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An abstract setting for boundary problems with affine symmetries

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Fisica matematica. — *An abstract setting for boundary problems with affine symmetries.* Nota (*) di TULLIO VALENT, presentata dal Socio G. Grioli.

ABSTRACT. — Two symmetries of affine type for any mapping acting between Banach spaces are described and studied. These symmetries translate certain structural properties of boundary value problems for differential operators to an abstract setting.

KEY WORDS: Symmetries for operators; Affine representations of Lie groups; Boundary value problems; Elasticity.

RIASSUNTO. — *Una formulazione astratta di problemi al contorno con simmetrie affini*. Vengono descritte e studiate due simmetrie di tipo affine per un operatore agente tra spazi di Banach. Tali simmetrie traducono, in un contesto astratto, delle proprità strutturali di problemi al contorno per operatori differenziali, come viene mostrato attraverso vari esempi.

PREFACE

For a mapping $A: U \subseteq X \to Y$, with X, Y real Banach spaces and U an open subset of X, we consider two symmetries of affine type (in the sense that they are defined starting from two affine representations of a Lie group G – one on X and the other on Y – related by a linear mapping $\tau: X \to Y$, which, when A «describes» a boundary problem, can have the meaning of a «trace mapping»).

The second symmetry seems unlike the first one, because it involves the representations of G on X and Y through their differentials and depends on an inner product on Y. Nevertheless, after proposing and discussing, in Sect. 3, a definition of «potential» for A with respect to τ and an inner product on Y, in Sect. 4 we prove (see Theorem 4.1) that the second symmetry is a consequence of the first one when A has a «potential» and satisfies (2.5), and the representations of G on X and Y have two easily stated properties (one of which is expressed in a particularly simple manner using the notion of the derived algebra of a Lie algebra, and is trivially satisfied if the Lie group G is semi-simple).

In Sect. 5 we examine closely a situation which is typical when the mapping A «describes» a boundary value problem for a differential operator and τ is a «trace mapping». Finally, in Sect. 6 we present some relevant examples arising from the study of Neumann's problems, and, in particular, of the «traction problem» in linear and nonlinear elastostatics. We observe that a consequence of Theorem 4.1 applied to Example 2 in Sect. 6 is that for any hyperelastic material (without internal structure) the principle of the material frame-indifference implies the symmetry of the Cauchy stress: however, as we remark in Sect. 6, this fact can be directly proved without difficulty.

In a subsequent article devoted to a perturbation problem in the presence of affine symmetries we shall be dealing with an equation of the type $A(x) + \varepsilon B(x) = 0$, where

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$A: U \to Y$ is an operator having the symmetries considered in this paper, $B: U \to Y$ is a given (perturbation) operator, and ε is a parameter; we shall show that a crucial role is played by the second symmetry of A when the first symmetry is present, in order to prove existence theorems.

1. Preliminaries

We shall deal with a mapping $A: U \subseteq X \to Y$, with X, Y real Banach spaces and U open in X. Let us denote by $\mathcal{A}(X)$ the (Banach) space of continuous, affine mappings from X into itself equipped with the norm

$$\psi \mapsto \|\psi(0)\|_x + \sup\{\|\psi(x) - \psi(0)\|_x : \|x\|_x \leq 1\},$$

where $\|\cdot\|_x$ is the norm of X, and by $\mathcal{L}(X)$ the subspace of $\mathcal{A}(X)$ whose elements are (continuous) linear mappings. The symbols $\mathcal{A}(Y)$ and $\mathcal{L}(Y)$ have an evident, analogous meaning.

In order to introduce affine symmetries on the operator A, we need to consider a Lie group G, and affine representations $g \mapsto \varrho_g$ of G on X and $g \mapsto \tilde{\varrho}_g$ of G on Y (this means that $g \mapsto \varrho_g$ is a homomorphism of group G into the group of invertible elements of $\mathcal{A}(X)$, and $g \mapsto \tilde{\varrho}_g$ is a homomorphism of group G into the group of invertible elements of $\mathcal{A}(Y)$). We will suppose that mappings $g \mapsto \varrho_g$ and $g \mapsto \tilde{\varrho}_g$ are of class C^1, and denote by $v \mapsto R_v$ and $v \mapsto \tilde{R}_v$ their differentials at the identity element e of group G; so $v \mapsto R_v$ and $v \mapsto \tilde{R}_v$ are continuous, linear mappings from the tangent space $T_e G$ to manifold G at e into $\mathcal{A}(X)$ and $\mathcal{A}(Y)$, respectively. For any $g \in G$, let l_g be the linear part of ϱ_g, and $\tilde{l}_g$ be the linear part of $\tilde{\varrho}_g$, so that

