essentially semigroup methods and an estimate, due to Bolley, Camus, P. The Lai (see [2]), of the solution of the elliptic boundary value problem depending on a parameter obtained applying formally the Laplace transform with respect to t. This estimate is reported in Theorem 1.

The new proof of this optimal regularity result (Theorem 2) which is here given can be extended in various directions; for example one can consider general boundary value problems, and broader classes of data (just to give an example, one can show that Theorem 2 can be extended to the case $\theta \in]0, 1[\cup]1, 2[$). But this requires, first of all, an extension of the result given in Theorem 1 and exhibits some new technical difficulties; so the most general case will be treated somewhere else and here we shall limit ourselves to the linear case treated in [6]. We add only that the result given in Theorem 2 is in fact of optimal regularity, as the assumptions of Theorem 2 are necessary and sufficient to get the desired regularity of the solution. This is not clear from [6].

We introduce now some notations we shall use in the sequel; if Ω is a bounded open subset of $\boldsymbol{R}^n$, with boundary of class $C^{1+\alpha}$, for some nonnegative α, we shall indicate with $\|\cdot\|_{\xi,\overline{\Omega}}$ and with $\|\cdot\|_{\xi,\partial\Omega}$ the norms in, respectively, the space $C^{\xi}(\overline{\Omega})$ and $C^{\xi}(\partial\Omega)$, for a certain $\xi \in [0, 1+\alpha]$; through the formula $f(t)(x) := f(t, x)$ we shall identify scalar valued mappings of domain Q with functions of domain $[0, T]$ with values in functional spaces on $\overline{\Omega}$ or $\partial\Omega$; so, for example, if E is a space of such a type on $\overline{\Omega}$ or $\partial\Omega$, we shall indicate with $B([0, T]; E)$ $\{f\colon [0, T] \to E \mid f$ is bounded with values in $E\}$. Analogous notations will be used for functions which are continuous, hölder continuous, etc. with values in E; each of these classes will be equiped with a natural norm.

If A is a linear operator in a Banach space, we shall indicate with $\varrho(A)$ and with $\sigma(A)$ its resolvent set and its spectrum respectively.

If E and F are Banach spaces, we shall indicate with $\mathcal{L}(E, F)$ the Banach space of linear bounded operators from E to F; if $E = F$, we shall simply write $\mathcal{L}(E)$.

We shall use some elements of real interpolation theory (see for example [7, ch. 1]). Assume that E_0 and E_1 are Banach spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$; if $\alpha \in]0,1[$, we indicate with $(E_0, E_1)_{\alpha,\infty}$ the corresponding interpolation space. If E_1 is the domain of an operator A in E_0 such that $\boldsymbol{R}^+ \subseteq \varrho(A)$ and $\|(\xi - A)^{-1}\|_{\mathcal{L}(E_0)} = O(\xi^{-1})$ as $\xi \to +\infty$, one can show that $(E_0, E_1)_{\alpha,\infty}$ coincides with the set of elements x in E_0 such that $\|A(\xi - A)^{-1}x\|_0 = O(\xi^{-\alpha})$ as $\xi \to +\infty$. If E is a Banach space such that $E_1 \subseteq E \subseteq E_0$ and $\alpha \in]0,1[$ we shall write $E \in J_\alpha(E_0, E_1)$ if there exists $C > 0$ such that for any $x \in E_1$ $\|x\|_E \leqslant C\|x\|_0^{1-\alpha}\|x\|_1^{\alpha}$.

Finally, we shall use quite loosely the symbol C to indicate a constant that we are not interested to specify and may be different from time to time.

The problem

We start by introducing the main assumptions of this *Note*; let $\theta \in]0,1[$; we shall say that the conditions (H_θ) are satisfied if:

(I) Ω is an open bounded subset of $\boldsymbol{R}^n$, lying on one side of its topological boundary $\partial\Omega$, which is a submanifold of $\boldsymbol{R}^n$ of dimension $n-1$ and class $C^{2+\theta}$;

(II) $\mathcal{A} = \mathcal{A}(x, \partial_x) = \sum_{|\alpha| \leqslant 2} a_\alpha(x)\partial_x^\alpha$ is a strongly elliptic operator of order two (that is, $\mathrm{Re} \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geqslant \nu|\xi|^2$ for some $\nu > 0$ and for any $(x,\xi) \in \overline{\Omega} \times \boldsymbol{R}^n$ with coefficients of class $C^\theta(\overline{\Omega})$).

