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Recent results on the Boltzmann equation

Memoria (*) di Carlo Cercignani

Abstract. — In the last few years the theory of the nonlinear Boltzmann equation has witnessed a veritable turrent of contributions, spurred by the basic result of DiPerna and Lions. Here we wish to survey these results with particular attention to some recent developments.

Key words: Kinetic theory; Boltzmann equation; Rarefied gases.

Riassunto. — *Recenti risultati sull'equazione di Boltzmann*. Negli ultimi anni la teoria dell'equazione di Boltzmann nonlineare ha registrato una vera folla di contributi, stimolati dal risultato fondamentale di DiPerna e Lions. In questa *Memoria* vogliamo passare in rassegna questi risultati dedicando un'attenzione particolare ad alcuni tra gli sviluppi più recenti.

1. Introduction

In this paper, an attempt will be made to survey the known results on the Cauchy and the mixed problems which arise when we consider the time evolution of a rarefied gas either in the entire space $\Re^3$ or in a subset of it which may be either a vessel Ω (Ω is a bounded open set of $\Re^3$ with a sufficiently smooth boundary $\partial\Omega$), endowed with unit normal $n(x)$ (pointing into Ω) or the region outside a solid body.

The evolution equation for the distribution function $f(x, \xi, t)$ is the Boltzmann equation [1-3]

$$\Lambda f = Q(f,f) \quad \text{in } \mathcal{D}\,(\mathcal{D} = \Omega \times \Re^3 \times (0, T))\,. \tag{1.1}$$

In eq. (1.1) Λf is the free streaming operator defined by

$$\Lambda f = \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} \tag{1.2}$$

and $Q(f, f)$ the collision term

$$Q(f, f)(x, \xi, t) = \int_{\Re^3} \int_{\mathcal{B}^+} (f' f'_* - f f_*)\, B(V, n)\, d\xi_*\, dn \qquad (V = \xi - \xi_*)\,. \tag{1.3}$$

Here $B(V, n)$ is a kernel containing the details of the molecular interaction, f', f'_*, f_* are the same as f, except for the fact that the velocity argument ξ ($\in \Re^3$) is replaced by ξ', ξ'_*, ξ_*, respectively, ξ_* being an integration variable (having the meaning of the velocity of a molecule colliding with the molecule of velocity ξ, whose path we are following). ξ' and ξ'_* are the velocities of two molecules entering a collision that will bring them to have velocities ξ and ξ_*, whereas the unit vector n describes a hemisphere $\mathcal{B}^+$ of the unit sphere $\mathcal{B}$. The relations between ξ', ξ'_*, on one hand, and ξ, ξ_*, on the

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other hand, read as follows:

$$\begin{cases} \xi' = \xi - n[(\xi - \xi_*)\cdot n], \\ \xi'_* = \xi_* + n[(\xi - \xi_*)\cdot n]. \end{cases} \tag{1.4}$$

We shall assume that the gain and loss parts of $Q(f, f)$ denoted by $Q^{\pm}$, are separately meaningful (a.e.). Please remark that the loss term $Q^-(f, f)$ equals f times an expression linear in f that we shall denote by $\nu(f)$. Occasionally we simply write Q or $Q^{\pm}$ to denote $Q(f, f)$ or $Q^{\pm}(f, f)$.

Equation (1.1) must be solved with an initial condition

$$f(x, \xi, 0) = f_0(x, \xi) \tag{1.5}$$

and, in the case of mixed problems, with the boundary condition

$$\gamma^+ f(x, \xi, t) = K\gamma^- f \quad \text{on } E^+ \tag{1.6}$$

(where, as explained below, K is a linear integral operator and $\gamma^{\pm} f$ denote the traces of f on $E^{\pm} = \{(t, x, \xi) \in \partial\Omega \times \mathfrak{R}^3 \times (0, T) | \pm \xi\cdot n(x) > 0\}$).

If we work in spaces of sufficiently regular solutions, we can introduce the semigroup $U(t)$ (actually a group) associated with the collisionless evolution ($\Lambda f = 0$). Then the above initial-boundary value problem reduces to solving the following integral equation

$$f(t) = U(t) f_0 + \int_0^t U(t-s) Q(f(s), f(s))\, ds. \tag{1.7}$$

It is easy to prove local existence of this equation in several spaces, typically:

$$X_{\alpha,\beta} = \{f | (1 + |\xi|^2)^{\alpha/2} \exp(-\beta|\xi|^2) f \in L^\infty(\Omega \times \mathfrak{R}^3)\} \tag{1.8}$$

with norm

$$\|f\|_{\alpha,\beta} = \|(1 + |\xi|^2)^{\alpha/2} \exp(-\beta|\xi|^2) f\|_{L^\infty(\Omega \times \mathfrak{R}^3)}. \tag{1.9}$$

A case in which one can say a lot more about the solutions of initial boundary value problems is the case when the data are compatible with a solution close to a uniform Maxwellian distribution M, given by

$$M = A \exp(-\beta|\xi - u|^2) \tag{1.10}$$

where β is the inverse temperature and u the average velocity (frequently assumed to be zero). Then techniques from rigorous perturbation theory [4] can be used. To this end, let us introduce the perturbation h such that

$$f = M + M^{1/2} h \tag{1.11}$$

and assume (for simplicity) that, in the case of mixed problems, M coincides with the wall Maxwellian, so that eq. (1.6) is satisfied by the restrictions of M to $E^{\pm}$. Equations (1.1)-(1.3) can then be rewritten in the following way:

$$\Lambda h = Lh + \Gamma(h, h) \quad \text{in } \mathcal{D}, \tag{1.12}$$

$$\gamma^+ h(x, \xi, t) = \hat{K}\gamma^- h \quad \text{on } E^+ \tag{1.13}$$

$$h(x, \xi, 0) = h_0(x, \xi) \tag{1.14}$$

where L and Γ are two suitable operators, linear and quadratic, respectively, whereas

$$\hat{K} = M^{-1/2} K M^{1/2} . \tag{1.15}$$

Now the global solution can be found by perturbation techniques, provided the linearized operator

$$B = -\xi \cdot \partial / \partial x + L \tag{1.16}$$

with the boundary conditions (1.13) generates a semigroup $T(t)$ with a nice decay.

One finds two types of result: the first of them with a decay like $t^{-\alpha}$ $(\alpha > 0)$ applies to unbounded domains (in particular, to the Cauchy problem) whereas the second, with an exponential decay, applies to bounded domains.

There are several papers dealing with the proofs of the behaviors indicated above. The case of a bounded domain has been considered by Guiraud [5] in the case of diffuse reflection and by Shizuta and Asano [6] in the case of specular reflection, both assuming that Ω is convex. The case of unbounded domains exterior to a bounded convex obstacle was treated by several Japanese authors [7, 8].

When we want to deal with the more difficult case of arbitrarily large data, we can hope for some significant results only if we use the L^1 framework and the techniques introduced by DiPerna and Lions [9] to deal with the Cauchy problem, as discussed with more detail in the next few sections.

2. The results of DiPerna and Lions

A turning point in the existence theory of the Boltzmann equation occurred in 1987, when Golse *et al.* [10] were able to prove certain results that have become known as «velocity averaging lemmas». Also in 1987, DiPerna and Lions used these lemmas and other estimates to prove the first general global existence theorem for the Boltzmann equation [9]. Their result, with proof, is given in the latter paper, in a review article by Gérard [11] and in a recent book [3]. Here we shall merely outline the ideas of the proof as well as the meaning and limitations of the result.

We begin by fixing some notation. If $\Omega \subset \Re^3$ is open, $L^p_{\text{loc}}(\Omega) = \{f\colon \Omega \to \Re, f|_U \in L^p(U)$ for all $U \subset \Omega$ which are open and relatively compact$\}$. If $\Omega \subset \Re^3$, $\Omega' \subset \Re^l$, the space of all measurable functions on $\Omega' \times \Omega$ whose restrictions to $\Omega' \times U$ is in $L^p(\Omega' \times U)$ for each open and relatively compact $U \subset \Omega$ will be denoted by $L^p(\Omega' \times \Omega_{\text{loc}})$. $S(\Re^3)$ denotes the Schwartz space of rapidly decreasing C^∞-functions on $\Re^3$. For each $s \in \Re_+$, $H^s(\Re^3)$ is the usual Sobolev space, *i.e.* the completion of $S(\Re^3)$ with respect to the norm

$$\|f\|_{H^s} := \left(\int (1 + |z|^{s/2})^2 \, |\hat{f}(z)|^2 \, dz \right)^{1/2} . \tag{2.1}$$

As in Section 1, we shall use Λ as an abbreviation for the transport operator given by eq. (1.2).

In fact, we first discuss some results about a generalized Boltzmann equation

$$(2.2)\qquad (\partial_t + \xi\cdot\nabla_x) f = \int_{\Re^3}\int_{S^2} q(x, V, n)[f'f'_* - ff_*]\,dn\,d\xi_*$$

with the convention $f_* = f(\cdot, \xi_*, \cdot)$, $f' = f(\cdot, \xi', \cdot)$, etc. Of course, for eq. (1.1) $q(\dots) = B(V, n)$.

LEMMA 2.1. Suppose that q is a nonnegative measurable function in $L^\infty_{\mathrm{loc}}(\Re^3 \times \Re^3 \times S^2)$, which depends only on x, V and $|V\cdot n|$ and grows at most polynomially with respect to x and V. Then, if $f \in C^1(\Re_+, S(\Re^3 \times \Re^3))$ is a positive solution of (1.3) such that $|\ln f|$ grows at most polynomially in (x, ξ), uniformly on compact time intervals in $\Re_+$, we have

$$(2.3)\qquad \int\int f\,dx\,d\xi = \int\int f_0\,dx\,d\xi,$$

$$(2.4)\qquad \int\int f|\xi|^2\,dx\,d\xi = \int\int f_0|\xi|^2\,dx\,d\xi,$$

$$(2.5)\qquad \int\int f|x - t\xi|^2\,dx\,d\xi = \int\int f_0|x|^2\,dx\,d\xi,$$

$$(2.6)\qquad \int\int f\ln f\,dx\,d\xi + \int_0^t\int\int e(f)(\cdot, s)\,dx\,d\xi\,ds = \int\int f_0\ln f_0\,dx\,d\xi,$$

where

$$(2.7)\qquad e(f)(x, \xi, t) =$$

$$= \frac{1}{4}\int\int (f'f'_* - ff_*)\ln\left(\frac{f'f'_*}{ff_*}\right)(x, \xi, \xi_*, n, t)\cdot q(x, \xi - \xi_*, n)\,d\xi_*\,dn.$$

These identities imply the estimates

$$(2.8)\qquad \int\int f(1 + |x|^2 + |\xi|^2)\,dx\,d\xi \le \int\int f_0(1 + 2|x|^2 + (2t^2 + 1)|\xi|^2)\,dx\,d\xi$$

and

$$(2.9)\qquad \int\int f|\ln f|\,dx\,d\xi + \int_0^\infty\int\int e(f)(\cdot, s)\,ds\,dx\,d\xi \le$$

$$\le \int\int f_0(|\ln f_0| + 2|\xi|^2 + 2|x|^2)\,dx\,d\xi + C$$

where C is a purely numerical constant.

