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The growth of solutions of algebraic differential equations

Memoria (*) di WALTER K. HAYMAN

ABSTRACT. — Suppose that $f(z)$ is a meromorphic or entire function satisfying $P(z, f, f', \ldots, f^{(n)}) = 0$ where P is a polynomial in all its arguments. Is there a limitation on the growth of f, as measured by its characteristic $T(r, f)$? In general the answer to this question is not known. Theorems of Gol'dberg, Steinmetz and the author give a positive answer in certain cases. Some illustrative examples are also given.

KEY WORDS: Differential equations; Entire; Meromorphic; Growth.

RIASSUNTO. — *Sulla crescita delle soluzioni di equazioni differenziali algebriche*. Sia $f(z)$ una funzione meromorfa o intera dell'equazione $P(z, f, f', \ldots, f^{(n)}) = 0$, dove P è un polinomio in tutti i suoi termini. Esiste una limitazione della crescita di f, considerata rispetto alla sua caratteristica $T(r, f)$? La risposta a tale questione non è in generale nota. L'autore e i Teoremi Gol'dberg e Steinmetz danno una risposta positiva in alcuni casi. Vengono anche forniti alcuni esempi.

1. INTRODUCTION

Suppose that f is meromorphic in the complex plane C and satisfies there an equation of the type

$$P(z, f, f', \ldots, f^{(n)}) = 0, \tag{1}$$

where P is a polynomial in all its arguments. In other words

$$P = \sum_{\lambda \in I} a_\lambda(z) f^{i_0} (f')^{i_1} \ldots (f^{(n)})^{i_n} \tag{2}$$

where I consists of finite multi-indices of the form $\lambda = (i_0, i_1, \ldots, i_n)$, the i_j are positive integers or zero, and the $a_\lambda(z)$ are polynomials in z.

In this lecture I should like to talk about the following classical conjectures (c.c.) Steve Bank and Lee Rubel told me about. I owe all my information about c.c. to discussions with them. The conjecture is related to a false conjecture of Borel [5]. It is made by Bank [2, p. 1] for the case $n = 2$.

CONJECTURE. *If $f(z)$ is a solution of* (1) *and $T(r, f)$ denotes the Nevanlinna characteristic of f we have*

$$T(r, f) < a \exp_{n-1}(br^c), \qquad 0 \leq r < \infty \tag{3}$$

where a, b, c are positive constants and $\exp_l(x)$ is the l times iterated exponential, i.e.

$$\exp_0(x) = x, \qquad \exp_1(x) = e^x, \qquad \exp_l(x) = \exp_1\{\exp_{l-1}(x)\}.$$

(*) Gli argomenti contenuti in questa *Memoria* furono presentati nella conferenza del Simposio Matematico, tenutosi presso l'Accademia dei Lincei l'8 febbraio 1996.

We recall the definition of $T(r, f)$. We write

$$m(r, f) = (1/2\pi) \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta, \qquad \text{and} \qquad \log^+ x = \max(\log x, 0).$$

Further let $n(t, f)$ denote the number of poles of $f(z)$ in $|z| \leqslant t$, counting multiplicity and write

$$N(r, f) = \int_0^r \{n(t, f) - n(0, f)\} \frac{dt}{t} + n(0, f) \log r.$$

Then:

$$T(r, f) = m(r, f) + N(r, f).$$

We remark that if f is entire, *i.e.* $N(r, f) = 0$, we have for $0 < r < R$:

$$T(r, f) \leqslant \log^+ M(r, f) \leqslant ((R + r)/(R - r)) \, T(R, f).$$

Here

$$M(r, f) = \max_{|z| = r} |f(z)|$$

is the maximum modulus.

In particular:

$$T(r, f) \leqslant \log^+ M(r, f) \leqslant 3T(2r, f).$$

It follows that in this case we can replace $T(r, f)$ by $\log M(r, f)$ in (3) with the same constant c (but different a, b). If $n = 1$ the inequality (3) reduces to

$$T(r, f) < a(1 + br^c). \tag{4}$$

Functions f satisfying (4) are said to have finite order. The lower bound ϱ of possible numbers c is called the order of f.

The standard elementary exponential and trigonometric functions, $\sin z$, $\cos z$, $\tan z$, e^z and the gamma function $\Gamma(z)$ have order 1. The Weierstrass elliptic function $\wp(z)$, which satisfies

$$\wp'(z)^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \tag{5}$$

has order 2. The function e^{e^z} has infinite order, but satisfies (3) with $n = 2$, $c = 1$. My principal general references will be to Laine [8].

