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Semiclassical states of nonlinear Schrödinger equations with bounded potentials

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Analisi matematica. — *Semiclassical states of nonlinear Schrödinger equations with bounded potentials.* Nota di ANTONIO AMBROSETTI, MARINO BADIALE e SILVIA CINGOLANI, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — Using some perturbation results in critical point theory, we prove that a class of nonlinear Schrödinger equations possesses semiclassical states that concentrate near the critical points of the potential V.

KEY WORDS: Nonlinear Schrödinger equations; Critical point theory; Homoclinic solutions.

RIASSUNTO. — *Stati semiclassici di equazioni di Schrödinger con potenziali limitati.* Usando dei risultati di perturbazione nella teoria dei punti critici, si prova che alcune equazioni di Schrödinger nonlineari hanno stati semiclassici che si concentrano vicino ai punti critici del potenziale V.

1. MAIN RESULTS

In this *Note* we outline some results that are discussed in a more general and complete form in [1]. We deal with the existence of stationary waves $\psi(x,t)$ of the nonlinear Schrödinger (NLS, in short) equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \Delta \psi + V(x)\psi - |\psi|^{p-1}\psi, \qquad x \in \mathbb{R}^N,$$

($\hbar$ denotes the Planck constant) that concentrate near the critical points of the potential V as $\hbar \to 0$ (semiclassical states). Up to a translation, we can (and will) assume that the critical point of interest is 0. Setting $\psi(t,x) = \exp(-i\omega t/\varepsilon)\varphi(x)$, $\hbar = \varepsilon$, and making the change of variable $y = x/\varepsilon$, φ must satisfy

$$\begin{cases} -\Delta\varphi + V(\varepsilon y)\varphi = \omega\varphi + |\varphi|^{p-1}\varphi, & y \in \mathbb{R}^N, \\ \lim_{|y|\to\infty} \varphi(y) = 0. \end{cases} \tag{1}$$

In the limit as $\varepsilon \to 0$, equation (1) becomes

$$-\Delta\varphi + V(0)\varphi = \omega\varphi + |\varphi|^{p-1}\varphi. \tag{2}$$

Hereafter we assume that $p < p^*$, where $p^* = (N+2)/(N-2)$ if $N > 2$ and $p^* = +\infty$ if $N = 1, 2$. In such a case (2) has, for each $\omega < V(0)$, a unique, positive, radial solution z_0, see [7].

We will consider potentials V satisfying

(V) $V \in C^\infty(\mathbb{R}^N, \mathbb{R})$ is bounded and has either a local minimum or a local maximum at 0; precisely, there exists an even integer $m > 0$ such that $D^k V(0) = 0$ for all $k < m$ and $D^m V(0)$ is either positive or negative defined.

We shall prove:

(*) Nella seduta del 13 giugno 1996.

THEOREM 1. *Suppose that* (V) *holds and let* $\omega < \inf V$. *Then for all* $\hbar = \varepsilon > 0$ *sufficiently small, the NLS equation has a positive solution* φ_ε *such that*

$$\varphi_\varepsilon(x) \sim z_0(x/\varepsilon). \tag{3}$$

The meaning of (3) will be made precise at the end of the next section.

Theorem 1 improves the result of [4, 8]: the former deals with $N = 1$ and $p = 3$; the latter deals with any $N \geq 1$ and $1 < p < p^*$, under the hypothesis that V does not oscillate rapidly at infinity. In both the papers it is used a Lyapunov-Schmidt procedure and this requires to assume that V is non degenerate. The existence of solutions of (1) with $\varepsilon = 1$, has been also proved in [10] by means of the *Mountain-Pass* theorem, provided $V(x) \to +\infty$ as $|x| \to \infty$. In spite of its global nature, this approach has been adapted in [11] to show that the *Mountain-Pass* solutions concentrate near minima of V as $\varepsilon \to 0$. When the present *Note* and [1] were completed, we also became aware that in [3, 6] a similar question is addressed, together with the existence of «multi-bump» solutions. However, although V can be possibly not bounded above, their arguments only work for minima of V.