For the proof we refer to the papers and book quoted above. We also remind the reader that $e(f)$ is always nonnegative (Boltzmann inequality) [1-3]. It is also well-known that weight functions ψ which will lead to conservation equations are 1, ξ_i, $1 \le i \le 3$ (momentum), $|\xi|^2$, $x_i\xi_j - x_j\xi_i$, $1 \le i \le j \le 3$ (angular momentum), $x_i - t\xi_i$, $1 \le i \le 3$ (centre of mass) and $|x - t\xi|^2$ (moment of inertia).

Let us now specify the assumptions on the collision kernel $q(V, n)$ for which a general existence result will be proved. Notice that we assume no dependence of q on x;

such a dependence only enters when we construct approximate solutions a little later.

Suppose that $q \in L^1_{\text{loc}}(\mathfrak{R}^3 \times S^2)$, $q \geq 0$, and that q depends only on $|V|$ and $|V \cdot n|$. Let

$$A(V) = \int_{S^2} q(V, n)\, dn \,. \tag{2.10}$$

Suppose, furthermore, that for every $R > 0$

$$1/(1 + |\xi|^2) \int_{|\xi_*| \leq R} A(\xi - \xi_*)\, d\xi_* \to 0 \quad \text{as } |\xi| \to \infty \tag{2.11}$$

and that

$$A \in L^\infty_{\text{loc}}(\mathfrak{R}^3) \tag{2.12}$$

(this last assumption was not made by DiPerna and Lions [9], but, as noticed by Gérard [11], it simplifies certain technicalities of the proof).

We now split $Q(f, f) = Q_+(f, f) - Q_-(f, f)$ and write $Q_-(f, f) = f\nu(f)$. Note that $\nu(f) = A * f$, where $*$ denotes a convolution product in velocity space.

We do not know whether a global classical solution of the Boltzmann equation exists. One of the crucial steps in the paper by DiPerna and Lions [9] is to introduce weaker solution concepts which lighten the burden of proof, but are still strong enough to guarantee that the collision terms are defined. As before, we write $U(t)$ $(t \in \mathfrak{R})$ for the one-parameter family of operators defined by

$$U(t) g(x_i, \xi_i) = g(x - \xi t, \xi) \tag{2.13}$$

for each measurable g on $\mathfrak{R}^3 \times \mathfrak{R}^3$. First we reformulate the mild solution concept, with minimal integrability constraints on the collision terms.

Definition 2.2. A measurable function $f = f(x, \xi, t)$ on $[0, \infty) \times \mathfrak{R}^3 \times \mathfrak{R}^3$ is a mild solution of the Boltzmann equation to the (measurable) initial value $f_0(x, \xi)$ if for almost all (x, ξ) $U(-s) Q_\pm(f, f)(x, \xi, s)$ are in $L^1_{\text{loc}}[0, \infty)$, and if for each $t \geq 0$ eq. (1.7) holds.

One of the key ideas of DiPerna and Lions was to introduce a new concept of solution, such that the bounds (2.8) and (2.9) could be put to best use, and then to regain mild solutions via a limiting procedure. They called the relaxed solution concept «renormalized solution».

Definition 2.3. A function $f = f(x, \xi, t) \in L^1_+(\mathfrak{R}^+_{\text{loc}} \times \mathfrak{R}^3 \times \mathfrak{R}^3)$ is called a renormalized solution of the Boltzmann equation if

$$(Q_\pm(f, f))/(1 + f) \in L^1_{\text{loc}}(\mathfrak{R}_+ \times \mathfrak{R}^3 \times \mathfrak{R}^3) \tag{2.14}$$

and if for every Lipschitz continuous function $\beta : \mathfrak{R}_+ \to \mathfrak{R}$ which satisfies $|\beta'(t)| \leq$ $\leq C/(1 + t)$ for all $t \geq 0$ one has

$$\Lambda\beta(f) = \beta'(f) Q(f, f) \tag{2.15}$$

in the sense of distributions.

We remark that the division by $1+f$ is natural inasmuch as it leads to a «quasi-linearization» of $Q(f, f)$

LEMMA 2.4. Let $f \in (L^1_{\mathrm{loc}} \times \mathfrak{R}^3 \times \mathfrak{R}^3)$.

i) If f satisfies (2.14) and (2.15) with $\beta(t) = \ln(1+t)$, then f is a mild solution of the Boltzmann equation.

ii) If f is a mild solution of the Boltzmann equation and if $(Q_\pm(f, f))/(1+f) \in L^1_{\mathrm{loc}}(\mathfrak{R}_+ \times \mathfrak{R}^3 \times \mathfrak{R}^3)$, then f is a renormalized solution.

PROOF. See [3, 9, 10].

The basic result of DiPerna and Lions is given by

THEOREM 2.5 [9, 3, 10]. Suppose that $f_0 \in L^1_+(\mathfrak{R}_+ \times \mathfrak{R}^3 \times \mathfrak{R}^3)$ is such that $\int\int f_0(1+|x|^2+|\xi|^2)\,dx\,d\xi$ and $\int\int f_0|\ln f_0|\,dx\,d\xi$ are bounded. Then there is a renormalized solution of the Boltzmann equation such that $f \in C(\mathfrak{R}_+, L^1(\mathfrak{R}^3 \times \mathfrak{R}^3))$, $f|_{t=0} = f^0$, and (2.8), (2.9) hold.

The renormalized solution f is found as a limit of functions solving truncated equations. For some $\delta > 0$ and some modified nonnegative collision kernel $\bar{q} \in C_0^\infty(\mathfrak{R}^3 \times S^2)$ such that $\bar{q}$ vanishes for $V \cdot n < \delta$, let

$$\bar{Q}(g, g) = \int\int \bar{q}(g' g'_* - g g_*)\,dn\,d\xi_* \tag{2.16}$$

and

$$\tilde{Q}(g, g) = \left(1 + \delta \int |g|\,d\xi\right)^{-1} \bar{Q}(g, g). \tag{2.17}$$

LEMMA 2.6. Let $f_0 \in S(\mathfrak{R}^3 \times \mathfrak{R}^3)$ be nonnegative such that $|\ln f_0|$ grows at most polynomially. Then the Cauchy problem

$$\Lambda f = \tilde{Q}(f, f), \qquad f|_{t=0} = f_0 \tag{2.18}$$

has a unique global solution f which satisfies the hypotheses of Lemma 2.1. It also satisfies the estimates (2.8) and (2.9).

This assertion can be proved by contraction mapping.

Let now $q_n \in C^\infty_{0,+}(\mathfrak{R}^3 \times S^2)$ satisfy (2.11) and (2.12) (uniformly for all n) and suppose that $q_n \to q$ a.e. Furthermore, we approximate f_0 in $L^1_+(\mathfrak{R}^3 \times \mathfrak{R}^3)$ by a sequence $\{f_0^n\}_n \subset S(\mathfrak{R}^3 \times \mathfrak{R}^3)$ such that

$$\forall n \; f_0^n \geq \mu_n e^{-|x|^2-|\xi|^2} \qquad (\mu_n > 0), \tag{2.19}$$

$$\int\int f_0^n(1+|x|^2+|\xi|^2)\,dx\,d\xi \to \int\int f_0(1+|x|^2+|\xi|^2)\,dx\,dv, \tag{2.20}$$

$$\int\int f_0^n|\ln f_0^n|\,dx\,d\xi \to \int\int f_0|\ln f_0|\,dx\,d\xi. \tag{2.21}$$

Let $\delta_n \searrow 0$, and let Q^n be $\tilde{Q}$ (from (2.17)) with $\delta = \delta_n$, $\bar{q} = q_n$. Then, Lemma 2.6 assures us that there is a sequence $\{f^n\}$ such that $Tf^n = Q^n(f^n, f^n)$, $f^n|_{t=0} = f_0^n$, and (by (2.8) and (2.9))

$$\forall T > 0 \sup_{t \in [0,T]} \sup_n \int\int f^n (1 + |x|^2 + |\xi|^2)\, dx\, d\xi < \infty , \tag{2.22}$$

$$\forall T > 0 \sup_{t \in [0,T]} \sup_n \int\int f^n |\ln f^n|\, dx\, d\xi < \infty , \tag{2.23}$$

$$\sup_n \int_0^\infty \int\int e_n(f^n)\, dx\, d\xi\, dt < \infty , \tag{2.24}$$

where

$$e_n(f^n) = \frac{1}{4}\left(1 + \delta_n \int f^n\, d\xi\right)^{-1} \cdot \int\int (f^{n\prime} f_*^{n\prime} - f^n f_*^n) \ln\left(\frac{f^{n\prime} f_*^{n\prime}}{f^n f_*^n}\right) q_n\, d\xi_*\, dn . \tag{2.25}$$

We recall now the Dunford-Pettis criterion for weak compactness in L^1: Let $\{f_n\}_{n \in \mathcal{N}} \subset \subset L^1(\mathfrak{R}^3)$. Then the following *i*) and *ii*) are equivalent.

i) $\{f_n\}$ is contained in a weakly sequentially compact set of $L^1(\mathfrak{R}^3)$.

iia) $\{f_n\}$ is bounded in $L^1(\mathfrak{R}^3)$.

iib) $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that $\forall E \subset \mathfrak{R}^3$ (E measurable) with $\lambda(E) < \delta$, $\sup_n \int_E |f_n|\, dx \leqslant \varepsilon$.

iic) $\forall \varepsilon > 0 \; \exists K$ compact, $K \subset \mathfrak{R}^3$, such that $\sup_n \int_{\mathfrak{R}^3 - K} |f_n|\, dx \leqslant \varepsilon$.

We will apply the criterion to the following situation. If $h \in C(\mathfrak{R}_+, \mathfrak{R}_+)$ and $w \in L^\infty_{\mathrm{loc}}(\mathfrak{R}^3, \mathfrak{R}_+)$ are such that $h(t)/t \to \infty$ $(t \to \infty)$ and $w(x) \to \infty$ $(|x| \to \infty)$, then the inequality

$$\sup_n \int_{\mathfrak{R}^3} [h(|f_n|) + |f_n|(1 + w)]\, dx < \infty \tag{2.26}$$

implies that $\{f_n\}_{n \in \mathcal{N}}$ satisfies *ii*).

