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On asymptotics of solutions and eigenvalues of the boundary value problem with rapidly alternating boundary conditions for the system of elasticity

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Analisi matematica. — *On asymptotics of solutions and eigenvalues of the boundary value problem with rapidly alternating boundary conditions for the system of elasticity.* Nota (*) di OLGA A. OLEINIK e GREGORY CHECHKIN, presentata dal Socio O. A. Oleinik.

ABSTRACT. — Boundary value problems for the system of linear elasticity with rapidly alternating boundary conditions are studied and asymptotic behavior of solutions is considered when a small parameter, which defines the oscillation of the boundary conditions, tends to zero. Estimates for the difference between such solutions and solutions of the limit problem are given.

KEY WORDS: Homogenization; Linear elasticity system; Alternating boundary conditions.

RIASSUNTO. — *Sul comportamento asintotico delle soluzioni e degli autovalori del problema ai limiti per il sistema dell'elasticità con condizioni ai limiti rapidamente alternanti.* Vengono studiati i problemi ai limiti per il sistema dell'elasticità lineare con condizioni ai limiti rapidamente alternanti. Si considera inoltre il comportamento asintotico della soluzione quando un piccolo parametro, che definisce l'oscillazione delle condizioni al limite, tende a zero. Vengono calcolate stime per la differenza tra tali soluzioni e le soluzioni del problema ai limiti.

0. – The problem of the asymptotic behavior of solutions of boundary value problems with rapidly alternating boundary conditions for second order elliptic equations was studied in many papers [1, 3, 5-8, 13, 14, 20]. For the elasticity system this problem was considered in papers [2, 4, 5] and the convergence to a solution of a limit problem was proved. In this paper we give the estimates for the deviation in $H^1(\Omega)$ norm of solutions of the considered problem from the limit problem solutions. The problem of vibration is also studied here. Some theorems of this kind are formulated in [15].

1. – Let Ω be a smooth domain in $\boldsymbol{R}^n$, $n \geq 2$ and let $\partial\Omega$ be its boundary. We suppose that $\partial\Omega = \Gamma_\varepsilon \cup \gamma_\varepsilon$ and consider the boundary value problem:

$$L_k(u_\varepsilon) \equiv \frac{\partial}{\partial x_i}\left(a_{kl}^{ij} \frac{\partial u_\varepsilon^l}{\partial x_j}\right) = f_k(x) \quad \text{in } \Omega, \quad k = 1, \dots, n, \tag{1}$$

$$u_\varepsilon = 0 \quad \text{on } \gamma_\varepsilon, \tag{2}$$

$$\sigma(u_\varepsilon) \equiv A^{ij}(x) \frac{\partial u_\varepsilon}{\partial x_j} \nu_i = 0 \quad \text{on } \Gamma_\varepsilon, \tag{3}$$

where $u_\varepsilon = (u_\varepsilon^1, \dots, u_\varepsilon^n)$, $L(u) = (L_1(u), \dots, L_n(u))^* \equiv (\partial/\partial x_i)(A^{ij}(x)(\partial u_\varepsilon/\partial x_j))$, A^{ij} are $(n \times n)$-matrices with elements a_{kl}^{ij}, which are bounded measurable functions, $a_{kl}^{ij}(x) = a_{lk}^{ji}(x) = a_{il}^{kj}(x)$,

$$\kappa_1 \xi_{ki}\xi_{ki} \leq a_{kl}^{ij}(x)\, \xi_{ki}\xi_{lj} \leq \kappa_2 \xi_{ki}\xi_{ki}, \qquad \kappa_1, \kappa_2 = \text{const} > 0, \qquad x \in \Omega, \tag{4}$$

(*) Pervenuta all'Accademia il 24 ottobre 1995.

$\{\xi_{ki}\}$ are real symmetric matrices, $\nu = (\nu_1, \ldots, \nu_n)$ is an outward normal vector to the boundary $\partial\Omega$, $f(x) = (f_1(x), \ldots, f_n(x))^* \in (L_2(\Omega))^n$, Γ_ε consists of the sets Γ_ε^k, $k = 1, \ldots, N_\varepsilon$, $\operatorname{diam} \Gamma_\varepsilon^k \leqslant \varepsilon$, and the distance between them is greater or equal than 2ε, ε is a small positive parameter, $\gamma_\varepsilon = \partial\Omega \backslash \Gamma_\varepsilon$. Here and throughout we use the usual convention of repeated indices.

