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Curry algebras $N_1$


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Logica matematica. — Curry algebras $N_1$. Nota di Jair Minoro Abe, presentata (*) dal Socio A. Bressan.

Abstract. — In [6] da Costa has introduced a new hierarchy $N_i$, $1 \leq i \leq w$ of logics that are both paraconsistent and paracomplete. Such logics are now known as non-alethic logics. In this article we present an algebraic version of the logics $N_i$ and study some of their properties.

Key words: Algebraic logic; Paraconsistent logic; Paracomplete logic; Non-alethic logic.

Riassunto. — Le algebre «Curry» $N_1$. Nell’articolo [6] da Costa ha introdotto una nuova gerarchia $N_i$, $1 \leq i \leq w$, di logiche che sono al tempo stesso paraconsistenti e paracomplete. Tali logiche sono adesso conosciute come logiche nonaletiche. In questo articolo presentiamo una versione algebrica della logica $N_i$ e studiamo alcune proprietà.

1. Introduction

In recent years, a number of different kinds of logic have been proposed with the aim of avoiding the property that from a contradiction anything may be deduced. Roughly speaking, these logics (called paraconsistent logics) allow formulas of the form $A \land \lnot A$ to be applied in a non-trivial manner in deductions. One such type of logic are the logics $C_n$ of da Costa (see, e.g. [4, 5]). Their «duals», in a precise sense, are the logics known as paracomplete logics (systems $P_n$, see, e.g. [7]). A logic is called paracomplete if, according to it, a proposition and its negation can be both false.

In [6], da Costa describes a new hierarchy $N_i$, $1 \leq i \leq w$ of logics which are simultaneously paraconsistent and paracomplete. These logics were dubbed non-alethic by F. M. Quesada.

The aim of the present Note is to present an algebraic version of the logic $N_1$, developing some ideas of the authors and to study some of the main properties of this algebraic version of $N_1$.

2. The calculus $N_1$

We now present $N_1$ formally. We begin with a language $L$ consisting of a denumerable number of sentential variables closed as usual under $\lnot$ (negation), $\rightarrow$ (implication), $\lor$ (disjunction), and $\land$ (conjunction); the symbol $\leftrightarrow$ (for equivalence) is introduced as usual, and we have three new defined symbols: $A^0$ is an abbreviation for $\lnot(A \land \lnot A)$, $A^*$ is an abbreviation for $A \lor \lnot A$, and $\lnot A$ is an abbreviation for $\lnot A \land A^0$ (called strong negation). Capital latin letters are metalinguistic schematic variables.

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The postulates (axiom schemata and primitive rules of inference) of \( N_1 \) are those of classical positive logic plus the following:

(I) \( A^* \& B^0 \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow \neg B) \rightarrow \neg A) \),

(II) \( A^0 \& B^0 \rightarrow (A \rightarrow B)^0 \& (A \& B)^0 \& (A \vee B)^0 \& (\neg A)^0 \),

(III) \( A^* \& B^* \rightarrow (A \rightarrow B)^* \& (A \& B)^* \& (A \vee B)^* \& (\neg A)^* \),

(IV) \( A^0 \rightarrow (A \rightarrow \neg \neg A) \& (A \rightarrow (\neg A \rightarrow B)) \),

(V) \( A^* \rightarrow (\neg \neg A \rightarrow A) \),

(VI) \( A^0 \vee A^* \).

The concepts of proof, of deduction, etc. are defined as in Kleene [8].

**Definition 2.1.** In \( N_1 \): \( A \leq B \triangleq \text{def} \vdash A \rightarrow B \), \( A \equiv B = \text{def} \vdash A \leq B \) and \( B \leq A \).

**Theorem 2.2.** \( \leq \) is a quasi-order, and \( \equiv \) is an equivalence relation.

**Theorem 2.3.** In \( N_1 \), \( \neg \) is not compatible with the equivalence relation \( \equiv \), and we have \( \vdash A \land \neg A \rightarrow B \).

**Theorem 2.4.** Adding the principle of the excluded middle, \( A \vee \neg A \), to \( N_1 \), we get \( C_1 \).

**Theorem 2.5.** Adding the principle of contradiction, \( \neg (A \& \neg A) \) to \( N_1 \), we get \( P_1 \).

**Theorem 2.6.** Adjoining to \( N_1 \) the schemata \( A \vee \neg A \) and \( \neg (A \& \neg A) \), we obtain the classical propositional calculus.

**Theorem 2.7.** The strong negation possesses all properties of the classical negation; for instance

\[ \vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A), \quad \vdash A \rightarrow (\neg A \rightarrow B), \quad \vdash A \& \neg A \rightarrow B, \]
\[ \vdash A \rightarrow \neg \neg A, \quad \vdash \neg \neg A \rightarrow A. \]

3. **The Curry algebras \( N_1 \)**

The algebraic structures considered here are those seen in [2, 3].