A major problem with weak convergence is the well-known fact that nonlinear functions are in general not weakly continuous. A useful property is, however, the fact that convex functions are at least lower semi-continuous. If $F\colon \mathfrak{R} \to \mathfrak{R}$ is convex and if $f_n \underset{w}{\to} f$ in L^1, then

$$\int F \circ f\, dx \leqslant \liminf_{n \to \infty} \int F \circ f_n\, dx . \tag{2.27}$$

Also, if one of the factors in a product converges a.e. and the other factor converges weakly, then the product is compact in the weak topology. Specifically, let $f_n \underset{w}{\to} f$ in L^1 let $\{g_n\} \subset L^\infty$ be bounded and let $g_n \to g$ a.e., then

$$f_n \cdot g_n \underset{w}{\to} fg \text{ in } L^1 . \tag{2.28}$$

This follows because for every $\varepsilon > 0$ there is a compact set K such that

$$\sup_n \int_{\mathfrak{R}^3 \setminus K} (|f_n g_n| + |fg|)\, dx \leqslant \varepsilon ,$$

and by Egorov's theorem, there is a set $E \subset K$ such that

$$\sup_n \int_E |f_n| \, dx \leqslant \varepsilon$$

and such that $g_n \to g$ uniformly on $K \backslash E$.

We can now work with the «approximating sequence of solutions to modified equations» given above. Q_+^n, Q_-^n, A_n all refer to this situation. The collision kernel in $Q_\pm^n$ is really x- (and t-) depend, and given by

$$q_n(x, V, n) = \left(1 \Big/ \left(1 + \delta_n \int f_n \, d\xi\right)\right) q_n(V, n).$$

Lemma 2.7. For all $T > 0$, $R > 0$ the sequences

$$(Q_+^n(f_n, f_n))/(1 + f_n) \quad \text{and} \quad (Q_-^n(f_n, f_n))/(1 + f_n) \tag{2.29}$$

are contained in weakly compact subsets of $L^1((0, T) \times \mathfrak{R}^3 \times B_R)$, where $B_R = \{\xi \in \in \mathfrak{R}^3;\ \|\xi\| \leqslant R\}$.

Since $\{f^n\}$ has uniformly bounded entropy and second moments, (2.26) implies that we can extract a subsequence (again denoted by $\{f^n\}$) which converges weakly in $L^1((0, T) \times \mathfrak{R}^3 \times \mathfrak{R}^3)$,

$$f^n \underset{w}{\to} f. \tag{2.30}$$

Let $g_\delta^n := (1/\delta) \ln(1 + \delta f^n)$. The uniform bounds on entropy and second moments for f^n easily imply that

$$\sup_{t \in [0, T]} \sup_n \|f^n - g_\delta^n\|_{L^1(\mathfrak{R}^3 \times \mathfrak{R}^3)} \to 0 \qquad \text{as } \delta \to 0. \tag{2.31}$$

Also, since

$$\Lambda g_\delta^n = (1/(1 + \delta f^n)) Q^n(f^n, f^n), \tag{2.32}$$

then

$$U(-t-h) g_\delta^n - U(-t) g_\delta^n = \int_t^{t+h} \frac{U(-s) Q^n(f^n, f^n)(s)}{1 + \delta U(-s) f_n(s)} \, ds. \tag{2.33}$$

By the compactness insured by Lemma 2.7, then $\forall \delta > 0$ $\forall T > 0$ $\forall R > 0$

$$\sup_{t \in [0, T]} \sup_n \|U(-t-h) g_\delta^n - U(-t) g_\delta^n(t)\|_{L^1(\mathfrak{R}^3 \times B_R)} \to 0$$

as $h \to 0$. We next estimate, by (2.31) and (2.8),

$$\sup_t \|U(-t-h) f^n - U(-t) f^n\|_{L^1(\mathfrak{R}^3 \times \mathfrak{R}^3)} \leqslant \leqslant 0(\delta) + \sup_t \|U(-t-h) g_\delta^n - U(-t) g_\delta^n\|_{L^1(\mathfrak{R}^3 \times B_R)}. \tag{2.34}$$

This easily entails

$$\sup_{t \in [0, T]} \sup_n \|U(-t-h) f^n - U(-t) f^n\|_{L^1(\mathfrak{R}^3 \times \mathfrak{R}^3)} \underset{h \to 0}{\longrightarrow} 0, \tag{2.35}$$

and a standard equicontinuity argument shows that the (weak) limit f must then satisfy

$$U(-t) f \in C(\mathfrak{R}_+; L^1(\mathfrak{R}^3 \times \mathfrak{R}^3)) \tag{2.36}$$

and, for all $T > 0$

(2.37) $$\sup_{t \in [0,T]} \|U(-t-h)f - U(-t)f\|_{L^1} \underset{h \to 0}{\longrightarrow} 0 .$$

Actually, by using an elementary argument from integration theory,

(2.38) $$f \in C(\mathfrak{R}_+ ; L^1(\mathfrak{R}^3 \times \mathfrak{R}^3)) .$$

Also, by using the convexity of the function $x \cdot \max(\ln x, 0)$,

(2.39) $$\forall t \int\int f|\ln f|\, d\xi\, dx + \limsup_n \int_0^t \int\int e_n(f^n)\, d\xi\, dx \leqslant$$
$$\leqslant \int\int f_0(|\ln f_0| + 2|x|^2 + 2|\xi|^2)\, d\xi\, dx + C ,$$

and

(2.40) $$\forall t \int\int f(1 + |x|^2 + |\xi|^2)\, d\xi\, dx \leqslant \int\int f_0(1 + 2|x|^2 + (2t^2+1)|\xi|^2)\, d\xi\, dx .$$

By now, we have a weakly convergent sequence $f_n \to f$, and the limit f is in $C([0,T]; L^1)$. Subsequences of $(Q^n_{+,-}(f^n, f^n))/(1+f^n)$ will also converge weakly (by Lemma 2.7), but we cannot say *a priori* whether the limits will by $(Q_{+,-}(f,f))/(1+f)$, because nonlinear functionals are in general not weakly continuous. This problem was first overcome by DiPerna and Lions by a skillful use of results known as «velocity averaging lemmas» (see [10]). We present these below (actually, we confine our discussion to a simplified situation, which is all we need).

LEMMA 2.8. Let $u \in L^2(\mathfrak{R} \times \mathfrak{R}^3 \times \mathfrak{R}^3)$ have compact support, and suppose that $\Lambda u \in L^2(\mathfrak{R} \times \mathfrak{R}^3 \times \mathfrak{R}^3)$. Then $\int u\, d\xi \in H^{1/2}(\mathfrak{R} \times \mathfrak{R}^3)$, and the $H^{1/2}$-norm of $\int u\, d\xi$ is bounded in terms of $\|u\|_{L^2}$, $\|\Lambda u\|_{L^2}$ and the support of u.

We will use Lemma 2.8 to pass from weak to strong convergence in L^1-settings. The next lemma is the crucial step.

LEMMA 2.9. Suppose that $\{g_n\} \subset L^1((0,T) \times \mathfrak{R}^3 \times \mathfrak{R}^3)$ is weakly relatively compact, and that $\{\Lambda g_n\}$ is weakly relatively compact in $L^1_{\text{loc}}((0,T) \times \mathfrak{R}^3 \times \mathfrak{R}^3)$. Then, if $\{\psi_n\}$ is a bounded sequence in $L^\infty((0,T) \times \mathfrak{R}^3 \times \mathfrak{R}^3)$ which converges a.e., then $(\int g_n \psi_n\, d\xi)$ is compact in the norm topology in $L^1((0,T) \times \mathfrak{R}^3)$.

We note an immediate

COROLLARY. Under the hypotheses of Lemma 2.9, if $g_n \underset{w}{\to} g$ in $L^1((0,T) \times \mathfrak{R}^3 \times \times \mathfrak{R}^3)$ and $\psi_n \to \psi$ a.e., then

(2.41) $$\left\| \int g_n \psi_n\, d\xi - \int g\psi\, d\xi \right\|_{L^1((0,T) \times \mathfrak{R}^3)} \to 0 .$$

We also have:

LEMMA 2.10. Let $\{f_n\}$ be a relatively compact sequence in $L^1((0,T) \times \mathfrak{R}^3 \times \mathfrak{R}^3)$, and suppose that there is a family of real-valued uniformly Lipschitz continuous functions $\{\beta_\delta\}_{\delta > 0}$, $\beta_\delta(0) = 0$ for all δ, such that

i) $\beta_\delta(s) \to s$ as $\delta \to 0$, uniformly on compact subsets of $\mathfrak{R}_+$, *ii*) the sequence $\{T(\beta_\delta(f^n))\}$ is, for every δ, weakly relatively compact in $L^1_{\text{loc}}((0,T)\times\mathfrak{R}^3\times\mathfrak{R}^3)$. Then, if $f^n \underset{w}{\to} f$ in L^1_+, $\{\psi_n\}_n$ is bounded in $L^\infty((0,T)\times \mathfrak{R}^3\times\mathfrak{R}^3)$ and $\psi_n \to \psi$ a.e.,

$$\lim_{n\to\infty}\left\|\int f^n\psi_n\,d\xi - \int f\psi\,d\xi\right\|_{L^1} = 0. \tag{2.42}$$

We return now to our sequence. We have:

LEMMA 2.11. Let $\{f^n\}$ be the sequence of solutions to approximating problems as above. There is a subsequence such that for each $T>0$

i) $\int f^n d\xi \to \int f d\xi$ a.e. and in $L^1((0,T)\times\mathfrak{R}^3)$,

ii) $A_n * f^n \to A * f$ in $L^1((0,T)\times\mathfrak{R}^3\times B_R)$ for all $R>0$, and a.e.,

iii) for each compactly supported function $\varphi\in L^\infty((0,T)\times\mathfrak{R}^3\times\mathfrak{R}^3)$,

$$\left(\frac{\int Q^n_\pm(f^n,f^n)\,\varphi\,d\xi}{1+\int f^n d\xi}\right)\to\left(\frac{\int Q_\pm(f,f)\,\varphi\,d\xi}{1+\int f d\xi}\right) \tag{2.43}$$

in $L^1((0,T)\times\mathfrak{R}^3)$.

Consider now Λ^{-1}, defined by $u=\Lambda^{-1}g$, *i.e.* $\Lambda u = g$ with $u|_{t=0}=0$:

$$\Lambda^{-1}g(x,\xi,t) = \int_0^t g(x-(t-s)\,\xi,\xi,s)\,ds. \tag{2.44}$$

Λ^{-1} is, as one checks immediately, continuous and weakly continuous from $L^1((0,T)\times\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}})$ into $C([0,T]);\,L^1(\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}})$, and if $g\geqslant 0$, also $\Lambda^{-1}g\geqslant 0$. We use Λ^{-1} to rewrite the Boltzmann equation in yet another form.

Suppose that $F\in C([0,T];\,L^1(\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}}))$, $\Lambda F\geqslant 0$. The operator defined as follows: $\Lambda_F^{-1} := e^{-F}\Lambda^{-1}e^F$ is then continuous (and weakly continuous) from $L^1((0,T)\times\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}})$ into $C([0,T];\,L^1(\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}}))$.