We will study the limit behavior of solutions of problem (1)-(3), when ε tends to zero and $N_\varepsilon \to \infty$. Existence and uniqueness of the solutions u_ε of problem (1)-(3) in space $(H^1(\Omega, \gamma_\varepsilon))^n$ can be proved using functional methods [9]. The space $H^1(\Omega, \gamma_\varepsilon)$ is defined as the completion of the functions from the space $C^\infty(\overline{\Omega})$, vanishing in a neighborhood of γ_ε, with respect to the norm

$$\|u\|_{H^1(\Omega)} \equiv \left(\int_\Omega (u^2 + |\nabla u|^2)\, dx \right)^{1/2}.$$

2. – Lemma 1. *For the function $u(x)$ from the space $H^1(\Omega, \gamma_\varepsilon)$ the following estimate*

$$\int_{\Omega_\eta} u^2\, dx \leqslant C\eta^2 \int_{\Omega_\eta} |\nabla u|^2\, dx \tag{5}$$

is valid, where the constant C does not depend on ε, η and u; $\Omega_\eta = \{x : x \in \Omega, \varrho(x, \partial\Omega) \leqslant \eta\}$, $\varrho(x, \partial\Omega)$ is equal to the distance between x and $\partial\Omega$, $\varepsilon \leqslant \eta$.

Proof. Let q_ε^k be a ball with radius ε and let $p_\varepsilon^k \subset \Gamma_\varepsilon^k$ be a center of the ball q_ε^k. Also let Q_ε^k be a ball with radius 2ε with the same center, $s_\varepsilon^k = \partial\Omega \cap q_\varepsilon^k$, $S_\varepsilon^k = \partial\Omega \cap Q_\varepsilon^k$. The function $u(x)$ is obviously equal to 0 on $S_\varepsilon^k \backslash s_\varepsilon^k$. The domain G_η^k, which is a union of the inward normals to the set S_ε^k with the length η, is considered. Since the boundary $\partial\Omega$ is smooth, the domains G_η^k are diffeomorphic for all k. Then the Friedrichs inequality for the domains G_η^k (see [16]) gives us inequality (5) in G_η^k with the constant C, which does not depend on ε, η and k. Since $u = 0$ on $\partial\Omega \backslash \left(\bigcup_k^{N_\varepsilon} S_\varepsilon^k \right)$, then, as usual, in the domain $\Theta_\eta = \Omega_\eta \backslash \left(\bigcup_k^{N_\varepsilon} G_\eta^k \right)$ we obtain inequality (5), using the representation of function $u(x)$ as an integral of its normal derivative. The summation of these inequalities gives us inequality (5).

Lemma 2. *For the function $u(x)$ from the space $H^1(\Omega, \gamma_\varepsilon)$ the estimate*

$$\int_\Omega u^2\, dx \leqslant C_1 \int_\Omega |\nabla u|^2\, dx, \tag{6}$$

is valid, where the constant C_1 does not depend on ε and $u(x)$.

Proof. By the mean-value theorem for an integral and (5) for $\eta = \varepsilon$, we obtain that there exists $\varepsilon_0 \leqslant \varepsilon$ such that

$$\int_{l_{\varepsilon_0}} u^2\, ds \leqslant \widetilde{C}\varepsilon \int_{\Omega_\varepsilon} |\nabla u|^2\, dx\,, \tag{7}$$

where $l_{\varepsilon_0} = \{x : x \in \Omega, \varrho(x, \partial\Omega) = \varepsilon_0\}$. In the framework of the imbedding theorem (see [17]), we obtain

$$\int_{\Omega \setminus \Omega_{\varepsilon 0}} u^2\, dx \leqslant C_2 \left(\int_{l_{\varepsilon 0}} u^2\, ds + \int_{\Omega \setminus \Omega_{\varepsilon 0}} |\nabla u|^2\, dx \right), \tag{8}$$

where the constant C_2 does not depend on ε and $u(x)$ because of the smoothness of the boundary $\partial\Omega$.

The summation of inequality (5) for $\eta = \varepsilon$ and (8) gives us inequality (6).

THEOREM 1 (Korn's type inequality). *For the function* $u(x) = (u^1(x), \ldots, u^n(x))^*$ *from the space* $(H^1(\Omega, \gamma_\varepsilon))^n$ *the inequality*

$$\int_\Omega \sum_{i=1}^n |\nabla u^i|^2\, dx \leqslant C_3 \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right)^2 dx \leqslant \tag{9}$$

$$\leqslant C_4 \int_\Omega \sum_{i,j,k,l=1}^n a_{kl}^{ij}(x) \frac{\partial u^l}{\partial x_j} \frac{\partial u^k}{\partial x_i}\, dx$$

is valid, where C_3, C_4 *do not depend on* $u(x)$ *and* ε.