In contrast to the preceding papers, our approach is based on some perturbation results in critical point theory, see [2]. We believe that, dealing with bounded potentials, this is the most natural frame for the existence of semiclassical states. Indeed, it permits to obtain in a neat way our existence theorem and to prove several additional properties, such as stability and necessary conditions for existence of stationary waves satisfying (3). Furthermore, the abstract approach can also be used to improve Theorem 1 as well as to show the existence of multiple homoclinics for a class of second order hamiltonian systems.

2. OUTLINE OF THE PROOF OF THEOREM 1

For the sake of simplicity, we will outline the proof in the case $N = 1$. The general case requires some more technicalities, only. We also take $V(0) = 0$, hence $\omega < 0$.

STEP 1. *The abstract setting*. Let us consider the Hilbert space $E = W^{1,2}(\mathbb{R})$, endowed with the scalar product

$$(u|v) = \int_{\mathbb{R}} (u'v' - \omega \cdot uv)\, dx$$

and norm $\|u\|^2 = (u|u)$, and let $J_\varepsilon : E \to \mathbb{R}$ denote the smooth functional

$$J_\varepsilon(u) = (1/2)\|u\|^2 + (1/2)\int_{\mathbb{R}} V(\varepsilon x) u^2(x)\, dx - (p+1)^{-1} \int_{\mathbb{R}} |u(x)|^{p+1}\, dx.$$

Critical points $u \in E$ of J_ε correspond to solutions of (1). As anticipated before, critical points of J_ε will be searched, for ε small, by a perturbation procedure, following the arguments discussed in [2]. Let us set $z_\theta(x) = z_0(x + \theta)$. Since (2) is autonomous, then z_θ is also a solution of (2), for all $\theta \in \mathbb{R}$. Correspondingly, $Z = \{z_\theta\}_{\theta \in \mathbb{R}}$ is a (one dimensional) manifold consisting of critical points of the unperturbed functional J_0. As a consequence of the results of [9], it turns out that Z is *non degenerate*, in the sense that

$$T_\theta Z = Ker\,[J_0''(z_\theta)]\,, \tag{4}$$

where $T_\theta Z = \operatorname{span}\{z_\theta'\}$ denotes the tangent space to Z at z_θ.

STEP 2. *Natural constraint for J_ε.* For $(\theta, \varepsilon) \in \mathbb{R} \times \mathbb{R}$ we search $w \in E$, $w \perp T_\theta Z$, verifying the equation $\nabla J_\varepsilon(z_\theta + w) \in T_\theta Z$, namely

$$\nabla J_\varepsilon(z_\theta + \omega) = \alpha z_\theta'\,, \tag{5}$$

for some $\alpha \in \mathbb{R}$.

LEMMA 2. *For all $\theta_0 > 0$ there exist $\varepsilon_0 > 0$, $\alpha = \alpha(\theta, \varepsilon)$ and $w = w(\theta, \varepsilon)$, depending smoothly on $(\theta, \varepsilon) \in T := \{|\theta| < \theta_0\} \times \{|\varepsilon| < \varepsilon_0\}$, such that $\alpha(\theta, 0) = 0$, $w(\theta, 0) = 0$ and (5) holds. Furthermore there results*

(i) *w is smooth with respect to θ and $\partial_w w(\theta, \varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly in θ, $|\theta| < \theta_0$;*

(ii) *for all $k < m$ there results: $\varepsilon^{-k} w(\theta, \varepsilon) \to 0$ as $\varepsilon \to 0$.*

For $(\theta, \varepsilon) \in T$ (the appropriate choice of θ_0 will be made in Step 3 below) we set $Z_\varepsilon = \{u \in E \colon u = z_\theta + w(\theta, \varepsilon)\}$. The preceding Lemma implies that Z_ε is, for ε small, a one dimensional manifold close to Z. Moreover one has