If $\{F_n\}$ is a bounded sequence in $C([0,T];\,L^1(\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}}))$ such that $\Lambda F_n\geqslant 0$, $F_n(x,\xi,t)\to F(x,\xi,t)$ for all t and almost all (x,ξ), and if $g_n \underset{w}{\to} g$ in $L^1((0,T)\times(\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}}))$, then, for all $t\in[0,T]$,

$$\Lambda_{F_n}^{-1}g_n(t)\xrightarrow{w}\Lambda_F^{-1}g(t) \tag{2.45}$$

in $L^1(\mathfrak{R}^3\times\mathfrak{R}^3_{\text{loc}})$. (This is easily proved by using the explicit solution formula for Λ^{-1}, given above.)

To use (2.45), let $F_n=\Lambda^{-1}(A_n * f^n)$, where f^n, A_n are from the modified Boltzmann equation, as used above. This equation can be written as

$$\Lambda f^n + (A_n * f^n)\,f^n = Q^n_+(f^n,f^n) \tag{2.46}$$

or, after multiplication with e^{F_n} and observing that

$$\begin{cases}\Lambda(f^n e^{F_n}) = (\Lambda f^n)\,e^{F_n} + f^n(\Lambda F_n)\,e^{F_n} = e^{F_n}Q^n_+(f^n,f^n),\\ f^n = f^n_0 e^{-F_n} + \Lambda_{F_n}^{-1}Q^n_+(f^n,f^n).\end{cases} \tag{2.47}$$

By Lemma 2.11 *ii*) and the above remarks, $\{F_n\}$ is a bounded sequence in

$C((0, T); L^1(\mathfrak{R}^3) \times \mathfrak{R}^3_{loc})$ and for all $t \in \mathfrak{R}_+$

$$F_n \to F = \Lambda^{-1}(A * f) \quad \text{a.e.} \tag{2.48}$$

Starting from these remarks a careful argument gives [9, 11, 3]:

LEMMA 2.12. For all $t \in \mathfrak{R}_+$, we have $\Lambda_F^{-1} Q_+(f, f) \in L^1(\mathfrak{R}^3 \times \mathfrak{R}^3_{loc})$ and

$$f = U(t) f_0 e^{-F} + \Lambda_F^{-1} Q_+(f, f). \tag{2.49}$$

Equation (2.49) is already saying that f satisfies the Boltzmann equation in some sense. We will now simply check that it satisfies the criteria for a renormalized solution (as given before).

First, it is easy to show that for every $T < \infty$

$$(Q_-(f, f))/(1 + f) \in L^1([0, T] \times \mathfrak{R}^3 \times \mathfrak{R}^3_{loc}) \tag{2.50}$$

$\Bigl($just use the condition on A and that

$$\sup_{t \in [0, T]} \sup_n \int\int f^n (1 + |x|^2 + |\xi|^2)\, dx\, d\xi < \infty \Bigr). \tag{2.51}$$

As for $(Q_+(f, f))/(1 + f)$, by an elementary inequality [3, 9, 10], we have

$$Q_\pm^n(f^n, f^n)\Bigl(1 + \delta \int f^n\, d\xi\Bigr)^{-1} \leqslant \tag{2.52}$$

$$\leqslant 2 Q_\mp^n(f^n, f^n)\Bigl(1 + \delta \int f^n\, d\xi\Bigr)^{-1} + \frac{4 e_n}{\ln 2}\Bigl(1 + \delta \int f^n\, d\xi\Bigr)^{-1}$$

and

$$\sup_n \int_0^\infty \int\int e_n(f^n)\, dx\, d\xi\, ds < \infty . \tag{2.53}$$

Because of the nonnegativity of $e_n(f^n)$ and (2.53), we can assume that $e_n(f^n)$ converges weakly (in $\mathcal{D}'$, or in the vague topology on the bounded measures) to a bounded, nonnegative measure μ by Lemma 2.11; we also know that the other two terms in (2.52) converge weakly in L^1 and so

$$\frac{Q_\pm(f, f)}{1 + \delta \int f\, d\xi} \leqslant \frac{2 Q_\mp(f, f)}{1 + \delta \int f\, d\xi} + \frac{4}{\ln 2\bigl(1 + \delta \int f\, d\xi\bigr)}\, \mu . \tag{2.54}$$

(2.54) remains true if we replace μ by its absolutely continuous part $e \in L^1((0, T) \times \times \mathfrak{R}^3 \times \mathfrak{R}^3)$, and by taking $\delta \to 0$ it follows that

$$Q_\pm(f, f) \leqslant 2 Q_\mp(f, f) + E \tag{2.55}$$

with $E \in L^1((0, T) \times \mathfrak{R}^3 \times \mathfrak{R}^3)$. (2.50) and (2.55) now entail that

$$(Q_+(f, f))/(1 + f) \in L^1([0, T] \times \mathfrak{R}^3 \times \mathfrak{R}^3_{loc}). \tag{2.56}$$

To show that $Q_+(f, f)(x, \xi, \cdot) \in L^1(0, T)$ for almost all (x, ξ), we use that by Lemma 2.12 for all t, $\Lambda_F^{-1} Q_+(f, f) \in L^1(\mathfrak{R}^3 \times \mathfrak{R}^3_{loc})$ and $F \in L^1(\mathfrak{R}^3 \times \mathfrak{R}^3_{loc})$. Explicitly, we see that

$$(2.57) \qquad \int_0^t U(-s)\,Q_+(f,f)\exp-(U(-t)F-U(-s)F)\,ds$$

is in $L^1(\mathfrak{R}^3\times\mathfrak{R}^3_{\mathrm{loc}})$ for all t, and because $U(-t)F$ is nonnegative, *increasing* with respect to t and in $L^1(\mathfrak{R}^3\times\mathfrak{R}^3_{\mathrm{loc}})$ with respect to (x,ξ), it follows that $U(-\,-t)\,Q_+(f,f)\in L^1(0,T)$ for almost all (x,ξ). For Q_-, the same assertion follows from (2.55). Now we can use Lemma 2.12 to conclude that f is a mild solution of the Boltzmann equation in the sense defined above.

The only remaining step is the verification of the entropy estimate (2.9) from (2.33). This is a consequence of the proof of Lemma 2.11, which, for all $\delta>0$, entails

$$\frac{f^n f^n_*}{1+\delta\int f^n\,d\xi}\underset{w}{\rightarrow}\frac{ff_*}{1+\delta\int f\,d\xi},$$

$$\frac{f^{n\prime}f^{n\prime}_*}{1+\delta\int f^n\,d\xi}\underset{w}{\rightarrow}\frac{f'f'_*}{1+\delta\int f\,d\xi},$$

in $L^1((0,T)\times\mathfrak{R}^3_x\times\mathfrak{R}^3_\xi\times S^2)$. Now, by using the convexity of the function

$$(x,y)\rightarrow(x-y)\ln(x/y)$$

on $\mathfrak{R}_+\times\mathfrak{R}_+$, we see that for all $T>0$

$$\int_0^T\!\!\int\!\!\int\frac{e(f)}{1+\delta\int f\,d\xi}\,dx\,d\xi\,dt\leqslant\liminf_{n\rightarrow\infty}\int_0^T\!\!\int\!\!\int\frac{e_n(f^n)}{1+\delta\int f^n\,d\xi}\,dx\,d\xi\,dt\,.$$

The entropy estimate (2.9) follows from this and the monotone convergence theorem in the limit $\delta\rightarrow0$.

Once the lemmas above are taken for granted, this completes the proof of the theorem of DiPerna and Lions.

This result is of the greatest importance for the theory of the Boltzmann equation, yet leaves a lots of problems open. Is the «renormalized solution» a weak solution in the usual sense as well? A positive answer in a particular case will be given in Section 7. Is the solution unique? (this question is open even for the case just mentioned). How regular is the solution? Thus, as any important new result in mathematics, the theorem of DiPerna and Lions solved a problem but opened a new chapter of research.

3. More on boundary conditions

As we indicated in Section 1 on $\partial\Omega$ we impose a linear boundary condition of a rather standard form [1-3]:

$$(3.1) \qquad \gamma^+f(t,x,\xi)=\int_{\xi'\cdot n<0}K(\xi'\rightarrow\xi;x,t)\,\gamma^-f(t,x,\xi')\,d\xi'\equiv K\gamma^-f$$

$$(x\in\partial\Omega,\ \xi\cdot n>0)$$

where $K(\xi' \to \xi; x, t)$ is a kernel (the boundary scattering kernel) such that:

$$K(\xi' \to \xi; x, t) \geqslant 0\,, \tag{3.2}$$

$$\int_{\xi\cdot n > 0} K(\xi' \to \xi; x, t)\,|\xi\cdot n|\,d\xi = |\xi'\cdot n|\,, \tag{3.3}$$

$$M_w(\xi) = \int_{\xi'\cdot n < 0} K(\xi' \to \xi; x, t)\,M_w(\xi')\,d\xi' \tag{3.4}$$

where M_w is the wall Maxwellian given by eq. (1.10) with $u = 0$ and $\beta = \beta_w$ (β_w being the inverse temperature of the wall), whereas $\gamma^{\pm}$ are the trace operators introduced in Section 1.

The case of an isothermal boundary has been treated by Hamdache [12]. In the case of non-isothermal data along $\partial\Omega$ the initial-boundary value problem possesses boundary data which are compatible, not with a Maxwellian, but rather with one of those steady solutions, whose theory is still in its infancy (for an example see the paper by Arkeryd *et al.* [13]); thus one cannot expect the solution the tend toward a Maxwellian when $t \to \infty$ as has been recently shown for other kinds of boundary conditions [14-17]. The main difficulties in tackling this problem seem to lie with large velocities. For this reason, Arkeryd and Cercignani [18] introduced a modified Boltzmann equation in which they cut off all the collisions such that the sum of the squares of two colliding molecules is larger than m^2 where m is an assigned positive constant. The only place where they used this cutoff was the entropy estimate and the need for the cutoff disappears when the temperature is constant. Thus their paper contains also a slightly different proof of Hamdache's theorem, with an extension to more general boundary conditions, to a more detailed study of the boundary behavior, and for the full class of collision operators of the DiPerna-Lions existence context [9].

A central observation for these proofs was an inequality introduced by Darrozès and Guiraud [19] in 1966 and subsequently discussed by several authors. We shall state this inequality in the form of a lemma.

LEMMA 3.1 [1, 19-22]. If eqs. (3.1), (3.2), (3.3) and (3.4) hold, then:

$$\int \xi\cdot n\gamma f \log \gamma f\,d\xi \leqslant -\beta_w \int \xi\cdot n\,|\xi|^2\gamma f\,d\xi \qquad (\text{a.e. in } t \text{ and } x \in \partial\Omega) \tag{3.5}$$

where β_w is the inverse temperature evaluated at the point $x \in \partial\Omega$. Unless the kernel in eq. (3.1) is a delta function, equality holds in eq. (3.9) if and only if the trace γf of f on $\partial\Omega$ coincides with M_w (the wall Maxwellian).