PROOF. We define the function $\psi(s) \in C^\infty(R^1)$ such that $\psi(s) = 0$, when $s \in [-\infty, 1]$, $\psi(s) = 1$, when $s \geqslant 1 + \sigma$, $0 < \sigma < 1/2$, $0 \leqslant \psi(s) \leqslant 1$. Let $\widetilde{\psi}_\varepsilon^k(x) = \psi(r_k/\varepsilon)$, where $(r_k, \theta_k^1, \ldots, \theta_k^{n-1})$ is a local system of polar coordinates, whose center is $p_\varepsilon^k \in \Gamma_\varepsilon^k$. Let

$$\widetilde{\psi}_\varepsilon(x) = \prod_{k=1}^{N_\varepsilon} \widetilde{\psi}_\varepsilon^k(x)\,.$$

For the function $u\widetilde{\psi}_\varepsilon$ the Korn inequality holds in Ω, if $u \in (H^1(\Omega, \gamma_\varepsilon))^n$, *i.e.*,

$$\int_\Omega \sum_{i=1}^n |\nabla(u^i \widetilde{\psi}_\varepsilon)|^2\, dx \leqslant C_5 \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial(u^i \widetilde{\psi}_\varepsilon)}{\partial x_j} + \frac{\partial(u^j \widetilde{\psi}_\varepsilon)}{\partial x_i} \right)^2 dx\,, \tag{10}$$

where the constant C_5 does not depend on ε and $u(x)$. It is easy to see, that

$$\sum_{i,j=1}^n \left(\frac{\partial(u^i \widetilde{\psi}_\varepsilon)}{\partial x_j} + \frac{\partial(u^j \widetilde{\psi}_\varepsilon)}{\partial x_i} \right)^2 = \sum_{i,j=1}^n \left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right)^2 \widetilde{\psi}_\varepsilon^2 + 2|u|^2 |\nabla \widetilde{\psi}_\varepsilon|^2 + \tag{11}$$

$$+2 \sum_{i,j=1}^n \frac{\partial \widetilde{\psi}_\varepsilon}{\partial x_i} \frac{\partial \widetilde{\psi}_\varepsilon}{\partial x_j} u^i u^j + 2 \sum_{i,j=1}^n \left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right) \widetilde{\psi}_\varepsilon \left(\frac{\partial \widetilde{\psi}_\varepsilon}{\partial x_i} u^j + \frac{\partial \widetilde{\psi}_\varepsilon}{\partial x_j} u^i \right).$$

Then by the Hölder inequality, we obtain from (11)

$$(12)\qquad \sum_{i,j=1}^{n}\left(\frac{\partial(u^i\widetilde{\psi}_\varepsilon)}{\partial x_j}+\frac{\partial(u^j\widetilde{\psi}_\varepsilon)}{\partial x_i}\right)^2\leqslant C_6\left(|u|^2|\nabla\widetilde{\psi}_\varepsilon|^2+\sum_{i,j=1}^{n}\left(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\right)^2\widetilde{\psi}_\varepsilon^2\right),$$

C_6 does not depend on ε and $u(x)$. By using the estimate (10)-(12), we deduce

$$(13)\qquad \int\limits_{\Omega\setminus\Omega_{2\varepsilon}}\sum_{i=1}^{n}|\nabla u^i|^2\,dx\leqslant\int\limits_{\Omega}\sum_{i=1}^{n}|\nabla(u^i\widetilde{\psi}_\varepsilon)|^2\,dx\leqslant$$

$$\leqslant C_6\int\limits_{\Omega}\sum_{i,j=1}^{n}\left(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\right)^2dx+C_7\int\limits_{\Omega}|u|^2|\nabla\widetilde{\psi}_\varepsilon|^2\,dx\,.$$

It is easy to see that $|\nabla\widetilde{\psi}_\varepsilon|\leqslant C_8/\varepsilon$ and $|\nabla\widetilde{\psi}_\varepsilon|=0$ in $\Omega\setminus\Omega_{2\varepsilon}$. Thus we have

$$(14)\qquad \int\limits_{\Omega\setminus\Omega_{2\varepsilon}}\sum_{i=1}^{n}|\nabla u^i|^2\,dx\leqslant C_6\int\limits_{\Omega}\sum_{i,j=1}^{n}\left(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\right)^2dx+C_9\frac{1}{\varepsilon^2}\int\limits_{\Omega_{2\varepsilon}}|u|^2\,dx\,.$$

Let us set

$$D(u,\Omega)\equiv\int\limits_{\Omega}\sum_{i=1}^{n}|\nabla u^i|^2\,dx\,,\qquad E(u,\Omega)\equiv\int\limits_{\Omega}\sum_{i,j=1}^{n}\left(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\right)^2dx\,.$$