If the conditions (H_θ) are satisfied, there exist $R \geqslant 0$, $\phi_0 \in]\pi/2, \pi[$ such that for any $\lambda \in \boldsymbol{C}$, with $|\lambda| \geqslant R$ and $|\mathrm{Arg}\,\lambda| \leqslant \phi_0$ the problem

$$(2)\qquad \begin{cases} \lambda u - \mathcal{A}u = f, \\ \gamma_0 u = g, \end{cases}$$

has for any $f \in C^\theta(\overline{\Omega})$, $g \in C^{2+\theta}(\partial\Omega)$ a unique solution u belonging to $C^{2+\theta}(\overline{\Omega})$ (see [7, ch. 3]); it is of fundamental importance for parabolic problems to estimate how the norms $\|u\|_{\theta,\overline{\Omega}}$ and $\|u\|_{2+\theta,\overline{\Omega}}$ depend on the data and the parameter λ; the following result is due to Bolley, Camus and P. The Lai (see [2, Theorem 1]):

THEOREM 1. *Assume that the assumptions (H_θ) are satisfied, for some $\theta \in]0,1[$; then, there exist $R \geqslant 0$, $\phi_0 \in]\pi/2, \pi[$, $M > 0$ such that for any $\lambda \in \boldsymbol{C}$, with $|\lambda| \geqslant R$ and $|\mathrm{Arg}\,\lambda| \leqslant \phi_0$ the solution u of problem (2) with $g = 0$ satisfies the estimate*

$$(3)\qquad |\lambda|^{1+\theta/2}\|u\|_{0,\overline{\Omega}} + |\lambda|\,\|u\|_{\theta,\overline{\Omega}} + \|u\|_{2+\theta,\overline{\Omega}} \leqslant M[\|f\|_{\theta,\overline{\Omega}} + |\lambda|^{\theta/2}\|\gamma_0 f\|_{0,\partial\Omega}].$$

We want to study the following mixed Cauchy-Dirichlet parabolic problem:

$$(4)\qquad \begin{cases} \partial_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & t \in [0,T],\ x \in \overline{\Omega}, \\ u(t,x') = g(t,x'), & t \in [0,T],\ x' \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \overline{\Omega}. \end{cases}$$

More specifically, we shall prove the following result:

THEOREM 2. *Assume that the assumptions* (H_θ) *are satisfied for some* $\theta \in]0, 1[$; *then problem* (4) *has a unique strict solution* u *belonging to* $B([0, T]; C^{2+\theta}(\overline{\Omega}))$ *such that* $\partial_t u \in B([0, T]; C^\theta(\overline{\Omega}))$ *if and only if the following conditions are satisfied*:

(*a*) $u_0 \in C^{2+\theta}(\overline{\Omega})$;

(*b*) $f \in C([0, T]; C(\overline{\Omega})) \cap B([0, T]; C^\theta(\overline{\Omega}))$;

(*c*) $g \in C([0, T]; C^2(\partial\Omega)) \cap B([0, T]; C^{2+\theta}(\partial\Omega)) \cap C^1([0, T]; C(\partial\Omega))$ *and* $\partial_t g \in B([0, T]; C^\theta(\partial\Omega))$;

(*d*) $\partial_t g - \gamma f \in C^{\theta/2}([0, T]; C(\partial\Omega))$;

(*e*) $\gamma_0 u_0 = g(0)$;

(*f*) $\partial_t g(0) - \gamma_0 f(0) = \gamma_0 \mathcal{A} u_0$.

We begin the proof of Theorem 2 verifying the necessity of the conditions (*a*)-(*f*):

LEMMA 1. *Assume that the assumptions* (H_θ) *are satisfied; then, if problem* (4) *has a strict solution* u *belonging to* $B([0, T]; C^{2+\theta}(\overline{\Omega}))$ *with* $\partial_t u \in B([0, T]; C^\theta(\overline{\Omega}))$, *the conditions* (*a*)-(*f*) *are all satisfied.*

PROOF. The only condition which is not obvious is (*d*); it is easily seen that one has $\partial_t g - \gamma_0 f = \gamma_0 \mathcal{A} u$; now, one can verify that u is Lipschitz continuous with values in $C^\theta(\overline{\Omega})$; as $C^2(\overline{\Omega}) \in J_{1-\theta/2}(C^\theta(\overline{\Omega}); C^{2+\theta}(\overline{\Omega}))$, we have that $u \in C^{\theta/2}([0, T]; C^2(\overline{\Omega}))$, which implies immediately the result.