$$\varrho_g(x) = l_g(x) + \varrho_g(0), \qquad \tilde{\varrho}_g(y) = \tilde{l}_g(y) + \tilde{\varrho}_g(0)$$

for all $x \in X$ and $y \in Y$. The differentials at e of mappings $g \mapsto l_g$ and $g \mapsto \tilde{l}_g$ will be denoted by $v \mapsto L_v$ and $v \mapsto \tilde{L}_v$; they are continuous, linear mappings from $T_e G$ into $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, respectively. As, for all $v \in T_e G$, $x \in X$, and $y \in Y$,

$$R_v(x) = \left(\frac{d}{d\lambda} \varrho_{\exp(\lambda v)}(x)\right)_{\lambda=0}, \qquad \tilde{R}_v(y) = \left(\frac{d}{d\lambda} \tilde{\varrho}_{\exp(\lambda v)}(y)\right)_{\lambda=0},$$

we have

$$R_v(x) = L_v(x) + R_v(0), \qquad \tilde{R}_v(y) = \tilde{L}_v(y) + \tilde{R}_v(0). \tag{1.1}$$

2. Symmetries for operator A

We shall consider symmetries for the operator $A: U \to Y$ expressed by the following property: *an affine representation $g \mapsto \varrho_g$ of a Lie group G on X, an affine representation $g \mapsto \tilde{\varrho}_g$ of G on Y, a continuous, linear mapping $\tau: X \to Y$, and an inner product $(y_1, y_2) \mapsto y_1 \bullet y_2$ on Y exist such that $\varrho_g(U) \subseteq U$, $\forall g \in G$, and*

$$\tilde{\varrho}_g \circ \tau = \tau \circ \varrho_g, \qquad \forall g \in G, \tag{2.1}$$

$$A(\varrho_g(x)) = \tilde{l}_g(A(x)), \qquad \forall g \in G \quad \text{and} \quad \forall x \in U, \tag{2.2}$$

$$A(x) \bullet \tilde{R}_v(\tau(x)) = 0, \qquad \forall v \in T_e G \quad \text{and} \quad \forall x \in U. \tag{2.3}$$

Note that from (2.1) it follows that, for any $v \in T_e G$, $x \in X$ and $\lambda \in \boldsymbol{R}$

$$\tilde{\varrho}_{\exp(\lambda v)}(\tau(x)) = \tau(\varrho_{\exp(\lambda v)}(x)),$$

which implies, after differentiating at $\lambda = 0$, that

$$\tilde{R}_v(\tau(x)) = \tau(R_v(x))\,; \tag{2.4}$$

thus (2.3) can be written in the form

$$A(x) \bullet \tau(R_v(x)) = 0\,, \qquad \forall v \in T_e G \quad \text{and} \quad \forall x \in U\,. \tag{2.3$'$}$$

Note also that, in view of (1.1), (2.3)′ implies

$$A(x) \bullet \tau(R_v(0)) = 0\,, \qquad \forall v \in T_e G \quad \text{and} \quad \forall x \in U\,. \tag{2.5}$$

We shall prove (see Theorem 4.1) that, if (2.5) holds and operator A has a potential with respect to τ and the inner product $\bullet$ on Y (in a sense that we shall make precise), then (2.2) implies (2.3)′ (and hence (2.3), if (2.1) holds) provided the representations $g \mapsto \varrho_g$ and $g \mapsto \tilde{\varrho}_g$ of G have the following properties (2.6) and (2.7):

(2.6) $\tilde{l}_g(y_1) \bullet \tilde{l}_g(y_2) = y_1 \bullet y_2\,, \qquad \forall g \in G$ *and* $\forall y_1,\ y_2 \in Y$,

(2.7) *the derived algebra of* $\{L_v : v \in T_e G\}$ *is pointwise dense in the subspace* $\{L_v : v \in T_e G\}$ *of* $\mathcal{L}(X)$.

Of course, in (2.7) $T_e G$ has to be regarded as a Lie algebra (with the Lie algebra structure induced by the Lie algebra of the Lie group G). Obviously, condition (2.7) is fulfilled if $T_e G$ coincides with its derived algebra; this occurs, for instance, when the Lie group G is *semisimple* (see, *e.g.*, [3, p. 313]).