Now adding $D(u,\Omega_{2\varepsilon})$ to the left and right sides of (14) and using Lemma 1, we obtain

$$(15)\qquad D(u,\Omega)\leqslant C_6E(u,\Omega)+C_{10}D(u,\Omega_{2\varepsilon})\,.$$

We consider the set $\Theta_{2\varepsilon}=\Omega_{2\varepsilon}\setminus\left(\bigcup\limits_{k}^{N_\varepsilon}G_{2\varepsilon}^k\right)$, which is defined in Lemma 1. The surface $\overline{\Theta_{2\varepsilon}}\cap\partial\Omega$ can be covered by open sets r_ε^j $(j=1,\dots,M_\varepsilon)$ in such a way that normals to r_ε^j of length 2ε inside of Ω and length 2ε outside of Ω form a domain R_ε^j which is star-shaped with respect to the ball b_ε^j of radius ε, which is outside of Ω. We define $u=0$ in $R_\varepsilon^j\setminus\Omega$. It is easy to see that $u\in H^1(R_\varepsilon^j)$.

Now we will use the following theorem from [10-12].

THEOREM. *If the domain G is star-shaped with respect to the ball Q, then the following Korn's type inequality*

$$D(u,G)\leqslant K(E(u,G)+D(u,Q))$$

is valid, where K is a constant, which does not depend on u.

This theorem gives us the following estimate

$$D(u,R_\varepsilon^j)\leqslant C_{11}(E(u,R_\varepsilon^j)+D(u,b_\varepsilon^j))\leqslant C_{11}E(u,R_\varepsilon^j)\,,$$

since $u=0$ in b_ε^j, $j=1,\dots,M_\varepsilon$.

The summation of these inequalities leads to the estimate

$$(16)\qquad D(u,\Theta_{2\varepsilon})\leqslant C_{12}E(u,\Theta_{2\varepsilon})\,.$$

It is not difficult to notice that $G_{2\varepsilon}^j$ can be covered by star-shaped domains with respect to balls, which belong to $\Theta_{2\varepsilon}$ ($j = 1, \ldots, N_\varepsilon$) if ε is sufficiently small. These balls do not intersect. Therefore, from the Theorem we obtain the following estimate

$$D\left(u_\varepsilon, \bigcup_{j=1}^{N_\varepsilon} G^j_{2\varepsilon}\right) \leq C_{13}\left(E\left(u_\varepsilon, \bigcup_{j=1}^{N_\varepsilon} G^j_{2\varepsilon}\right) + D(u_\varepsilon, \Theta_{2\varepsilon})\right). \tag{17}$$

Finally, from (4), (15), (16) and (17) we obtain (9).

LEMMA 3. *The solutions u_ε of the problem* (1)-(3) *are uniformly bounded with respect to ε in $H^1(\Omega)$.*

PROOF. The definition of the weak solution u_ε in $(H^1(\Omega, \gamma_\varepsilon))^n$ of problem (1)-(3) gives us the following integral identity

$$\int_\Omega \sum_{i,j,k,l=1}^n a^{ij}_{kl}(x) \frac{\partial u^l_\varepsilon}{\partial x_j} \frac{\partial v^k}{\partial x_i}\, dx = -\int_\Omega \sum_{k=1}^n f_k(x) v^k(x)\, dx$$

for all $v \in (H^1(\Omega, \gamma_\varepsilon))^n$. Taking $v^l = u^l_\varepsilon$, using Korn's inequality (Theorem 1) and the Friedrichs inequality (Lemma 2) we obtain that

$$\int_\Omega \sum_{k=1}^n |\nabla u^k_\varepsilon|^2\, dx \leq C_{14} \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial u^i_\varepsilon}{\partial x_j} + \frac{\partial u^j_\varepsilon}{\partial x_i}\right)^2 dx \leq$$

$$\leq C_{14} \frac{C_4}{C_3} \int_\Omega \sum_{i,j,k,l=1}^n a^{ij}_{kl}(x) \frac{\partial u^l_\varepsilon}{\partial x_j} \frac{\partial u^k_\varepsilon}{\partial x_i}\, dx \leq$$

$$\leq C_{14} \sum_{k=1}^n \|f_k\|_{L_2(\Omega)} \|u^k_\varepsilon\|_{L_2(\Omega)} \leq C_{15}\left(\int_\Omega \sum_{k=1}^n |\nabla u^k_\varepsilon|^2\, dx\right)^{1/2}.$$

Therefore

$$\left(\int_\Omega \sum_{k=1}^n |\nabla u^i_\varepsilon|^2\, dx\right)^{1/2} \leq C_{15}, \tag{18}$$

where the constant C_{15} does not depend on ε and u_ε. The uniform estimate of u_ε in $H^1(\Omega)$ follow from (6) and (18).