We set now

$$D(A) := \{u \in \bigcap_{1 \leq p < +\infty} W^{2,p}(\Omega) \mid \mathcal{A} u \in C(\overline{\Omega}), \gamma_0 u = 0\},$$

$Au = \mathcal{A} u$ for any $u \in D(A)$. It was proved by Stewart (see [11]) that A generates an analytic semigroup $\{T(t) \mid t \geq 0\}$ in $C(\overline{\Omega})$, which is not strongly continuous in 0. We use this fact to prove the uniqueness:

LEMMA 2. *Under the assumptions* (H_θ), *for any* $f \in C([0, T]; C(\overline{\Omega}))$, $g \in C([0, T]; C(\partial\Omega))$ *problem* (4) *has at most one strict solution.*

PROOF. Consider (4) with all data vanishing. A strict solution u of (4) clearly belongs (in this case) to $C([0, T]; D(A)) \cap C^1([0, T]; C(\overline{\Omega}))$; from [11] we have that necessarily $u(t) \equiv 0$.

The following lemma is the crucial step of the proof:

LEMMA 3. *Assume that the assumptions* (H_θ) *are satisfied for some* $\theta \in]0, 1[$ *and, moreover,* $f \in C([0, T]; C(\overline{\Omega})) \cap B([0, T]; C^\theta(\overline{\Omega}))$, $\gamma_0 f \in C^{\theta/2}([0, T]; C(\partial\Omega))$, $\gamma_0 f(0) = 0$. *Then, problem* (4) *with* $u_0 = 0$ *and* $g \equiv 0$ *has a strict solution* u *belonging to* $B([0, T]; C^{2+\theta}(\overline{\Omega}))$ *with* $\partial_t u$ *belonging to* $B([0, T]; C^\theta(\overline{\Omega}))$.

PROOF. We start by remarking that the assumptions of Lemma 3 are exactly conditions (a)-(f) in case $u_0 = 0$ and $g \equiv 0$. We set

$$u(t) := \int_0^t T(t-s) f(s)\, ds .$$

We recall that, for $t > 0$,

$$T(t) = (2\pi i)^{-1} \int_\gamma \exp(\lambda t)(\lambda - A)^{-1} d\lambda ,$$

where γ is the usual path lying in $\varrho(A)$, joining $+\infty e^{-i\theta_0}$ to $+\infty e^{i\theta_0}$ for some $\theta_0 \in]\pi/2, \pi[$. From Theorem 1 we have that there exists $C > 0$ such that for every $t \in [0, T]$, $f \in C^\theta(\overline{\Omega})$

$$\|T(t)f\|_{\theta,\overline{\Omega}} + t\|T(t)f\|_{2+\theta,\overline{\Omega}} \leqslant C[\|f\|_{\theta,\overline{\Omega}} + t^{-\theta/2}\|\gamma_0 f\|_{0,\partial\Omega}] . \tag{5}$$

We set also, for $t > 0$,

$$T^{(-1)}(t) := (2\pi i)^{-1} \int_0^t T(s)\, ds = (2\pi i)^{-1} \int_\gamma \exp(\lambda t) \lambda^{-1} (\lambda - A)^{-1} d\lambda ;$$

we have

$$\|T^{(-1)}(t)f\|_{\theta,\overline{\Omega}} + t\|T^{(-1)}(t)f\|_{2+\theta,\overline{\Omega}} \leqslant C[t\|f\|_{\theta,\overline{\Omega}} + t^{1-\theta/2}\|\gamma_0 f\|_{0,\partial\Omega}] . \tag{6}$$