2. Some examples

Example 1. The following simple example underlies c.c. (see [1, p. 55]).

We recall that $f_1(z) = e^z$ satisfies $f_1'(z) = f_1(z)$ and also (3) with $n = 1$. A similar conclusion holds for $f_n(z)$ defined inductively by

$$f_n(z) = \exp\left\{ \int f_{n-1}(z) \, dz \right\}.$$

In fact we have

(6) $\quad f_n'(z)/f_n(z) = f_{n-1}(z), \quad f_{n-1}' = (f_n'/f_n)', \quad f_{n-1}^{(p)} = (f_n'/f_n)^{(p)}.$

In particular: $f_2'(z)/f_2(z) = e^z$, so that $(f_2'/f_2)' = f_2'/f_2$ or

(7) $$f_2 f_2'' - (f_2')^2 = f_2 f_2'.$$

Suppose that we have shown that $P_n(f_n) = 0$, where P_n is a homogeneous polynomial with constant coefficients in $(f_n, f_n', \ldots, f_n^{(n)})$. Substituting

$$f_{n+1}'/f_{n+1} = f_n, \quad (f_{n+1}'/f_{n+1})' = f_n', \quad (f_{n+1}'/f_{n+1})^{(n)} = f_n^{(n)},$$

we obtain a similar conclusion for f_{n+1}.

Example 2. The following surprising example due to the late Steve Bank [3, Theorem 2, p. 178] shows that no conclusion on the growth follows if we allow the coefficients a_λ to be slowly growing functions instead of polynomials.

Consider the equation $f'(z)/f(z) = F(z)$ where

$$F(z) = \sum\left\{\frac{p_n}{z - a_n} - \frac{p_n}{z - b_n}\right\} = \sum \frac{p_n(a_n - b_n)}{(z - a_n)(z - b_n)}.$$

Here we may take the a_n to be complex numbers tending rapidly to infinity, the p_n to be positive integers also tending rapidly to infinity and then b_n to be so close to a_n that the series converges rapidly.

By making a_n tend to infinity very quickly we can ensure that $T(r, F(z))/\log r \to \infty$ as slowly as we please. (It must tend to infinity as soon as $F(z)$ is transcendental meromorphic.) In fact $N(r, F)$ depends only on the a_n, b_n and $m(r)$ can be made bounded.

However $f(z)$ has a zero of multiplicity p_n at a_n and a pole of multiplicity p_n at b_n. Thus $N(r, f) > p_n \log 2$ as soon as $r > 2|b_n|$. Since p_n can be chosen to tend to ∞ as rapidly as we please we see that $T(r, f)$ can also grow as rapidly as we please as a function of r. I suspect that one cannot provide entire solutions of (1) which grow rapidly, while the coefficients grow slowly. But we cannot even prove this for polynomial coefficients. No growth estimate is known in the direction of c.c. even for entire solutions. However, Bank [1, Lemma, p. 58] is able to estimate $T(r, f)$ in terms of $T(r, F)$ when f is entire.

Example 3. We might suppose that if you have a solution of (1) on the real axis, then this must have limited growth. This however is not the case. The following example is due to Vijayaraghavan [12]. Consider $f_m(z) = z^2\wp(mz)$, where $\wp(z)$ is the elliptic function with periods $1, i$. It is evident that $f_m(z)$ satisfies a second order differential equation independent of m. In fact we can eliminate m between the equations

(8) $\quad \{(f_m(z)/z)'\}^2 = 4m^2 P(f_m), \quad$ where $P(w) = (w - e_1)(w - e_2)(w - e_3)$

and the equation obtained by differentiating (8).

Suppose now that $m = 1 + in$, where n is positive and irrational. Then $(1 + in)z$ does not meet any lattice point $p + iq$, except 0, for integral p, q and real z, and so the solution f is analytic on the whole real axis. On the other hand suppose that n is a Liouville number, where a block of digits $a_1 a_2 \ldots a_s$ is followed by an enormously long block of zeros. Then n has a series of rational approximations of type

$$n - t/10^s = (1/10^s)\,\varepsilon(s)$$

where t is an integer and the sequence $\varepsilon(s)$ can tend to zero arbitrarily rapidly as $s \to \infty$, depending on the choice of n. Thus

$$10^s m = 10^s(1 + in) = 10^s + \theta it + i\theta\,\varepsilon(s) = z_0 + i\theta\,\varepsilon(s).$$

We note that $\wp(z_0 + \eta) = c(\eta)/\eta^2$ when η is small, where $c(\eta)$ is independent of z_0, and $c(\eta) \to 1$ as $\eta \to 0$. Thus $|f_m(10^s)| > 1/\{2\varepsilon(s)^2\}$ for a sequence of positive integers s, where $\varepsilon(s)$ can tend to zero as rapidly as we please with s (in particular more rapidly than $\exp_k(s)$ for every positive integer k).