3. – Let $u_0(x)$ be a weak solution of the problem

$$L(u_0) = f(x) \quad \text{in } \Omega, \tag{19}$$

$$u_0 = 0 \quad \text{on } \partial\Omega. \tag{20}$$

THEOREM 2. *For the solutions u_ε of problem* (1)-(3) *and the solution u_0 of problem* (19), (20) *the estimate*

$$\int_\Omega |\nabla(u_\varepsilon - u_0)|^2 \psi^2_\varepsilon(x)\, dx \leq C_{16} |\ln \varepsilon|^{-\delta} \tag{21}$$

is valid, where the constant C_{16} does not depend on ε, $0 < \delta < 2 - 2/n$, $N_\varepsilon =$

$= O(|\ln \varepsilon|^{(1-\delta/2)^{n-1}})$ *as* $\varepsilon \to 0$, N_ε *is the number of* Γ_ε^k *on the boundary* $\partial\Omega$, $\psi_\varepsilon(x) = \prod_{k=1}^{N_\varepsilon} \psi_\varepsilon^k(x)$, $\psi_\varepsilon^k(x) = \psi(|\ln \varepsilon| / |\ln r_k|)$, *where* $(r_k, \theta_k^1, \ldots, \theta_k^{n-1})$ *is a local system of polar coordinates, whose center is in* $p_\varepsilon^k \in \Gamma_\varepsilon^k$, $\psi(s)$ *is a function, defined in the proof of Theorem* 1.

PROOF. Subtracting the integral identity of the problem (19), (20) from the integral identity of problem (1)-(3) and setting $v = (u_\varepsilon - u_0)\psi_\varepsilon^2$, we obtain

$$\int_\Omega \sum_{i,j,k,l=1}^n a_{kl}^{ij}(x) \frac{\partial(u_\varepsilon^k - u_0^k)}{\partial x_i} \frac{\partial((u_\varepsilon^l - u_0^l)\psi_\varepsilon^2)}{\partial x_j} dx = 0$$

and therefore

$$\int_\Omega \sum_{i,j,k,l=1}^n a_{kl}^{ij}(x) \frac{\partial(u_\varepsilon^k - u_0^k)}{\partial x_i} \frac{\partial(u_\varepsilon^l - u_0^l)}{\partial x_j} \psi_\varepsilon^2 \, dx =$$

$$= -2 \int_\Omega \sum_{i,j,k,l=1}^n a_{kl}^{ij}(x) \frac{\partial(u_\varepsilon^k - u_0^k)}{\partial x_i} (u_\varepsilon^l - u_0^l)\psi_\varepsilon(x) \frac{\partial \psi_\varepsilon}{\partial x_j} dx .$$

From the Korn and the Hölder inequalities for $(u_\varepsilon - u_0)\psi_\varepsilon$ we obtain the following estimate

$$\int_\Omega \sum_{i=1}^n |\nabla(u_\varepsilon^i - u_0^i)|^2 \psi_\varepsilon^2 \, dx = \int_\Omega \sum_{i=1}^n |\nabla((u_\varepsilon^i - u_0^i)\psi_\varepsilon) - (u_\varepsilon^i - u_0^i)\nabla\psi_\varepsilon|^2 \, dx \leqslant \tag{22}$$

$$\leqslant 2 \int_\Omega \sum_{i=1}^n |\nabla((u_\varepsilon^i - u_0^i)\psi_\varepsilon)|^2 + 2\int_\Omega |u_\varepsilon - u_0|^2 \, |\nabla\psi_\varepsilon|^2 \, dx \leqslant$$

$$\leqslant C_{17}\left(\int_\Omega \sum_{i,j=1}^n \left(\frac{\partial(u_\varepsilon^i - u_0^i)}{\partial x_j} + \frac{\partial(u_\varepsilon^j - u_0^j)}{\partial x_i} \right)^2 \psi_\varepsilon^2 \, dx + \int_\Omega |u_\varepsilon - u_0|^2 \, |\nabla\psi_\varepsilon|^2 \, dx \right).$$