From the algebraic point of view, \( N_1 \) is a classical implicative lattice. By Theorem 2.3 it follows that this lattice has a first element. Due to (I)-(VI), we concluded that in this lattice there is an operator, denoted by \( \prime \), possessing some properties of the Boolean complement. Summarizing, \( N_1 \) is a Curry algebra \( N_1 \).
Definition 3.1. A Curry algebra $N_1$ is a classical implicative lattice $\langle S, =, \land, \lor, ' \rangle$ with greatest and smallest elements (not necessarily unique), 1 and 0, and with an operator $'$ satisfying the following properties, where $p^0 = (p \land p')'$ and $p^* = p \lor p'$:

1) $p^0 \land q^* \leq ((p \rightarrow q) \rightarrow ((p \rightarrow q') \rightarrow p'))$,
2) $p^0 \land q^0 \leq (p \rightarrow q)^0 \land (p \land q)^0 \land (p \lor q)^0 \land (q')^0$,
3) $p^* \land q^* \leq (p \rightarrow q)^* \land (p \land q)^* \land (p \lor q)^* \land (q')^*$,
4) $p^0 \leq (p \rightarrow p'') \land (p \rightarrow (p' \rightarrow q))$,
5) $p^* \leq p'' \rightarrow p$,
6) $p^0 \lor q^* \equiv 1$.

Theorem 3.2. Adjoining to a Curry algebra $N_1$ the postulate $(p \land p')' \equiv 1$, we obtain a CP$_1$-algebra (for these algebras see [1]), and adjoining the postulate $p \lor p' \equiv 1$ we get a C$_1$-algebra (for these algebras see [2]). Moreover if we add both postulates we obtain a Boolean algebra.

Theorem 3.3. A Curry algebra $N_1$ is distributive and has a greatest element.

Definition 3.4. Let $p$ be an element of a Curry algebra $N_1$. We put $-p = p' \land p^0$.

Theorem 3.5. In a Curry algebra $N_1$, $-p$ is a Boolean complement of $p$, so $p \lor -p \equiv 1$ and $p \land -p \equiv 0$.

Theorem 3.6. In a Curry algebra $N_1$, the structure composed by the underlying set and by operations $\land, \lor, -$, is a Boolean algebra.

Definition 3.7. Let $\langle S, =, \rightarrow, \land, \lor, ' \rangle$ be a Curry algebra $N_1$, and $\langle S, =, \land, \lor, - \rangle$ be the Boolean algebra obtained as in the above theorem. Any Boolean algebra that is isomorphic to the quotient algebra of $\langle S, =, \land, \lor, - \rangle$ by $\equiv$ is called Boolean algebra associated with the Curry algebra $N_1$.

Theorem 3.8 (Representation Theorem). Any Curry algebra $N_1$ is associated with a field of sets. Moreover, any Curry algebra $N_1$ is associated with the field of sets simultaneously open and closed of a totally disconnected compact Hausdorff space.

4. The completeness of the logic $N_1$

It is easy to introduce the concepts of filter, ultrafilter, and homomorphisms between Curry algebras $N_1$. All usual properties from classical algebra are as expected: for instance, the shell of a homomorphism is a filter.

We would like to mention only the following results.

Theorem 4.1 (Soundness). If $A$ is a provable formula of the logic $N_1$, then $h(A) \equiv 1$ for any homomorphism $h$ from the set of all formulas of the logic $N_1$ into any Curry algebra $N_1$.

Proof. By induction on the length of proofs.
THEOREM 4.2. Let \( U \) be an ultrafilter in \( F \) (the set of all formulas of \( N_1 \)). Then, there is a homomorphism \( h \) from \( F \) into \( 2 \) (where \( 2 = \{0, 1\} \) is the two-element Boolean algebra) such that the shell of \( h \) is \( U \).

THEOREM 4.3 (Completeness). Let \( F \) be the set of all formulas of \( N_1 \), and \( A \in F \). Let us suppose that \( h(A) \equiv 1 \) for any homomorphism \( h \) from \( F \) into an arbitrary Curry algebra \( N_1 \). Then, \( A \) is a provable formula of \( N_1 \).

PROOF. Similar to the classical case, taking into account the previous theorem.

The algebraic treatment for the logics \( N_i \), \( 1 \leq i \leq w \) does not offer difficulties. Besides the results presented here we would like to emphasize the importance of certain «pre-algebraical» structures (in the sense that the fundamental relation of the structure is an equivalence relation instead of equality). They were called Curry algebras not only as a homage to the american logician H. B. Curry, but because he is one of defendants of the use of pre-structures (see [3]). They are the central tool to deal algebraically with the majority of non-classical logics.

We hope to say something more about this in forthcoming papers.

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