For a proof see [21, 2, 22].

If the wall is moving, the above relations hold in the reference frame of the wall. Then, since the Maxwellian M_w has a drift velocity u_w, if we want to adopt a reference frame, with respect to which the wall moves, then ξ must be replaced by $\xi - u_w$.

In a paper by Arkeryd and Maslova [23] the work of Arkeryd and Cercignani [18] was extended to the noncutoff case at the price of introducing some restrictions on the kernel $K(\xi' \to \xi; x, t)$. Subsequently it was shown [24] that one of the conditions can be replaced by a more natural one thanks to the above inequality.

As for external problems there appears to be just one paper [25] dealing with the existence problem at the level of generality of the DiPerna and Lions paper [9].

All these results will be surveyed in the subsequent sections. Attention will be paid to some further results restricted to the case when the solution just depends on one space coordinate and to the problem of the trend to equilibrium which was mentioned above.

4. The results of Arkeryd and Cercignani

Before discussing the existence theorems for initial-boundary value problems, we need to recall some trace results giving the L^1 regularity of the trace of f on the boundary and study the semigroup generated by the free streaming operator, including a sort of Green's formula, that will be used in Section 6. This is done in detail by Arkeryd and Cercignani [18] and will not be repeated here.

In order to deal with the existence theorem in a vessel at rest, with a temperature that varies from one point to another, it is convenient to remark that there is a Maxwellian naturally associated with the problem at each point of the boundary, *i.e.* the wall Maxwellian M_w; an exception is offered by specularly reflecting boundaries, which will not be considered here because they have no temperature associated with the boundary. Equation (3.5), gives (for smooth solutions):

$$\int \xi\cdot n\gamma f \log \gamma f \, d\xi + \beta_w \int \xi\cdot n |\xi|^2 \gamma f \, d\xi \leqslant 0 \qquad (\text{a.e. in } t \text{ and } x \in \partial\Omega). \tag{4.1}$$

For this reason Arkeryd and Cercignani [18] consider an inverse temperature $\beta(x)$ with $\inf \beta(x) > 0$ which reduces to β_w at each point of the boundary and otherwise depends smoothly on x and the modified H-functional:

$$H = \int f \log f \, d\xi \, dx + \int \beta(x) |\xi|^2 f \, d\xi \, dx . \tag{4.2}$$

In general H will not decrease in time, as a consequence of the Boltzmann equation and inequality (4.1), because a simple calculation shows that

$$\frac{dH}{dt} \leqslant - \int \xi\cdot \frac{\partial\beta}{\partial x} |\xi|^2 f \, d\xi \, dx . \tag{4.3}$$

Thanks to the truncation for large speeds [18] $|\xi| \leqslant m$ and the right hand side of eq. (4.3) is bounded by a constant C given by

$$C = m^3 \left\| \frac{\partial\beta}{\partial x} \right\|_{L^\infty} \int f_0 \, d\xi \, dx . \tag{4.4}$$

Thus H is bounded by $H_0 + CT$ on $[0, T]$ if bounded initially by H_0.

This is the only point where the truncation for large speeds is needed; further, the truncation can be dispensed with, if β is constant because the left hand side of eq. (4.3) is bounded by 0.

Equation (4.2) implies [18] that both $\int f |\log f| \, d\xi \, dx$ and $\int |\xi|^2 f \, d\xi \, dx$ are separately bounded in terms of the initial data.

In the following we shall use the following notation:

$$\langle f, g\rangle = \int_{\mathcal{D}} fg\, dx\, d\xi\, dt\,, \tag{4.5}$$

$$\langle f, g\rangle_{\pm} = \int_{\partial E^{\pm}} fg\, d\sigma^{\pm}\,, \tag{4.6}$$

$$\langle f, g\rangle_t = \int_{\Omega\times R^3} f(t, x, \xi)\, g(t, x, \xi)\, dx\, d\xi\,, \tag{4.7}$$

$$(f, g)_{\pm} = \int_{\pm\xi\cdot n>0} fg\,|\xi\cdot n(x)|\, d\xi\,. \tag{4.8}$$

We also define the backward and forward stay times as

$$t^+ = t^+(x, \xi, t) = \inf\{s > 0; x - s\xi \in \partial\Omega\}, \tag{4.9}$$

$$t^- = t^-(x, \xi, t) = \inf\{s > 0; x + s\xi \in \partial\Omega\} \tag{4.10}$$

with the related quantities

$$s^+(x, \xi, t) = \min\{t, t^+(x, \xi, t)\}, \tag{4.11}$$

$$s^-(x, \xi, t) = \min\{T - t, t^-(x, \xi, t)\}. \tag{4.12}$$

We also use the mappings

$$R^s : \overline{\mathcal{D}} \to \overline{\mathcal{D}}$$

with

$$R^s(x, \xi, t) = (t + s, x + s\xi, \xi) \tag{4.13}$$

to define

$$f^{\#}(s, x, \xi, t) = f^{\#}(s) = f(R^s((x, \xi, t)))\,. \tag{4.14}$$

As hinted at in Section 1, in this and the next sections use will be made of the equivalent concepts of exponential, mild, and renormalized solutions as defined by DiPerna and Lions [9] for the Cauchy problem and such solutions will be found as limits of functions solving truncated equations.

The definitions of these solutions require some comment because of the boundary conditions, which are not satisfied exactly but only in the form of an inequality, as Hamdache [12] first pointed out. This aspect of the matter has been discussed by Arkeryd and Maslova [23] in some detail. The basic point is that, when approximating a solution with a sequence, we partially lose control upon the traces, that can only be shown to tend to measures $\mu^{\pm}$ of the spaces $\mathcal{M}^{\pm}$ of σ-finite measures defined on the σ-algebras $\mathcal{B}^{\pm}$ of Borel sets from $E^{\pm}$. Each of these measures can be decomposed into a part completely continuous $\mu_c^{\pm}$ with density $f^{\pm}$ with respect to the Lebesgue measure $d\sigma^{\pm}$, and a singular part $\mu_s^{\pm}$. The measures $\mu^{\pm}$ satisfy the boundary conditions:

$$\mu^+ = K\mu^- \tag{4.15}$$

where K can be defined for measures via

$$\langle \varphi, K\mu^-\rangle_+ = \langle K^*\varphi, \mu_-\rangle_- \tag{4.16}$$

where the adjoint operator K^* is defined by:

$$(4.17)\ K^*\varphi(x,\xi',t) = \int_{\xi\cdot n(x)>0} \varphi(x,\xi,t)K(\xi'\to\xi;t,x)|\xi\cdot n(x)|/|\xi'\cdot n(x)|\,d\xi,$$

and assumed to carry $C_0(E^+)$ into $C_b(E^-)$.

The traces of a solution $\gamma^\pm f$ will only satisfy

$$(4.18)\qquad \frac{d\mu^\pm}{d\sigma^\pm} \geq \gamma^\pm f.$$

Then we can introduce the following definitions:

DEFINITION 4.1. f is a mild solution of (1.1)-(1.3) if

$$(4.19)\qquad f\in L^1(\mathcal{D}),\quad f\geq 0,\quad (Q^\pm)^\# \in L^1([0,s_-(x,\xi,t)]),$$

$$(4.20)\qquad f^\#(s,x,\xi,t) = f^\#(\tau,x,\xi,t) + \int_\tau^s Q^\#(z,x,\xi,t)\,dz,\qquad 0\leq s\leq\tau\leq s^-(x,\xi,t),$$

$$(4.21)\qquad f = f_0(x,\xi)\quad \text{in } E^0\quad (E^0 = (x,\xi,t\in\overline{\mathcal{D}};t=0))$$

for a.e. $(x,\xi,t)\in E^\pm\cup E^0$, and there are $\mu^\pm\in\mathcal{M}^\pm$ satisfying (4.15) and (4.18).

DEFINITION 4.2. f is a solution in exponential multiplier form (or exponential solution for short) of (1.1)-(1.3) if

$$(4.22)\qquad f\in L^1(\mathcal{D}),\quad f\geq 0,\quad \nu(f)\in L^1_{\text{loc}}(\mathcal{D}),$$

$$(4.23)\qquad f^\#(s,x,\xi,t) = [f_0(x,\xi)\chi^0 + \gamma_+ f\chi^+]\exp\left(-\int_0^s(\nu(f))^\#(z,x,\xi,t)\,dz\right) +$$

$$+\int_0^s (Q^+)^\#(z,x,\xi,t)\exp\left(-\int_z^s(\nu(f))^\#(z',x,\xi,t)\,dz'\right)dz,\qquad 0\leq s\leq s^-(x,\xi,t)$$

for a.e. $(x,\xi,t)\in E^\pm\cup E^0$, and there are $\mu^\pm\in\mathcal{M}^\pm$ satisfying (4.15) and (4.18). Here χ^+ and χ^0 denote the characteristic functions of E^+ and E^0.

DEFINITION 4.3. f is a renormalized solution of (1.1)-(1.3) if

$$(4.24)\qquad f\in L^1(\mathcal{D}),\quad f\geq 0,\quad \nu(f)\in L^1_{\text{loc}}(\mathcal{D})$$

and f is a weak solution (with test functions vanishing in the neighborhood of $E^\pm$) of

$$(4.25)\qquad \Lambda\log(1+f) = (Q(f,f))/(1+f)\quad \text{in } \mathcal{D}$$

and there are $\mu^\pm\in\mathcal{M}^\pm$ satisfying (4.15) and (4.18).

The existence theorem proved by Arkeryd and Cercignani [18] reads as follows

THEOREM 4.1. Let $f^0\in L^1(\Omega\times\mathfrak{R}^3)$ be such that

$$(4.26)\qquad \int f^0(1+|\xi|^2)\,d\xi\,dx<\infty;\qquad \int f^0|\log f^0|\,d\xi\,dx<\infty.$$

Then there is a solution $f \in C(\mathfrak{R}_+, L^1(\Omega \times \mathfrak{R}^3))$ of the Boltzmann equation such that $f(\cdot, 0) = f^0$, which also satisfies mass conservation and has an H-functional with a bounded time derivative.

For the proof we refer to the original paper [18].

It is interesting to study the boundary condition satisfied by these solutions and prove

THEOREM 4.2. There is a solution as in Theorem 4.1, which satisfies

$$\gamma^+(f) \geqslant K(\gamma^- f) \quad \text{a.e. on } E^+ . \tag{4.27}$$

For the proof, we refer again to the original paper. We remark that the fact that we obtain an inequality is a consequence of the fact that we can only expect convergence of the traces of the approximating sequence to measures, as discussed above. In fact eq. (4.27) follows by taking the completely continuous part of eq. (4.16).