From inequalities (4), (22) and the Hölder inequality we deduce

$$\int_\Omega |\nabla(u_\varepsilon - u_0)|^2 \psi_\varepsilon^2(x) \, dx \leqslant \tag{23}$$

$$\leqslant C_{18}\left(\int_\Omega \sum_{i,j,k,l=1}^n a_{kl}^{ij}(x) \frac{\partial(u_\varepsilon^k - u_0^k)}{\partial x_i} (u_\varepsilon^l - u_0^l)\psi_\varepsilon \frac{\partial\psi_\varepsilon}{\partial x_j} dx + \int_\Omega |u_\varepsilon - u_0|^2 \, |\nabla\psi_\varepsilon|^2 \, dx \right) \leqslant$$

$$\leqslant \frac{1}{\delta} C_{19} \int_\Omega |u_\varepsilon - u_0|^2 \, |\nabla\psi_\varepsilon|^2 \, dx + C_{20}\delta \int_\Omega |\nabla(u_\varepsilon - u_0)|^2 \, \psi_\varepsilon^2 \, dx ,$$

where C_{19}, C_{20} do not depend on ε, δ is sufficiently small. The next inequality follows from (23)

$$\int_{\Omega} |\nabla(u_\varepsilon - u_0)|^2 \psi_\varepsilon^2(x)\, dx \leqslant C_{21} \sum_{k=1}^{N_\varepsilon} \int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^2 \, |\nabla \psi_\varepsilon^k|^2 \, dx \,, \tag{24}$$

where ω_ε^k is a ball with radius $\varepsilon^{1/(1+\sigma)}$, whose center is the point p_ε^k. Note that

$$|\nabla \psi_\varepsilon^k| \leqslant C_{22} \, |\ln \varepsilon| \, \frac{1}{\ln^2 r_k} \, \frac{1}{r_k} \,. \tag{25}$$

Let us consider the imbedding theorem of S. L. Sobolev [17]: space $H^1(\Omega)$ continuously imbeds in the space $L_q(\Omega)$, if the domain Ω is a finite union of star-shaped domains and $q \leqslant 2n/(n-2)$. Using this theorem, we can obtain the estimate of the right hand side of (24).

By using estimate (25) and the Hölder inequality, we deduce

$$\begin{aligned} &\int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^2 (|\ln \varepsilon| \, |\ln r_k|^{-2} r_k^{-1})^2 \, dx \leqslant \\ &\leqslant |\ln \varepsilon|^2 \left(\int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^{2p_1} dx \right)^{1/p_1} \left(\int_{\omega_\varepsilon^k} (|\ln r_k|^{-4} r_k^{-2})^{p_2} dx \right)^{1/p_2} , \end{aligned} \tag{26}$$

where $1/p_1 + 1/p_2 = 1$. We suppose that $2p_1 = q = 2n/(n-2), p_2 = n/2$. It is easy to see that

$$\left(\int_{\omega_\varepsilon^k} (|\ln r_k|^{-4} r_k^{-2})^{p_2} dx \right)^{1/p_2} \leqslant C_{23} (|\ln \varepsilon|^{1-2n})^{2/n} \,, \tag{27}$$

where the constant C_{23} does not depend on k and ε.

From inequalities (26) and (27) we obtain

$$\int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^2 \, |\Delta \psi_\varepsilon^k|^2 \, dx \leqslant C_{24} \, |\ln \varepsilon|^{2/n-2} \left(\int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^{2n/(n-2)} \, dx \right)^{(n-2)/n} , \tag{28}$$

where the constant C_{24} does not depend on ε and k. Thus, we have

$$\int_{\Omega} |\nabla(u_\varepsilon - u_0)|^2 \psi_\varepsilon^2(x)\, dx \leqslant C_{25} \sum_{k=1}^{N_\varepsilon} |\ln \varepsilon|^{2/n-2} \left(\int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^{2n/(n-2)} \, dx \right)^{(n-2)/n} . \tag{29}$$

Using the Hölder inequality and the imbedding Theorem, we obtain

$$\begin{aligned} &\sum_{k=1}^{N_\varepsilon} \left(\int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^{2n/(n-2)} \, dx \right)^{(n-2)/n} \leqslant \\ &\leqslant \left(\sum_{k=1}^{N_\varepsilon} 1 \right)^{2/n} \left(\sum_{k=1}^{N_\varepsilon} \int_{\omega_\varepsilon^k} |u_\varepsilon - u_0|^{2n/(n-2)} \, dx \right)^{(n-2)/n} \leqslant \\ &\leqslant (N_\varepsilon)^{2/n} \|u_\varepsilon - u_0\|^2_{L_q(\Omega)} \leqslant (N_\varepsilon)^{2/n} \|u_\varepsilon - u_0\|^2_{H^1(\Omega)} \,. \end{aligned} \tag{30}$$

Lemma 3 and the smoothness of the solution u_0 lead us to the conclusion that the norms $\|u_\varepsilon - u_0\|^2_{H^1(\Omega)}$ are uniformly bounded with respect to ε. Therefore, if we take $N_\varepsilon = O(|\ln \varepsilon|^{(1-\delta/2)n-1})$ as $\varepsilon \to 0$, then from (29) and (30) we obtain (21), where δ satisfies the inequality $0 < \delta < 2 - 2/n$. The theorem is proved.