We put

$$u_1(t) := \int_0^t T(t-s)[f(s) - f(t)]\, ds, \; u_2(t) := T^{(-1)}(t) f(t) .$$

From (5) and (6), as $C^2(\overline{\Omega}) \in J_{1-\theta/2}(C^\theta(\overline{\Omega}), C^{2+\theta}(\overline{\Omega}))$ we have

$$\|T(t-s)[f(s) - f(t)]\|_{2,\overline{\Omega}} \leqslant C(t-s)^{\theta/2-1} ,$$

which implies that $u_1 \in C([0, T]; C^2(\overline{\Omega}))$ and that

$$\|T^{(-1)}(t) f(t)\|_{2,\overline{\Omega}} \leqslant C(t^{\theta/2}\|f(t)\|_{\theta,\overline{\Omega}} + \|\gamma_0 f(t)\|_{0,\partial\Omega}) ,$$

so that $u_2 \in C([0, T]; C^2(\overline{\Omega}))$, taking into account the fact that $\gamma_0 f \in C([0, T]; C(\partial\Omega))$ and $\gamma_0 f(0) = 0$. So $u \in C([0, T]; C^2(\overline{\Omega}))$. Set now, for $\varepsilon \in]0, T[$, $t \in [\varepsilon, T]$,

$$u_\varepsilon(t) := \int_0^{t-\varepsilon} T(t-s) f(s)\, ds ;$$

one has that $u_\varepsilon \in C^1([\varepsilon, T]; C(\overline{\Omega}))$ and, for $t \in [\varepsilon, T]$,

$$u_\varepsilon'(t) = T(\varepsilon) f(t-\varepsilon) + \int_0^t AT(t-s) f(s)\, ds .$$

It is easily seen that $\|u(t) - u_\varepsilon(t)\|_{C([\delta, T]; \overline{\Omega})} \to 0$ as $\varepsilon \to 0^+$ for every $\delta \in]0, T[$, and in the same spaces u'_ε converges to $T(\cdot) f(\cdot) + \int_0^{\cdot} T(\cdot - s) f(s)\, ds$; it follows that $u \in C^1(]0, T]; C(\overline{\Omega}))$ and for every $t \in]0, T]$

$$u'(t) = T(t) f(t) + \int_0^t T(t-s) f(s)\, ds .$$

As $\gamma_0 f(0) = 0$, $f(0)$ belongs to the closure of $D(A)$ in $C(\overline{\Omega})$; this implies that $\|T(t) f(t) - f(0)\|_{0, \overline{\Omega}} \to 0$ as $t \to 0^+$ and so $u \in C^1([0, T]; C(\overline{\Omega}))$. From what we have already seen it follows also that u is a strict solution of (4) with $u_0 = 0$ and $g(t) \equiv \equiv 0$, as clearly for every $t \in [0, T]$

$$\gamma_0 u(t) = \int_0^t \gamma_0 T(t-s) f(s)\, ds = 0 .$$

It remains to verify that $u \in B([0, T]; C^{2+\theta}(\overline{\Omega}))$ and $\partial_t u \in B([0, T]; C^{\theta}(\overline{\Omega}))$; the second condition can be easily drawn from the first, using the first equation in (4). Remark now, that the first condition can be obtained showing that $\mathcal{A} u \in \in B([0, T]; C^{\theta}(\overline{\Omega}))$. We have

$$\mathcal{A} u_2(t) = A T^{(-1)}(t) f(t) = T(t) f(t) - f(t),$$

and, from (5),

$$\|T(t) f(t)\|_{\theta, \overline{\Omega}} \leqslant C(\|f(t)\|_{\theta, \overline{\Omega}} + t^{-\theta/2} \|\gamma_0 f(t)\|_{0, \partial\Omega}) \leqslant C' ,$$

for some $C' \geqslant 0$. Finally, we want to estimate $\|\mathcal{A} u_1(t)\|_{\theta, \overline{\Omega}}$; to this aim, we recall that $(C(\overline{\Omega}), D(A))_{\theta/2, \infty}$ is a closed subspace of $C^{\theta}(\overline{\Omega})$ (see [1]); we shall show that $A u_1$ is bounded with values in $(C(\overline{\Omega}), D(A))_{\theta/2, \infty}$; now, with the usual trick of taking as new unknown quantity $e^{-\lambda t} u$ instead of u, we can assume that $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geqslant 0\} \subseteq \varrho(A)$, in such a way we can take γ equal to the counterclockwise oriented boundary of $\{z \in \in \mathbb{C} \,|\, |\mathrm{Arg}(z)| \geqslant \theta_0\}$ for a suitable $\theta_0 \in]\pi/2, \pi[$, and $\sup_{\xi > 0} \|\xi^{\theta/2} A(\xi - A)^{-1} f\|_{0, \overline{\Omega}}$ as norm in $(C(\overline{\Omega}), D(A))_{\theta/2, \infty}$. So we have, for $\xi > 0$, $t \in [0, T]$