Theorems 4.1 and 4.2 contain Hamdache's result [12] and extend it. The extension is of interest for the study of the solutions of the Boltzmann equation when the boundaries drive the gas out of equilibrium. In order to obtain a realistic result one needs to remove the cutoff, as done for the first time by Arkeryd and Maslova [23] and discussed in the next section.

5. THE RESULTS OF ARKERYD AND MASLOVA

In this section we study the results presented by Arkeryd and Maslova in a recent paper [23]. They introduce a class of boundary operators for which (1.6), (3.1)-(3.3) hold, by restricting the adjoint operator K^*, but are able to avoid the cutoff for large velocities.

A better control of mass, energy and entropy for the distributions emerging from the wall are provided by the following conditions:

(5.1) There exists $K_2 > 0$ such that $K^* \, |\xi \cdot n(x)| \geqslant K_2$ (spreading condition).

(5.2) There exists $K_3 \leqslant \infty$ such that $K^* \, |\xi|^2 \leqslant K_3$ (energy condition).

(5.3) There exists $K_4 < \infty$ and $\alpha \in [0, 1)$ such that, for every $f \in L^1(\Gamma^-)$ with $f \geqslant 0$, it holds $\langle Kf, \log (Kf/(f, 1)_-)\rangle_+ - \alpha \hat{H}^- \leqslant K_4 (q_2^- + q)$ (entropy condition).

Here

$$\begin{cases} \hat{H}^- = \langle f, \log (f/(f, 1)_-)\rangle_- , & q_j^\pm = \langle f, |\xi|^j\rangle_\pm , \\ q = \langle f, |\xi \cdot n|\rangle_+ + \langle f, |\xi \cdot n|\rangle_- . \end{cases} \tag{5.4}$$

These conditions are reasonable for a linear operator, except for (5.3), which appears a bit unusual, since it is nonlinear, albeit homogeneous of first degree, in f. In the next section we shall discuss how to dispense with that condition by using Lemma 3.1 in a suitable way.

The other conditions have the following role [23, Lemma 4.1]:

1) Equation (5.1) (together with a proper use of momentum balance) gives a control on the mass flow hitting the boundary.

2) Equation (5.2) (together with 1)) gives a control on q_2^+.

3) Using 1) and 2) together with energy balance one obtains an a priori bound upon the energy without using the entropy estimates.

At this point Arkeryd and Maslova [23] use eq. (5.3) to bound entropy and entropy source. They also obtain bounds on $\langle f, |\log(f/(f, 1)_-)|\rangle_\pm$ which are related to entropy flows. To bound the latters, however, one should remove the dominator $(f, 1)_-$, which does not appear to be an easy matter.

The following lemma holds [23]:

LEMMA 5.1. Assume eqs. (3.2)-(3.3) and (5.1)-(5.3), together with

$$(f_0, \log f_0) \in L^1(\Omega), \tag{5.5}$$

$$(Q, \log f) \leqslant 0, \quad (Q, \psi) = 0 \quad \text{for } \psi = 1, \xi, |\xi|^2. \tag{5.6}$$

Then f satisfies the inequality

$$-\langle Q, \log f\rangle + H(T) + \langle f, |\log(f/(f, 1)_-)|\rangle_\pm \leqslant C(T), \tag{5.7}$$

with $C(T) > 0$ depending only on f_0 and on K_2, K_3, K_4.

Having these a priori bounds they proceed more or less as in the paper by Arkeryd and Cercignani [18], the main change being that they prefer to avoid the semigroups that were used there, and finally arrive at

THEOREM 5.2. Assume that

$$(1 + |\xi|^2) f_0, \quad f_0 \log f_0 \in L^1(\Omega \times R^3), \quad f_0 \geqslant 0 \tag{5.8}$$

and eqs. (5.1)-(5.3). Then there exists an exponential solution of (1.1)-(1.3) satisfying

$$\begin{cases} f \in C([0, T], L^1(\Omega \times R^3)), \quad f \geqslant 0, \quad \langle f, 1\rangle_t = \langle f_0, 1\rangle_0 ; \\ (1 + |\xi|^2)\gamma^\pm f \in L^{1\pm} ; \end{cases} \tag{5.9}$$

$$\sup_{t \leqslant T} [\langle f, \ln f\rangle_t + \langle f, |\xi|^2\rangle_t] + \langle e(f), 1\rangle \leqslant C(T). \tag{5.10}$$

Here, in agreement with the notation in Section 2:

$$e(f) = (1/4) \int_{R^3}\int_{B^+} (f'f'_* - ff_*) \log(f'f'_*/(ff_*)) B(\xi - \xi_*, n)\, d\xi_*\, dn. \tag{5.11}$$

6. Another result and a generalization

In this section, following a paper of the author [24], we want to prove a result which relaxes one of the assumptions of Arkeryd and Maslova [23] as well as a generalization to the case of moving boundaries.

The result alluded to is that Lemma 5.1 and (as a consequence) Theorem 5.2 hold without assuming (5.3), but only the compatibility with a Maxwellian, eq. (3.4). According to Lemma 3.1, the inequality of Darrozès and Guiraud [19] then holds. We shall prove

LEMMA 6.1. Assume eqs. (3.2)-(3.4) and (5.1)-(5.2), together with $\beta_w \leqslant C_0(T)$ and

$$(f_0, \log f_0) \in L^1(\Omega), \tag{6.1}$$

$$(Q, \log f) \leqslant 0, \quad (Q, \psi) = 0 \quad \text{for } \psi = 1, \xi, |\xi|^2. \tag{6.2}$$

Then f satisfies the inequality

$$-\langle Q, \log f\rangle + H(T) \leqslant C(T), \tag{6.3}$$

with $C(T) > 0$ depending only on f_0 and on K_2, K_3.

PROOF. Using Green's formula [6, 15, 20] and approximation

$$-\langle Q, \log f\rangle + H(T) + \langle f, \log f\rangle_- \leqslant H(0) + \langle f, \log f\rangle_+ . \tag{6.4}$$

Because of Lemma 3.1 this becomes

$$-\langle Q, \log f\rangle + H(T) + \langle \beta_w f, |\xi|^2\rangle_+ \leqslant H(0) + \langle \beta_w f, |\xi|^2\rangle. \tag{6.5}$$

Thanks to the fact that β_w is bounded, energy balance now gives [23, Lemma 4.1] that the last term is bounded by some $C(T)$. Hence

$$-\langle Q, \log f\rangle + H(T) + \langle \beta_w f, |\xi|^2\rangle_+ \leqslant C(T)$$

which implies that the three quantities in the left hand side are separately bounded. In particular, eq. (6.3) follows.

The only part of the thesis of Lemma 5.1, which does not follow from the new assumptions is the boundedness of the entropy flows $\langle f, |\log(f/(f, 1)_-)|\rangle_\pm$. This part of the lemma is never used in the proof of Theorem (5.2) and thus we can prove

THEOREM 6.2. Assume that

$$(1 + |\xi|^2) f_0, \quad f_0 \log f_0 \in L^1(\Omega \times R^3), \quad f_0 \geqslant 0,$$

and eqs. (3.2)-(3.4) and (5.1)-(5.2). Then there exists an exponential solution of (1.1)-(1.3) satisfying

$$\begin{cases} f \in C([0, T], L^1(\Omega \times R^3)), \quad f \geqslant 0, \quad \langle f, 1\rangle_t = \langle f_0, 1\rangle_0 ; \\ (1 + |\xi|^2)\gamma^\pm f \in L^{1\pm} ; \end{cases} \tag{6.6}$$

$$\sup_{t \leqslant T} [\langle f, \ln f\rangle_t + \langle f, |\xi|^2\rangle_t] + \langle e(f), 1\rangle \leqslant C(T) \tag{6.7}$$

where $e(f)$ is given by eq. (5.11).

REMARK. The above result applies also to the inhomogeneous boundary condition

$$\gamma^+ f(x, \xi, t) = \alpha f_+ + (1-\alpha) K \gamma^- f \quad (0 \leqslant \alpha \leqslant 1) \tag{6.8}$$

where $f_+ \geqslant 0$ is assigned.

We pass now to problems with moving boundaries. These are of some interest and do not seem to have been considered so far. The main difference is that in eq. (1.6) E^+ varies with time, because $\partial\Omega$ does. As remarked in Section 2, all the relations concerning the kernel K hold in the reference frame of the wall. Then, when the Maxwellian M_w has a drift velocity u_w, if we want to adopt a reference frame, with respect to which the wall moves, then ξ must be replaced by $\xi - u_w$. In particular, the indices + and − refer now to $(\xi - u_w)\cdot n > 0$ and $(\xi - u_w)\cdot n < 0$, respectively.

When we integrate the Boltzmann equation to obtain a priori inequalities, we obtain a factor $(\xi - u_w)\cdot n$ in place of $\xi \cdot n$, so that most of the changes compensate. The main differences arises in the entropy inequality, where a factor $|\xi - u_w|^2$ appears in place of simply ξ^2. The extra terms can be easily controlled, however, by means of the momentum balance equation (after scalar multiplication by a smooth vector-valued function $u(x, t)$, which reduces to u_w on the wall).

Then we have the following

COROLLARY 6.3. Theorem 6.2 holds in the presence of moving walls as well.

7. IMPROVED RESULTS IN THE CASE OF A SLAB

In a recent paper [26], R. Illner and the author proved a new result on the initial-boundary value problem for the nonlinear Boltzmann equation in the interval $\Omega = [0, 1]$ in one-dimensional spatial geometry, with general diffusive boundary conditions at $x = 0$ and $x = 1$. Thus in this section $x \in \Re$; in addition, the x-, y- and z-component of the velocity $\xi \in \Re^3$ will be denoted by ξ, η and ζ respectively; in order to avoid confusion we shall replace, when needed, the notation for the velocity vector $\xi \in \Re^3$ by v. The Boltzmann equation reads as follows

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = Q(f,f) \tag{7.1}$$

the remaining part of the notation being as before.

Cercignani and Illner [26] needed some truncations on the collision kernel B, in order to obtain more advanced results on the solution, in particular to replace the renormalized solutions of DiPerna and Lions by the more powerful result that the solution is a weak one in the usal sense. This line was started by the author [27-29]. In order to present their result, we assume that there is an $\varepsilon > 0$ such that

$$B(\ldots) = 0 \text{ if } |v - v_*| \leqslant \varepsilon\,. \tag{7.2}$$

$$B \text{ is bounded}\,. \tag{7.3}$$

A third and less serious assumption on B is that the ratio r between

$$\int_S [n\cdot(v - v_*)]^2 B(n\cdot(v - v_*), |v - v_*|)\, dn$$

and

$$|v - v_*|^2 \int_S B(n\cdot(v - v_*), |v - v_*|)\, dn$$

is bounded from below.

The assumption (7.2) can be summarized as saying that «collisions with small relative speed are disregarded» and is therefore physically more reasonable than the assumptions made by Arkeryd in [30].