Thus the c.c. is certainly not true for real analytic solutions of (1). Bank [2, p. 53] obtains increasing solutions of a third order equation, which have arbitrarily rapid growth on the positive axis. This is the integral of the example $(2 - \cos z - \cos nz)^{-1}$, where n is as above. This example is due to Basu, Bose and Vijayaraghavan [4].

3. Positive results, first order

I should like to devote the rest of this talk to positive results that are known. A full solution of the conjecture is only known for first order equations. I should like to quote the following important and beautiful

Theorem A ([6], [8, p. 223]). *Suppose that* (1) *is a first order equation,* i.e. P *is a polynomial in* f, f' *only. Then all meromorphic solutions of* (1) *have finite order.*

Gol'dberg considered various cases and obtained bounds for the growth of solutions which were all sharp. He showed that in some cases $T(r, f) < c(\log r)^2$, where c is a constant. The corresponding results for functions in a finite disk and near ∞ were also obtained. A typical example is the Riccati equation [8, p. 165]:

$$w' = a_0 w^2 + a_1 w + a_2$$

all of whose solutions are meromorphic, if a_0, a_1, a_2 are entire.

4. Positive results for second order

Unfortunately for higher order equations our knowledge is very fragmentary. We have seen that $f(z) = e^{e^z}$ satisfies a homogeneous second order differential equation. In this connection we have the following

Theorem B ([9], [8, p. 248]). *Suppose that in* (1) P *is homogeneous in* f, f' *and* f''. *Then all meromorphic solutions of* (1) *take the form* $w(z) = (g_1(z)/g_2(z)) \exp\{g_3(z)\}$, *where* g_1, g_2, g_3 *are entire functions of finite order.*

Steinmetz [10] (see also [8, p. 251]) obtained information about which equations

(1) admit meromorphic solutions but in this case he had to confine himself to the case when P in (1) is linear or quadratic in w''.

Perhaps the simplest case not covered by the results of Steinmetz is the equation

$$ww'' - w'^2 = a_2 w'' + a_1 w' + a_0 w + b \tag{9}$$

where the a_j and b are rational in z or even constants, and are not all identically zero. It seems unbelievable that this equation could have entire solutions of arbitrarily rapid growth but as far as I know this possibility has not been excluded so far. It is fair to conjecture that entire solutions of (9) all have finite order.

5. Entire solutions of general equations: the Wiman-Valiron method

We need a fairly precise version of the Wiman-Valiron method, which enables us to estimate the size of derivatives of an entire function at its point of maximum modulus. We write

$$f(z) = \sum_0^\infty a_n z^n, \qquad \mu(r) = \max_n |a_n| r^n,$$

$$N(r) = \sup\{n \mid |a_n| r^n = \mu(r)\}, \qquad M(r) = \max_{|z|=r} |f(z)|.$$

The following inequalities hold for all r except for a set E of finite logarithmic measure, *i.e.* such that

$$\int_E \frac{dt}{t} < \infty .$$

We say that the results hold for *normal* r. We have for normal r, if $\delta > 0$,

$$\mu(r) < M(r) < \mu(r) N(r)^{1/2} \{\log N(r)\}^{1/2+\delta} .$$

Also outside the countable set where $N(r)$ increases

$$r \frac{d}{dr} \log \mu(r) = N(r), \quad \text{so that } \log \mu(r) = \int_1^r N(t) \frac{dt}{t} + \text{constant} . \tag{10}$$

We can now state the fundamental results of Wiman-Valiron theory in the form that we need them for applications to differential equations (see *e.g.* [7, Theorem 12]); somewhat weaker results are given by [8, Chapter 3].

Lemma. *Suppose that r is normal and that $|z| = r$, $|f(z)| = M(r, f)$. Then for a fixed positive integer q and a positive δ, we have*

$$\{z/N(r)\}^q f^{(q)}(z) = f(z)\{1 + O[N(r)^{\delta - 1/2}]\} \quad \text{as } r \to \infty . \tag{11}$$

Before we can state the application to entire solutions of (1), we need some more terminology. Consider the terms in the differential polynomial P in (2). The degree of a term is defined to be $|\lambda| = i_0 + i_1 + \ldots + i_n$ and the weight $\|\lambda\|$ is defined by $\|\lambda\| = i_0 + 2i_1 + \ldots + (n+1) i_n$.