3. Potentials for a mapping $A: U \subseteq X \to Y$

In this section: X, Y are real linear spaces, U is any subset of X, $\tau: X \to Y$ is a linear mapping, β_Y is an inner product on Y, $((H, \beta_H), \varphi_H)$ is a Hilbert completion of (Y, β_Y) (*i.e.* H is a Hilbert space with inner product β_H, and φ_H is a linear isometry of (Y, β_Y) onto a dense subspace of H), and j_H denotes the canonical isomorphism of H onto ints dual H'.

We will say that a function $f: H \to \boldsymbol{R}$ is a *potential for a mapping* $A: U \to Y$ *with respect to* τ *and* β_Y if f is Gâteaux-differentiable and

$$j_H \circ \varphi_H \circ A = f' \circ \varphi_H \circ \tau|_U\,, \tag{3.1}$$

where f' is the Gâteaux-differential of f.

Remark 3.1. *Let* $((K, \beta_K), \varphi_K)$ *be another Hilbert completion of* (Y, β_Y), *let* j_K *be the canonical isomorphism of* K *onto its dual* K', *and let* φ *be the canonical linear isometry of the Hilbert space* K *onto the Hilbert space* H *so that* $\varphi \circ \varphi_K = \varphi_H$. *If* $f: H \to \boldsymbol{R}$ *is a potential for* A *with respect to* τ *and* β_Y, *then also the function* $f \circ \varphi: K \to \boldsymbol{R}$ *is a potential for* A *with respect to* τ *and* β_Y.

Proof. Let $f: H \to \boldsymbol{R}$ be a potential for A with respect to τ and β_Y, and let

$g = f \circ \varphi$. Then

$$g'(\varphi_K(y)) = f'(\varphi(\varphi_K(y))) \circ \varphi = f'(\varphi_H(y)) \circ \varphi,$$

for all $y \in Y$, and hence

$$(g' \circ \varphi_K \circ \tau)(x) = f'(\varphi_H(\tau(x))) \circ \varphi, \qquad \forall x \in X.$$

Thus, in view of (3.1), we obtain, for $x \in U$, $(g' \circ \varphi_K \circ \tau)(x) = j_H(\varphi_H(A(x))) \circ \varphi$.

Therefore g is a potential for A with respect to τ and β_Y (*i.e.*, $j_K \circ \varphi_K \circ A = g' \circ \varphi_H \circ \tau|_U$) if and only if $j_H(\varphi_H(A(x))) \circ \varphi = j_K(\varphi_K(A(x)))$, $\forall x \in U$.

Well, this equality is true because

$$j_H(\varphi_H(y)) \circ \varphi = j_K(\varphi_K(y)), \qquad \forall y \in Y,$$

and the last equality follows from the fact that, for all $k \in K$, we have

$$\begin{cases} (j_H(\varphi_H(y)) \circ \varphi)(k) = j_H(\varphi_H(y))(\varphi(k)) = \beta_H(\varphi_H(y), \varphi(k)), \\ (j_K(\varphi_K(y)))(k) = \beta_K(\varphi_K(y), k) = \beta_H(\varphi(\varphi_K(y)), \varphi(k)) = \beta_H(\varphi_H(y), \varphi(k)). \end{cases} \qquad ■$$

In concrete cases arising when the pair (A, τ) describes a boundary problem, τ is one-to-one and $\tau(X)$ is dense in Y when Y has the topology defined by the inner product β_Y (but not for the original topology of Y). In this situation we can find other, equivalent definitions of potential for A with respect to τ and β_Y. To this end, *we consider on X the inner product β_X defined by putting*

$$\beta_X(x_1, x_2) = \beta_Y(\tau(x_1), \tau(x_2)) \tag{3.2}$$

for all $x_1, x_2 \in X$, and we observe that to any Hilbert completion $((H, \beta_H), \varphi_H)$ of (Y, β_Y) one can associate the Hilbert completion $((H, \beta_H), \psi_H)$ of (X, β_X) where $\psi_H = \varphi_H \circ \tau$; conversely to any Hilbert completion $((H, \beta_H), \psi_H)$ of (X, β_X) one can associate a Hilbert completion $((H, \beta_H), \varphi_H)$ of (Y, β_Y) such that $\psi_H = \varphi_H \circ \tau$, by taking as φ_H the continuous, linear extension to Y of the linear isometry $\tau(x) \mapsto \psi_H(x)$ from the dense subspace $\tau(X)$ of (Y, β_Y) into (H, β_H).