THEOREM 3. *For the solution u_ε of problem* (1)-(3) *and the solution u_0 of problem* (19), (20) *the estimate*

$$\int_\Omega |u_\varepsilon - u_0|^2\, dx \leq C_{26}\, |\ln \varepsilon|^{-\delta}, \qquad 0 < \delta < 2 - 2/n, \tag{31}$$

is valid, if $N_\varepsilon = O(|\ln \varepsilon|^{(1-\delta/2)n-1})$ as $\varepsilon \to 0$.

PROOF. From Lemma 1 with $\eta = \varepsilon^{1/(1+\sigma)}$ and Lemma 3 it follows, that

$$\int_{\Omega_\eta} u_\varepsilon^2\, dx \leq C_{27}\, \varepsilon^{2/(1+\sigma)}. \tag{32}$$

Since $u_0 = 0$ on $\partial\Omega$,

$$\int_{\Omega_\eta} u_0^2\, dx \leq C_{28}\, \varepsilon^{2/(1+\sigma)}. \tag{33}$$

Therefore, the mean-value theorem for integrals gives us the conclusion that β exists such that $\beta \leq \eta$, and

$$\int_{l_\beta} |u_\varepsilon - u_0|^2\, ds \leq C_{29}\, \varepsilon^{1/(1+\sigma)}. \tag{34}$$

It is easy to see, that

$$\int_{l_\eta} |u_\varepsilon - u_0|^2\, ds \leq C_{30}\left(\int_{l_\beta} |u_\varepsilon - u_0|^2\, ds + \varepsilon^{1/(1+\sigma)} \int_{\Omega_\eta} |\nabla(u_\varepsilon - u_0)|^2\, dx\right). \tag{35}$$

From inequalities (34) and (35) we get

$$\int_{l_\eta} |u_\varepsilon - u_0|^2\, ds \leq C_{31}\, \varepsilon^{1/(1+\sigma)}, \tag{36}$$

since $\int_\Omega |\nabla(u_\varepsilon - u_0)|^2\, dx$ is uniformly bounded with respect to ε.

By the imbedding theorem (8) for $(u_\varepsilon - u_0)$, we obtain the estimate

$$\int_{\Omega\setminus\Omega_\eta} |u_\varepsilon - u_0|^2\, dx \leq C_{32}\left(\int_{\Omega\setminus\Omega_\eta} |\nabla(u_\varepsilon - u_0)|^2\, dx + \int_{l_\eta} |u_\varepsilon - u_0|^2\, ds\right), \tag{37}$$

where Ω_η contains the support of $\nabla\psi_\varepsilon$, and the constant C_{32} does not depend on ε, because of the smoothness of $\partial\Omega$. From estimates (36) and (37) we obtain

$$\int_\Omega |u_\varepsilon - u_0|^2\, dx \leq C_{33}\left(\int_{\Omega\setminus\Omega_\eta} |\nabla(u_\varepsilon - u_0)|^2\, dx + \varepsilon^{1/(1+\sigma)}\right) + \int_{\Omega_\eta} |u_\varepsilon - u_0|^2\, dx.$$

By using estimates (32) and (33), Theorem 2, we get (31). The theorem is proved.

4. – We study the limit behavior of the spectrum of problem (1)-(3) as $\varepsilon \to 0$. The question about the behavior of the spectrum of a boundary value problem, when the boundary conditions are perturbed, was considered in [18]. The case, when the sets Γ_ε^k are disposed in a periodic way, was considered in [3]. In the present paper on the basis of the theorem on the limit behavior of spectrum of the abstract operators sequence, which is proved in [19] (see, also [16]), we study a nonperiodic case.

Consider the spectral problems, which correspond to boundary value problems (1)-(3) and (19), (20):

$$L(u_\varepsilon^k) + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad \text{in } \Omega, \tag{38}$$

$$u_\varepsilon^k = 0 \quad \text{on } \gamma_\varepsilon, \tag{39}$$

$$\sigma(u_\varepsilon) = 0 \quad \text{on } \Gamma_\varepsilon, \quad k = 1, 2, \dots \tag{40}$$

and

$$L(u_0^k) + \lambda_0^k u_0^k = 0 \quad \text{in } \Omega, \tag{41}$$

$$u_0^k = 0 \quad \text{on } \partial\Omega, \quad k = 1, 2, \dots . \tag{42}$$

Here $u_\varepsilon^k \in H^1(\Omega, \gamma_\varepsilon)$ and $u_0^k \in H^1(\Omega, \partial\Omega)$, $k = 1, 2, \dots$. The sets $\{\lambda_\varepsilon^k\}$, $\{\lambda_0^k\}$, $k = 1, 2, \dots$, are eigenvalues such that $\lambda_\varepsilon^1 \leqslant \lambda_\varepsilon^2 \dots \leqslant \lambda_\varepsilon^k \leqslant \dots$, $\lambda_0^1 \leqslant \lambda_0^2 \leqslant \dots \leqslant \lambda_0^k \leqslant \dots$ and the eigenvalues are repeated according to their multiplicities.