$$\|\xi^{\theta/2} A(\xi - A)^{-1} \mathcal{A} u_1(t)\|_{0, \overline{\Omega}} =$$

$$= \left\| (2\pi i)^{-1} \int_0^t \left(\int_\gamma \exp(\lambda(t-s)) \lambda (\lambda - \xi)^{-1} A(\lambda - A)^{-1} [f(s) - f(t)]\, d\lambda \right) ds \right\|_{0, \overline{\Omega}} .$$

From

$$\|A(\lambda - A)^{-1} [f(s) - f(t)]\|_{0, \overline{\Omega}} \leqslant$$

$$\leqslant C[|\lambda|^{-\theta/2} \|f\|_{B([0, T]; C^{\theta}(\overline{\Omega}))} + (t-s)^{\theta/2} \|\gamma_0 f\|_{C^{\theta/2}([0, T]; C(\partial\Omega))}]$$

we have, for a certain $\alpha > 0$,

$$\|\mathcal{A}u_1(t)\|_{(C(\overline{\Omega}),D(A))_{\theta/2,\infty}} \leqslant$$

$$\leqslant C\left[\xi^{\theta/2}\int_0^t\left(\int_0^{+\infty}\exp(-\alpha r(t-s))r^{1-\theta/2}(\xi+r)^{-1}dr\right)ds\|f\|_{B([0,T];C^\theta(\overline{\Omega}))}+\right.$$

$$\left.+\xi^{\theta/2}\int_0^t\left(\int_0^{+\infty}\exp(-\alpha r(t-s))r(\xi+r)^{-1}dr\right)(t-s)^{\theta/2}ds\|\gamma_0 f\|_{C^{\theta/2}([0,T];C(\partial\Omega))}\right].$$

We have

$$\xi^{\theta/2}\int_0^t\left(\int_0^{+\infty}\exp(-\alpha r(t-s))r^{1-\theta/2}(\xi+r)^{-1}dr\right)ds=\Phi(t\xi),$$

$$\xi^{\theta/2}\int_0^t\left(\int_0^{+\infty}\exp(-\alpha r(t-s))r(\xi+r)^{-1}dr\right)(t-s)^{\theta/2}\,ds=\Psi(t\xi),$$

with

$$\Phi(\tau)=\tau^{\theta/2}\int_0^1\left(\int_0^{+\infty}e^{-\alpha\varrho}\varrho^{1-\theta/2}(\tau\sigma+\varrho)^{-1}d\varrho\right)\sigma^{\theta/2-1}\,d\sigma,$$

$$\Psi(\tau)=\tau^{\theta/2}\int_0^1\left(\int_0^{+\infty}e^{-\alpha\varrho}\varrho(\tau\sigma+\varrho)^{-1}d\varrho\right)\sigma^{\theta/2-1}\,d\sigma,$$

and it is not difficult to verify that Φ and Ψ are bounded in $\boldsymbol{R}^+$.

Proof of Theorem 2. Let $N\in\mathcal{L}(C(\partial\Omega),C(\overline{\Omega}))$ be such that $\gamma_0 Ng=g$ for any $g\in C(\partial\Omega)$ and for every $\theta'\in[0,2+\theta]$ $N_{|C^{\theta'}(\partial\Omega)}\in\mathcal{L}(C^{\theta'}(\partial\Omega),C^{\theta'}(\overline{\Omega}))$; an operator with these properties is constructed in [8]. Set $v(t):=u_0+N(g(t)-\gamma_0u_0)$; then $v\in C^1([0,T];C(\overline{\Omega}))\cap C([0,T];C^2(\overline{\Omega}))\cap B([0,T];C^{2+\theta}(\overline{\Omega}))$ and $\partial_t v\in B([0,T];C^\theta(\overline{\Omega}))$; subtracting v from u one reduces oneself to the situation treated in Lemma 3.

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