Now, if $f_H: H \to \boldsymbol{R}$ is a potential for A with respect to τ and β_Y, and we set

$$f_Y = f_H \circ \varphi_H, \tag{3.3}$$

then, denoting by f'_H and f'_Y the Gâteaux-differentials of f_H and f_Y, we have

$$f'_Y(\tau(x)) = f'_H(\varphi_H(\tau(x))) \circ \varphi_H, \qquad \forall x \in X,$$

and hence, as $f'_H(\varphi_H(\tau(x))) = j_H(\varphi_H(A(x)))$, $\forall x \in U$, we obtain

$$f'_Y(\tau(x)) = j_H(\varphi_H(A(x))) \circ \varphi_H, \qquad \forall x \in U. \tag{3.4}$$

Conversely, if a function $f_Y: Y \to \boldsymbol{R}$ satisfies (3.4) and a Gâteaux-differentiable function $f_H: Y \to \boldsymbol{R}$ is related to f_Y by (3.3), then clearly $j_H \circ \varphi_H \circ A = f'_H \circ \varphi_H \circ \tau|_U$, *i.e.*, $f_H: H \to \boldsymbol{R}$ is a potential for A with respect to τ and β_Y.

Furthermore, if $f_H: H \to \boldsymbol{R}$ is a potential for A with respect to τ and β_Y, and the function $f_X: X \to \boldsymbol{R}$ is defined by

$$f_X = f_H \circ \psi_H, \tag{3.5}$$

then $f'_X(x) = f'_H(\psi_H(x)) \circ \psi_H$, $\forall x \in X$ namely (as $\psi_H = \varphi_H \circ \tau$),

$$f'_X(x) = f'_H(\varphi_H(\psi(x))) \circ \psi_H\,, \qquad \forall x \in X\,,$$

which gives

$$f'_X(x) = j_H(\varphi_H(A(x))) \circ \psi_H\,, \qquad \forall x \in U\,. \tag{3.6}$$

Conversely, if a function $f_X\colon X \to \boldsymbol{R}$ satisfies (3.6) and a Gâteaux-differentiable function $f_H\colon H \to \boldsymbol{R}$ is related to f_X by (3.5), then evidently f_H is a potential for A with respect to τ and β_Y.

Thus (when τ is one-to-one and $\tau(X)$ is dense in (Y, β_Y)) *any Gâteaux-differentiable function* $f_X\colon X \to \boldsymbol{R}$ *satisfying* (3.6) *and any Gâteaux-differentiable function* $f_Y\colon Y \to \boldsymbol{R}$ *satisfying* (3.4) *can be called a potential for* A *with respect to* τ *and* β_Y.

Often, when X and Y are function spaces, the pair $((H, \beta_H), \psi_H)$ is a functional completion of (X, β_X), so that ψ_H is the identity function from X onto a dense subspace of H. In this case the condition (3.6) takes the simpler form

$$f'_X\,|_U = j_H \circ \varphi_H \circ A\,.$$

4. The main result

We are now in a position to prove the main result of this paper, which has been presented in Sect. 2.

Theorem 4.1. *Let X, Y be real Banach spaces, let U be an open subset of X, let $\tau\colon X \to Y$ be a continuous, linear mapping, let $(y_1, y_2) \mapsto y_1 \bullet y_2$ be an inner product on Y defining a topology weaker than the topology of Y, and let $A\colon U \to Y$ be a C^1-mapping. If A admits a potential with respect to τ and the inner product $\bullet$ having a symmetric second Gâteaux-differential, and* (2.1), (2.5), (2.6), (2.7) *hold, then symmetry* (2.2) *implies* (2.3).

Proof. Let $f\colon H \to \boldsymbol{R}$ be a potential with respect to τ and $\bullet$ having a symmetric second Gâteaux-differential f'', and let (2.1), (2.2), (2.5), (2.6), (2.7) be fulfilled. From (2.2) it follows that, for all $v \in T_e G$, $\lambda \in \boldsymbol{R}$, and $x \in U$, $A(\varrho_{\exp(\lambda v)}(x)) = \tilde{l}_{\exp(\lambda v)}(A(x))$, which yields

$$A'(x)(R_v(x)) = \tilde{L}_v(A(x))\,. \tag{4.1}$$

On the other hand, using (2.6) we obtain $y_2 \bullet \tilde{l}_{\exp(\lambda v)}(y_1) = y_1 \bullet \tilde{l}_{\exp(\lambda v)}(y_2)$ for all $v \in T_e G$, $y_1, y_2 \in Y$ and $\lambda \in \boldsymbol{R}$, and this easily gives

$$\tilde{L}_v(A(x)) \bullet \tau(x_1) = -\tilde{L}_v(\tau(x_1)) \bullet A(x) \quad \text{for all } v \in T_e G,\ x \in U\,, \quad \text{and } x_1 \in X\,. \tag{4.2}$$