LEMMA 3. *For ε small, $u \in Z_\varepsilon$ is a critical point of J_ε iff it is a critical point of J_ε constrained on Z_ε.*

PROOF (Sketch). Let $u_\varepsilon = z_\theta + w(\theta, \varepsilon) \in Z_\varepsilon$ be a critical point of J_ε constrained on Z_ε, then $\nabla J_\varepsilon(u_\varepsilon) \perp T_{u_\varepsilon} Z_\varepsilon$. The definition of Z_ε yields that $\nabla J_\varepsilon(u_\varepsilon) \in T_\theta Z$. As $\varepsilon \to 0$ one has that $z_\theta + w(\theta, \varepsilon) \to z_\theta$ and hence $T_{u_\varepsilon} Z_\varepsilon$ is close to $T_\theta Z$. Therefore $\nabla J_\varepsilon(u_\varepsilon) = 0$. ■

STEP 3. *Critical points of J_ε on Z_ε.* In the sequel we will assume that (V) holds with $a = D^m V(0) > 0$. The other case can be handled with obvious changes. According to the preceding Lemma we are led to study J_ε on Z_ε, namely the real valued function

$$\Phi_\varepsilon(\theta) = J_\varepsilon(z_\theta + w(\theta, \varepsilon))\,.$$

Using Lemma 2 there results

$$\Phi_\varepsilon(\theta) = (1/2)\|z_\theta\|^2 + (z_\theta\,|w) + (1/2)\int_{\mathbb{R}^N} V(\varepsilon x)(z_\theta + w)^2 - \\ -(p+1)^{-1}\int_{\mathbb{R}^N} |z_\theta + w|^{p+1} + o(\varepsilon^m)\,.$$

Let

$$\Gamma(\theta) = \int_{\mathbb{R}} a x^m z_\theta^2 \, dx .$$

Using again Lemma 2 one has

$$\int_{\mathbb{R}} V(\varepsilon x)(z_\theta + w)^2 \, dx = \int_{\mathbb{R}} V(\varepsilon x) z_\theta^2 \, dx + o(\varepsilon^m) = \varepsilon^m \int_{\mathbb{R}} (V(\varepsilon x)/\varepsilon^m) z_\theta^2 \, dx + o(\varepsilon^m) . \tag{6}$$

An application of the Dominated Convergence Theorem yields

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} (V(\varepsilon x)/\varepsilon^m) z_\theta^2 \, dx = (m!)^{-1} \Gamma(\theta) . \tag{7}$$

On the other side, from $-z_\theta'' + \lambda z_\theta = z_\theta^p$, it follows that

$$(z_\theta \,|\, w) = \int_{\mathbb{R}^N} |z_\theta|^p w . \tag{8}$$

Substituting (7) and (8) into the expression of Φ_ε we find

$$\Phi_\varepsilon(\theta) = b + (2m!)^{-1} \varepsilon^m \Gamma(\theta) + o(\varepsilon^m) \tag{9}$$

where

$$b = J_0(z_\theta) = (1/2) \|z_\theta\|^2 - (p+1)^{-1} \int_{\mathbb{R}^N} |z_\theta|^{p+1} .$$

On the other side, there results

$$\Gamma(\theta) = \int_{\mathbb{R}} a(x - \theta)^m z_0^2 \, dx , \qquad a > 0 ,$$

and hence Γ has a minimum at $\theta = 0$. Then we infer from (9) that there exists $\theta_0 > 0$ such that $\Phi_\varepsilon(0) < \inf\{\Phi_\varepsilon(\theta) \colon |\theta| = \theta_0\}$, provided ε is small enough. This implies

LEMMA 4. *Φ_ε attains a minimum in the interval $|\theta| \leq \theta_0$ at $\theta = \theta(\varepsilon)$, with $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $u_\varepsilon = z_{\theta(\varepsilon)} + w(\theta(\varepsilon), \varepsilon)$ is a critical point of J_ε on Z_ε.*

We are now in position to complete the proof of Theorem 1. Indeed, according to the discussion carried out in Steps 1 and 2, u_ε gives rise to a solution of (1) and $\varphi_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ yields a stationary wave of the NLS equation. Finally, the properties of $w(\theta, \varepsilon)$ and $\lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$ imply that

$$\varphi_\varepsilon(x) \simeq z_0(x/\varepsilon + \theta(\varepsilon)) ,$$

namely that φ_ε concentrates near $x = 0$ as claimed in (3).