For $x \in \partial\Omega$, *i.e.*, $x \in \{0, 1\}$, and $\omega = (-1)^x$, we impose the usual boundary conditions discussed in Sections 1 and 3.

The objective of [26] was to show that under reasonable assumptions on the diffuse boundary condition, and with the truncations on the collision kernel B made in (7.2) and (7.3), the initial-boundary value problem for the Boltzmann equation has a global weak solution in the usual sense. The main step in [26] was a proof that the gain and loss terms of the collision term $Q(f, f)$ are in $L^1([0, 1] \times \mathfrak{R}^3 \times [0, T])$ for any positive time $T > 0$. Cercignani and Illner [26] also showed that the boundary conditions are satisfied as identities in the weak sense, and obtained uniform bounds (for a given time interval) on the second moment (the kinetic energy) of f.

The assumptions on the boundary kernels are the same made in the previous section and thus exclude specular and bounce-back reflection.

In order to discuss the results of [26], we need some additional notation. For each $x \in [0, 1]$ and $t \geqslant 0$, let

$$\text{(7.4)} \qquad \begin{cases} \varrho(x, t) = \int f(x, v, t)\, dv\,, \\ m(t) = \int \varrho(x, t)\, dx\,, \\ j(x, t) = \int \xi f(x, v, t)\, dv\,, \\ p(x, t) = \int \xi^2 f(x, v, t)\, dv\,, \\ q(x, t) = \int \xi v^2 f(x, v, t)\, dv\,. \end{cases}$$

We call ϱ the mass density, $m(t)$ the total mass, j the mass flux (or momentum) in x-direction, p the momentum flux, and q the energy flux. At the boundaries we will need the ingoing and outgoing parts of these quantities. We use the abbreviations

$$
\begin{cases}
\varrho_+ = \displaystyle\int\limits_{\xi>0} f\,dv\,, & \varrho_- = \displaystyle\int\limits_{\xi<0} f\,dv\,, \\
j_+ = \displaystyle\int\limits_{\xi>0} \xi f\,dv\,, & j_- = \displaystyle\int\limits_{\xi<0} |\xi| f\,dv\,,
\end{cases}
\tag{7.5}
$$

etc., such that $\varrho = \varrho_+ + \varrho_-$, $j = j_+ - j_-$, $p = p_+ + p_-$ and $q = q_+ - q_-$.

Introducing an extension of the work of Bony [31] for discrete velocity models to the continuous velocity case, the following functional was considered [26-29]:

$$
I[f](t) = \underbrace{\int\int}_{x<y} \int\limits_{v} \int\limits_{v_*} (\xi - \xi_*) f(x, v, t) f(y, v_*, t)\, dv_*\, dv\, dx\, dy
\tag{7.6}
$$

where the first double integral is over the triangle $0 \leqslant x < y \leqslant 1$. One can then prove the following relation

$$
\int\limits_0^T \int\limits_0^1 \int\limits_v \int\limits_{v_*} (\xi - \xi_*)^2 f(x, v_*, t) f(x, v, t)\, dv\, dv_*\, dx\, dt =
\tag{7.7}
$$

$$
= I[f](0) - I[f](T) + \int\limits_0^T \left(p(0,t) \int\limits_0^1 \varrho(x,t)\, dx + p(1,t) \int\limits_0^1 \varrho(x,t)\, dx \right) dt\,;
$$

and show that the left-hand side of (7.7) is bounded for any finite time interval, though it may grow exponentially in time.

As remarked in [26], boundedness of the left-hand side of (7.7) follows, if we can obtain bounds on

$$
\int\limits_0^1 j_\pm(x,t)\, dx\,, \qquad \int\limits_0^T p(0,t)\, dt\,, \qquad \text{and on} \int\limits_0^T p(1,t)\, dt\,.
$$

Such bounds were obtained in [26] by a series of estimates, analogous to those used in [23, 24], which lead to

LEMMA 7.1. If f is a sufficiently smooth solution of the initial-boundary value problem for eq. (7.1) and the boundary conditions of the kind considered in the previous section, with initial value f_0, then

$$
E(t)\,, \qquad \int\limits_0^t (p(1,\tau) + p(0,\tau))\, d\tau\,, \qquad \int\limits_0^1 (j_+ + j_-)(x,t)\, dx
$$

and

$$
\int\limits_0^t \int\limits_0^1 \int\limits_v \int\limits_{v_*} (\xi - \xi_*)^2 f(x, v_*, \tau) f(x, v, \tau)\, dv\, dv_*\, dx\, d\tau
$$

can grow at most exponentially in time.

The next objective is then to show that the collision terms themselves can also grow at most exponentially in time. The method used in [26] to this end is the same as in [27-29].

Following largely the notation of [26-29] let

$$d\mu = dn\, dv_*\, dv\, dx \tag{7.8}$$

and, for $0 \leqslant \tau \leqslant T$,

$$\Delta(\tau, T) = \int_{[0,1]\times\mathfrak{R}^6\times S^2\times[\tau,T]} B(n\cdot(v-v_*), |v-v_*|)\, f(x,v,t)\, f(x,v_*,t)\, d\mu\, dt\,. \tag{7.9}$$

LEMMA 7.2. If the solution of the initial-boundary value problem for (7.1), defined as above exists as a classical solution for $t \in (0, \infty)$, and if the initial value f_0 has a finite H-functional $H[f_0]$ and finite energy

$$E(0) = \int_0^1 \int v^2 f_0(x,v)\, dv\, dx\,,$$

then there is a constant K (depending on the initial data, and ε and growing at most exponentially with T) such that

$$\Delta(\tau, T) \leqslant K\,. \tag{7.10}$$

The proof of Lemma 7.2 is a simple consequence of the next two lemmas.

LEMMA 7.3. Let u_1 be the x-component of the bulk velocity

$$u_1 = \int \xi f\, d\xi \Big/ \int f\, d\xi\,. \tag{7.11}$$

Then

$$\int_{\mathfrak{R}^3\times\mathfrak{R}^3\times[0,T]\times\mathfrak{R}} (\xi - u_1)^2 f(x,v,t)\; f(x,v_*,t)\, dx\, dt\, dv\, dv_* < K_0\,, \tag{7.12}$$

where K_0 is a constant, which only depends on the initial data, and can grow at most exponentially with T.

In fact, the integral in eq. (7.12) is nothing else than the integral in Lemma 7.1 (except for a factor 2) suitably rearranged. It is enough to expand the squares in both integrals and replace $\int \xi f d\xi$ by $u_1 \int f d\xi$.

Using an argument from [28, 29], one can prove:

LEMMA 7.4. Under the above assumptions

$$\int_{\mathfrak{R}^3\times\mathfrak{R}^3\times S^2\times[0,T]\times\mathfrak{R}} |v-u|^2 f(x,v,t)\, f(x,v_*,t)\,\cdot \tag{7.13}$$

$$\cdot B(n\cdot(v-v_*), |v-v_*|)\, dt\, dx\, dv\, dv_*\, dn < K_0\,,$$

where K_0 is a constant, which only depends on the initial data and can grow at most exponentially with T.

A simple rearrangement leads to:

LEMMA 7.5. Under the assumptions of Lemma 7.3, we have, for smooth solutions:

$$(7.14)\qquad \int_{\mathfrak{R}^3\times\mathfrak{R}^3\times S^2\times[0,T]\times\mathfrak{R}} |v-v_*|^2 f(x,v,t)\,f(x,v_*,t)\cdot B(n\cdot(v-v_*),|v-v_*|)\,d\mu\,dt < K_0\,,$$

where K_0 is the same constant as in Lemma 7.4 and hence can grow at most exponentially with T.

Lemma 7.2 now follows thanks to eq. (7.14) and the fact that $B(\cdot,\cdot)$ is zero for $|v-v_*|\leqslant\varepsilon$. Then we have

$$(7.15)\qquad \int_{\mathfrak{R}^3\times\mathfrak{R}^3\times S^2\times[0,T]\times\mathfrak{R}} f(x,v,t)\,f(x,v_*,t)\cdot B(n\cdot(v-v_*),|v-v_*|)\,dt\,dx\,dv\,dv_*\,dn < K_0/\varepsilon^2\,.$$

As in [26] the above estimates imply the existence of a global weak solution for the initial-boundary value problem, with the boundary conditions satisfied as equalities and not as inequalities. This can be stated in the form of a theorem as follows

THEOREM 7.6. Let $f_0\in L^1([0,1]\times\mathfrak{R}^3)$ be such that

$$(7.16)\qquad \int f_0(\cdot)(1+|x|^2+|v|^2)\,dv\,dx<\infty\,;\qquad \int f_0\,|\ln f_0(\cdot)|\,dv\,dx<\infty\,.$$

Also, assume that the collision kernel B and the boundary conditions satisfy the conditions made above. Then there is a global weak solution $f(x,v,t)$ of the initial-boundary value problem for eq. (7.1) such that $f\in C(\mathfrak{R}_+, L^1([0,1]\times\mathfrak{R}^3))$, $f(\cdot,0)=f_0$. This solution also satisfies the boundary conditions (7.4) a.e.

PROOF. See [26].

8. TREND TO EQUILIBRIUM

We shall now deal with the asymptotic trend for $t\to\infty$ when the boundary conditions are not incompatible with an equilibrium state. Discussions of equilibrium states in kinetic theory are as old as the theory itself; actually these states were discussed even before the basic evolution equation of the theory, *i.e.* the Boltzmann equation, was formulated. The recent work on the mathematical aspects of kinetic theory has led to new results on this problem as well.

In the papers discussed in the previous sections, it was generally assumed that there is a boundary Maxwellian M_w, which may vary along the boundary itself. This Maxwellian is uniquely determined except for very special boundary conditions, such as specular and reverse reflection, which we have frequently excluded. If we assume that the Maxwellian is the same at each point of the boundary, as in the paper by Hamdache [12], we can conjecture that the solution will tend asymptotically in time toward the nondrifting Maxwellian M_w. A proof of this was provided in several papers. In fact, the circumstance that there is a weak limit when time tends to infinity and that this limit is a Maxwellian was discussed by Desvillettes [14] and Cercignani [15] (see also [3]), starting from a remark by DiPerna and Lions [32]. Subsequently L. Arkeryd [16] proved that f actually tends to a Maxwellian in a strong sense for a periodic box, but his argument works in other cases as well; his proof uses techniques of nonstandard analysis and, as such, is outside the scope of this paper. Then P.-L. Lions [17] obtained the same result without resorting to nonstandard analysis. The author [33], following the approach of [17] gave a proof that is particularly suitable to deal with the solutions in a slab discussed in the previous section. The main differences from Lions's proof [17] are: *a*) his assumption that $B > 0$ a.e. is-not needed; *b*) the Maxwellian is uniquely determined, thanks to the boundary conditions which allow a unique Maxwellian.