Define the operator $A_\varepsilon : L_2(\Omega) \to H^1(\Omega, \gamma_\varepsilon)$, setting $A_\varepsilon f = -u_\varepsilon$, where u_ε is the solution of problem (1)-(3). The operator $A_0 : L_2(\Omega) \to H^1(\Omega, \partial\Omega)$ is defined by the formula $A_0 f = -u_0$, where u_0 is the solution of the problem (19), (20). Let $H_\varepsilon = H_0 = L_2(\Omega)$, $V = H^1(\Omega, \partial\Omega)$ and let R_ε be the identity operator in $L_2(\Omega)$.

Let us verify the conditions of Theorem 1.4 (ch. 3) from [16] (see also [19]). The condition C1 is fulfilled automatically. It is easy to establish the positiveness, self-adjointness and compactness of the operators A_ε and A_0. The norms $\|A_\varepsilon\|_{L(H_\varepsilon)}$ are uniformly bounded with respect to ε by virtue of Lemma 3.

In view of Theorem 3 the condition C3 holds. If a sequence $\{A_\varepsilon f_\varepsilon\}$ is bounded in $H^1(\Omega, \gamma_\varepsilon)$, therefore, it is compact in $L_2(\Omega)$. Because of Lemma 3 the condition C4 is fulfilled.

Consider the spectral problems

$$A_\varepsilon u_\varepsilon^k = \mu_\varepsilon^k u_\varepsilon^k, \qquad \mu_\varepsilon^1 \geqslant \mu_\varepsilon^2 \geqslant \dots, \qquad k = 1, 2, \dots$$

and

$$A_0 u_0^k = \mu_0^k u_0^k, \qquad \mu_0^1 \geqslant \mu_0^2 \geqslant \dots, \qquad k = 1, 2, \dots .$$

It is obvious, that $\mu_\varepsilon^k = (\lambda_\varepsilon^k)^{-1}$, $\mu_0^k = (\lambda_0^k)^{-1}$. Theorem 1.4 (ch. 3) from [16] gives us:

$$|\mu_\varepsilon^k - \mu_0^k| \leqslant C_{34} \sup_{\substack{u \in N(\mu_0^k, A_0) \\ \|u\|_{H_0} = 1}} \|A_\varepsilon u - A_0 u\|_{H_\varepsilon}, \tag{43}$$

$k = 1, 2\ldots$ where $N(\mu_0^k, A_0) = \{u: u \in H_0, A_0 u = \mu_0^k u\}$. Thus the following theorem follows from (43) and Theorem 3:

THEOREM 4. *There exists a constant* C_{35}, *which does not depend on* ε *and such, that for eigenvalues* λ_ε^k *and* λ_0^k *of the problems* (38)-(40) *and* (41), (42), *respectively, the estimate* $|(\lambda_\varepsilon^k)^{-1} - (\lambda_0^k)^{-1}| \leqslant C_{35} |\ln \varepsilon|^{-\delta}$ *for sufficiently small* ε *is valid, where* $0 < \delta < 2 - 2/n$, $N_\varepsilon = O(|\ln \varepsilon|^{(1-\delta/2)n-1})$ *as* $\varepsilon \to 0$.

5. – In the same way we considered also the elliptic equations and the stationary linear elasticity system in perforated domains with rapidly alternating boundary conditions. Let $\Omega_\varepsilon = \Omega \setminus \left\{ \bigcup_k T_k \right\}$ where the domain T_k has a diameter ε, and we consider the equation in Ω_ε with the boundary conditions (2), (3) and the Dirichlet boundary conditions on ∂T_k. Then the theorems, which are similar to Theorems 1-3 are valid. Moreover, we considered the problem when the Dirichlet condition is given on the boundary of some domains T_k and the condition of the form (3) is given on the boundary of the other T_k. In addition, we suppose in this case that the function $u \in H^1(\Omega_\varepsilon)$ can be extended in $H^1(\Omega)$ in such a way that $\|u\|_{H^1(\Omega)} \leqslant C_{36} \|u\|_{H^1(\Omega_\varepsilon)}$, where the constant C_{36} does not depend on ε.

Similar results are proved in the case when we set on γ_ε some other type of coercive boundary conditions.

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