We shall consider the terms of highest degree in (1) and among all these the terms of highest weight. Let Λ be the set of all these indices λ. Then we have the following.

THEOREM C. *Suppose that in the equation* (1), Λ *is defined as above. Let d be the maximum degree of all the polynomials* $a_\lambda(z)$ *and suppose that*

$$\sum_{\lambda \in \Lambda} a_\lambda(z) \neq 0 .$$

Then all entire solutions of (1) *have finite order* ϱ, $\varrho \leq \max\{2d, 1+d\}$.

The result appears to be new, but the technique goes back to Valiron [11, pp. 221-223] and Wittich [13, pp. 70-71].

To prove this result we apply (11). We note that by (11)

$$f^{i_0}(f')^{i_1}\dots(f^{(n)})^{i_n} = f(z)^p\{N(r)/z\}^q\{1+O(N(r)^{\delta-1/2})\}$$

where $p = i_0 + \dots + i_n$, $q = i_1 + 2i_2 + \dots + ni_n$. Suppose now that $f(z)$ has order greater than ϱ. It follows from (10) that for normal r: $N(r) = O\{\log \mu(r)\}^2 = O\{\log |f|\}^2$.

Also since f is transcendental entire $|f|$ is larger than any fixed power of r for large r. Thus if $p = |\lambda|$ the terms of degree less than p have order $|f|^{p-1+\delta}$, for every positive δ. Consider now the terms of highest degree $p = |\lambda|$ and greatest weight $p+q = \|\lambda\|$. For these terms of index λ in Λ we have

$$\sum_{\lambda \in \Lambda} a_\lambda(z) f^{i_0}(f')^{i_1}\dots(f^{(n)})^{i_n} = f(z)^p\{N(r)/z\}^q\left\{\sum a_\lambda(z) + O(r^d N(r)^{\delta-1/2})\right\}$$

where d is the largest degree among the polynomials $a_\lambda(z)$. If the order is greater than ϱ we can choose r, so that $N(r) > r^\varrho$. The terms of weight less than $p+q$ contribute at most

$$O\{|f|^p\{N(r)/r\}^q r/N(r)\cdot r^d\} = O\{|f|^p\{N(r)/r\}^q r^{1+d-\varrho}\}$$

so these are dominated by the leading terms if $\varrho > 1+d$. Suppose also that $\varrho > 2d$. Then $r^d N(r)^{\delta-1/2} = O\{r^{d-(1/2-\delta)\varrho}\} \to 0$ if δ is chosen sufficiently small. Thus in this case we have

$$P = f(z)^p\{N(r)/r\}^q\left\{\sum a_\lambda(z) + o(1)\right\} \neq 0$$

by hypothesis, and this contradicts the equation (1). Thus the Theorem is proved.

EXAMPLE 4. Consider second order, second degree equations. Clearly Theorem C holds if there is only one term of highest weight and degree, and this is always the case for first order equations. For second order, second degree, the terms of degree 2 are $(f'')^2$, $f''f'$, $f''f$, f'^2, ff', f^2 whose weight is respectively 6, 5, 4, 4, 3, 2. Thus the solutions have finite order unless the equation is

$$a_1(z)(ff'' - f'^2) + a_2 ff' + a_3 f^2 + b_1 f'' + b_2 f_2' + b_3 f + b_4 = 0 .$$

In fact we saw earlier that the equation

$$ff'' - f'^2 - ff' = 0 \tag{7}$$

has the solution $f(z) = e^{e^z}$, which has infinite order.

EXAMPLE 5. To illustrate the bound on ϱ consider the equation $f' = 2zf$. Here $d = 1$ and the equation has the solution $f(z) = e^{z^2}$ which has order $2 = 2d = 1 + d$.

EXAMPLE 6. Theorem C does not extend to meromorphic solutions. The function $y = \tan e^z$ is meromorphic of infinite order and satisfies

$$y''y^2 - 2y'^2y - y'y^2 + y'' - y' = 0,$$

but $\sum a_\lambda(z) = 1 - 2 = -1$.

For more information on the area I would like to refer the reader to [8, in particular to Chapters 11 to 13].

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