We also point out that recently Arkeryd and Nouri [34] have sketched a proof of the fact that for boundary conditions satisfying the restriction of [23] and $B > 0$, the Maxwellian is uniquely determined (for renormalized solutions). This had already been pointed out for the weak limit in [15] (see also [3]).

We remark that the new point with respect to the general case is that we have constants in our estimates in place of functions which may grow exponentially in time [12, 33]. We then obtain:

THEOREM 8.1. *Let f be a solution of the initial boundary value problem* (1.3), (1.6), *where a suitably vanishing B is allowed. Then, when t tends to infinity, $f(\cdot, \cdot, t)$ converges strongly to the global Maxwellian $n_0 M_w$ where the constant factor n_0 is uniquely fixed by mass conservation.*

PROOF. It is enough to show that for every sequence t_n tending to ∞ there exists a subsequence t_{n_k} such that $f_{n_k}(x, v, t) = f(x, v, t + t_{n_k})$ converges in $L^1((0, 1) \times \mathfrak{R}^3 \times \times [0, T])$ to $n_0 M_w$ for any $T > 0$. The weak convergence of this sequence follows from the uniform boundedness of mass, energy and entropy.

Thus $f_n(x, v, t) = f(x, v, t + t_n)$ is weakly compact in $L^1(\Omega \times \mathfrak{R}^3 \times [0, T])$ for any sequence t_n of nonnegative numbers and any $T > 0$. If $t_n \to \infty$, then there exist a subsequence t_{n_k} and a renormalized solution $M(x, v, t)$ in $L^1(\Omega \times \mathfrak{R}^3 \times [0, T])$ such that f_{n_k} converges weakly to $M(x, v, t)$ in $L^1(\Omega \times \mathfrak{R}^3 \times [0, T])$ for any $T > 0$; in addition, the gain term $Q^+(f_n, f_n)$ converges a.e. to $Q^+(M, M)$. In order to prove that M is a Maxwellian, we remark that, since the integral $\int\int e(f)\, dv\, dt$ as given by (5.11) is finite, then

$$\int_{t_{nk}}^{T+t_{nk}} \int_{\mathfrak{R}^3} \int_{\Omega} \int_{S^2} \int_{\mathfrak{R}^3} [f(x, v', t)\, f(x, v'_*, t) - f(x, v, t)\, f(x, v_*, t)] \cdot$$

$$\cdot \log \frac{f(x, v', t)\, f(x, v'_*, t)}{f(x, v, t)\, f(x, v_*, t)} B(n \cdot (v - v_*), |v - v_*|)\, d\mu\, dt \to 0 \quad (k \to \infty)$$

and thus

$$(8.1)\qquad \int_0^T\int_{\mathfrak{R}^3}\int_{\Omega}\int_{S^2}\int_{\mathfrak{R}^3}[f_{n_k}(x,v',t)\,f_{n_k}(x,v'_*,t)-f_{n_k}(x,v,t)\,f_{n_k}(x,v_*,t)]\cdot$$

$$\cdot\log\frac{f_{n_k}(x,v',t)\,f_{n_k}(x,v'_*,t)}{f_{n_k}(x,v,t)\,f_{n_k}(x,v_*,t)}\,B(n\cdot(v-v_*),\,|v-v_*|)\,d\mu\,dt\to 0\qquad (k\to\infty)\,.$$

And, according to an argument by DiPerna and Lions [32] (see also [3]) we can pass to the limit and obtain

$$(8.2)\qquad \int_0^T\int_0^1\int_{\mathfrak{R}^3}\int_{S^2}\int_{\mathfrak{R}^3}[M(x,v',t)\,M(x,v'_*,t)-M(x,v,t)\,M(x,v_*,t)]\cdot$$

$$\cdot\log\frac{M(x,v',t)\,M(x,v'_*,t)}{M(x,v,t)\,M(x,v_*,t)}\,B(n\cdot(v-v_*),\,|v-v_*|)\,d\mu\,dt=0\,.$$

This implies

$$(8.3)\qquad M(x,v',t)\,M(x,v'_*,t)=M(x,v,t)\,M(x,v_*,t)$$

$$(\text{a.e. in } v_*,n,x,v_*,t \text{ for } B(V,n)\geqslant 0)\,.$$

In the case of the slab discussed in the previous section we have the unusual restriction on the relative speed which produces a vanishing kernel for $V<\varepsilon$. We use, however, the fact that one can use local arguments (in v, v_*) to deduce that $M(x,v,t)$ is a local (in x and t) Maxwellian. This was clear to Boltzmann [35-37] for twice differentiable functions and has been extended to the case when f is only assumed to be a distribution by Wennberg [38]. Then we conclude that $M(x,v,t)$ is a local Maxwellian.

But we have for all $K>1$

$$(8.4)\qquad |f'_{n_k}f'_{n_k*}-f_{n_k}f_{n_k}|\leqslant (K-1)\,f_{n_k},f_{n_k}+$$

$$+\frac{1}{\ln K}\,(f_{n_k}(x,v',t)\,f_{n_k}(x,v'_*,t)-f_{n_k}(x,v,t)\,f_{n_k}(x,v_*,t))\log\frac{f_{n_k}(x,v',t)\,f_{n_k}(x,v'_*,t)}{f_{n_k}(x,v,t)\,f_{n_k}(x,v_*,t)}\,.$$

Then, since $e(f_{n_k})$ converges to 0 a.e. and $Q^+(f_{n_k},f_{n_k})$ converges to $Q^+(M,M)$ a.e. we deduce that the loss term $Q^-(f_{n_k},f_{n_k})$ converges a.e. to $Q^-(M,M)$. Now, the loss term is of the form $f\nu(f)$ where $\nu(f)$ is a convolution product in velocity space. Then $f_{n_k}\nu(f_{n_k})\to M\nu(M)$ a.e.. Then either ϱ_M is zero, in which case f_{n_k} converges strongly to zero (a.e. in v), or is nonzero. In the second case $\nu(M)$ is also nonzero and if we let $u_{n_k}=\nu(f_{n_k})/\nu(M)$ we have that $u_{n_k}\to 1$ a.e. (by the averaging lemma). Then since $f_{n_k}u_{n_k}$ tends to $M(x,v,t)$ a.e., we conclude that $f_{n_k}\to M$ a.e.

But $M(x,v,t)$ must be a (renormalized and hence weak) solution of the Boltzmann equation or, since the collision term vanishes:

$$(8.5)\qquad \Lambda f=0\,.$$

In addition M must satisfy the boundary condition (1.6) [26].

Thus the solutions of the Boltzmann equation in a slab with the boundary conditions (1.6) tend (in the case of a boundary at constant temperature) to Maxwellians satisfying the free transport equation, $\Lambda f = 0$. These Maxwellians are well known since Boltzmann and are, *e.g.*, discussed in [3, Chapter III]. Now if we specialize this general solution to the case when $M(x, \cdot, t)$ is an L^1 function for any $t \geq 0$ and satisfies the boundary conditions, we see that M is a Maxwellian with no drift and constant temperature; this immediately implies that M is a uniform Maxwellian, which must coincide with M_w (which is an absolute nondrifting Maxwellian) except for a factor, which is fixed by mass conservation. Thus we have proved Theorem 8.1.

9. External problems

For applications to aerospace problems, the external case is of paramount importance. This aspect had been treated [6, 7] only in the perturbation framework described in Section 1, till P.-L. Lions [25] provided the relevant tools for dealing with external problems for large data in an L^1 framework.

In this situation $\Omega = \bar{\mathcal{O}}^c$ where $\mathcal{O}$ is a smooth bounded set in $\Re^3$. One provides an initial condition (1.5), boundary condition (1.6) on the boundary $\partial\Omega \times \Re^3 \times (0, T)$ and a condition at infinity

$$f \to M \quad \text{as} \quad |x| \to \infty, \tag{9.1}$$

where M is any assigned Maxwellian.

The results proved by Lions [25] are based on the following a priori estimates

$$\sup_{t \in [0,T]} \int_{\Omega \times \Re^3} dx\, d\xi \,(f \log(f/M) + M - f) \leq C, \tag{9.2}$$

$$\sup_{t \in [0,T]} \sup_{x_0 \in \Omega} \int_{\Omega \cap (x_0 + B^1)} dx \int_{\Re^3} d\xi\, f(1 + |v|^2 + |\log f|) \leq C, \tag{9.3}$$

whenever f_0 satisfies

$$\int_{\Omega \times \Re^3} dx\, d\xi \,(f \log(f_0/M) + M - f_0) \leq C_0, \tag{9.4}$$

for some positive constant $C = C(C_0, T)$. In the above B^1 is the unit ball.

Lions [25] gives a sketch of the proofs of the following results in the case when $\partial\Omega$ is specularly reflecting:

Theorem 9.1. Let $f_0 \geq 0$ satisfy (9.4). Then there exists a global renormalized solution satisfying eq. (9.1).

This solution is also a weak solution in a sense introduced in [17].

The result can be extended to other boundary conditions with the usual difficulty of the trace control. In fact we can apply the results of previous sections to $\Omega^R \times \Re^3$, where $\Omega^R = B^R \cap \Omega$ with B^R's radius R large enough and the following boundary condition on the artificial boundary:

$$\gamma^+ f = M \quad \text{on} \quad (\partial B^R \cap \Omega) \times \Re^3 \times (0, T), \tag{9.5}$$

which is a particular case of (6.8) for $\alpha = 1$ and $f^+ = M$. Then using the compactness properties of sequences of approximated solutions (see Theorem IV.2 in Lions's paper [21], with obvious changes) we can pass to the limit as R goes to ∞. This can be easily performed because the proof of the a priori bound (9.2) is only modified because of an additional boundary term of the form

$$\int_{\partial B^R \cap \Omega} d\sigma \int d\xi\, \xi \cdot n (f \log (f/M) + M - f) = \tag{9.6}$$

$$= - \int_{\partial B^R \cap \Omega} d\sigma \int_{\xi \cdot n < 0} d\xi\, |\xi \cdot n| (f \log (f/M) + M - f)$$

which is obviously negative, contributes to the time derivative. Then the estimates remain valid with constants independent of R.

10. Concluding remarks

We have surveyed the initial and initial-boundary value problems for the Boltzmann equation, with particular attention to recent results and the trend to equilibrium.

It appears that the subject has reached a certain maturity. Only difficult problems (such as smoothness, uniqueness, asymptotic behavior for long times or small mean free paths, steady solutions), which seem to require significantly new ideas, appear to remain open.

Another problem, which might be easier to solve, is to find conditions under which equality holds in (4.18). A simple but significant case, discussed in Section 7, is provided in the recent paper by Cercignani and Illner [26].

Finally we want to remark that we have not touched upon the interesting subject of the numerical simulation techniques used to solve the Boltzmann equation. For these we refer to the recent book of Bird [39] and to the survey paper by the author [40], devoted to the applications of the Boltzmann equation to aerospace problems.

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