3. FURTHER RESULTS

A sharper use of the preceding arguments yields several further results.

— *Stability*. Our variational approach allows us to know some informations about the

Morse index of the critical point u_ε. For example, if V has a local minimum at $x = 0$ then J_ε has a minimum on Z_ε at u_ε and such u_ε turns out to be a Mountain-Pass critical point of J_ε on E. This suggests that φ_ε is orbitally stable. Indeed, one can improve the results of [5, 9] showing that this is actually the case provided, *e.g.*, that $N = 1$ and $1 < p < 6$. Otherwise, φ_ε is unstable.

— *Necessary conditions for concentration*. This existence of semiclassical states concentrating near the critical points of V has been proved above. Conversely, one can show

THEOREM 5. *If there exists a solution $u_\varepsilon \in E$ of* (1) *such that $u_\varepsilon \to z_0(\cdot + \theta)$ as $\varepsilon \to 0$, then one must have $V'(0) = 0$. Furthermore, if* (V) *holds, then the solution must concentrate at $\theta = 0$.*

— *Improving Theorem* 1. Theorem 1 holds in a greater generality. Indeed, one can consider the NLS equation with a nonlinearity $g(u)$ which is possibly not homogeneous, and prove an existence result like Theorem 1.

— *Second order systems*. Consider the second order hamiltonian system in $\mathbb{R}^N$

$$-\ddot{u} + u + A(\varepsilon t)u = |u|^{p-1}u, \tag{10}$$

where $A(t) = [a_{i,j}(t)]$ is a symmetric matrix satisfying

(A) $a_{i,j}(t) > 0$ and there exists a positive (or negative) definite constant matrix $A_0 = [\alpha_{i,j}]$ and an even integer $m > 0$ such that $a_{i,j}(t) = \alpha_{i,j}t^m + o(t^m)$.

We look for a solution u of (10) such that $\lim_{|t|\to\infty} u(t) = 0$ (homoclinic solution), which correspond to critical points of

$$J_\varepsilon(u) = (1/2)\|u\|^2 + (1/2)\int_{\mathbb{R}} A(\varepsilon t)u\cdot u\,dt - (p+1)^{-1}\int_{\mathbb{R}} |u|^{p+1}\,dt$$

one E. It is easy to see that the preceding abstract approach can be used to handle this problem, too. In the present case, letting $\varepsilon = 0$ in (10), one immediately finds that $Z = \{\xi\cdot r_0(t+\theta) : \xi \in S^{N-1}, \theta \in \mathbb{R}\}$, where S^{N-1} denotes the unit sphere in $\mathbb{R}^N$ and r_0 satisfies

$$-\ddot{r} + r = r^p, \qquad \lim_{|t|\to\infty} r(t) = 0$$

and is symmetric around $t = 0$. The preceding arguments can be repeated an yield that homoclinic solutions of (10) can be found as the critical points of J_ε on the perturbed manifold $Z_\varepsilon \simeq Z$. Now $Z \simeq S^{N-1} \times \mathbb{R}$ is not contractible and this allows us to show the following multiplicity result.

THEOREM 6. *If* (A) *holds then, for $\varepsilon > 0$ small,* (10) *has at least two distinct homoclinic solutions which concentrate near $t = 0$ as $\varepsilon \to 0$.*

It is worth pointing out that this multiplicity result is different from the usual ones because A is not periodic in time.

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