From (4.1) and (4.2) it follows, for all $v \in T_e G$, $x \in U$, and $x_1 \in X$,

$$A'(x)(R_v(x)) \bullet \tau(x_1) = -\tilde{L}_v(\tau(x_1)) \bullet A(x)\,,$$

namely, by (2.4),

$$A'(x)(R_v(x)) \bullet \tau(x_1) = -\tau(L_v(x_1)) \bullet A(x)\,. \tag{4.3}$$

Since, in view of (3.1),

$$j_H \circ \varphi_H \circ A'(x) = f''(\varphi_H(\tau(x))) \circ \varphi_H \circ \tau\,, \qquad \forall x \in U\,,$$

we have, for any $x \in U$ and $\xi_1, \xi_2 \in X$,

$$A'(x)(\xi_1) \bullet \xi_2 = \beta_H(\varphi_H(A'(x)(\xi_1)), \varphi_H(\xi_2)) = \\ = \beta_H\big(j_H^{-1}(f''(\varphi_H(\tau(x))))(\varphi_H(\tau(\xi_1))), \varphi_H(\xi_2)\big) = \\ = \big(f''(\varphi_H(\tau(x)))(\varphi_H(\tau(\xi_1)))\big)(\varphi_H(\xi_2)),$$

where β_H denotes the inner product on the Hilbert space H, φ_H is a linear isometry of (Y, β_H) onto a dense subspace of H, and j_H denotes the canonical isomorphism of H onto its dual (see Sect. 3). Thus, for any $x \in U$ and any $v_1, v_2 \in T_e G$, we have

$$\begin{cases} A'(x)(R_{v_1}(x)) \bullet \tau(R_{v_2}(x)) = \big((f''(\varphi_H(\tau(x))))(\varphi_H(\tau(R_{v_1}(x))))\big)(\varphi_H(\tau(R_{v_2}(x)))), \\ A'(x)(R_{v_2}(x)) \bullet \tau(R_{v_1}(x)) = \big((f''(\varphi_H(\tau(x))))(\varphi_H(\tau(R_{v_2}(x))))\big)(\varphi_H(\tau(R_{v_1}(x)))), \end{cases}$$

and hence

$$A'(x)(R_{v_1}(x)) \bullet \tau(R_{v_2}(x)) = A'(x)(R_{v_2}(x)) \bullet \tau(R_{v_1}(x)), \tag{4.4}$$

because

$$\big(f''(\varphi_H(\tau(x))(\varphi_H(y_1)))(\varphi_H(y_2))\big) = \big(f''(\varphi_H(\tau(x))(\varphi_H(y_2)))(\varphi_H(y_1))\big),$$

for any $x \in U$ and any $y_1, y_2 \in Y$.

Combining (4.3) and (4.4) we obtain $A(x) \bullet \tau((L_{v_1} L_{v_2} - L_{v_2} L_{v_1})(x)) = 0$ for any $x \in U$ and any $v_1, v_2 \in T_e G$. Then, in view of (2.7), we have

$$A(x) \bullet \tau(L_v(x)) = 0, \quad \forall x \in U \text{ and } \forall v \in T_e G,$$

and this implies (2.3)′ because of (2.5). To conclude the proof it is sufficient to recall that, by (2.1), properties (2.3) and (2.3)′ are equivalent. ■

5. Remarks on the case when operator A describes a boundary problem

Mapping A can describe a boundary value problem for a differential operator, as we shall see in the next section. In this case Y is a product of Banach spaces, say $Y = Y_1 \times \times \ldots \times Y_r$, and spontaneously there are Banach spaces $X_1, \ldots, X_r$ in each of which X is a dense subset, and for each $j = 1, \ldots, r$ there is a continuous, linear mapping τ_j from X_j onto Y_j. In this section we place ourselves in this situation, and we put $\tau(x) = = (\tau_j(x))_{j=1,\ldots,r}$ $\forall x \in X$. The meaning of τ is that of a «trace mapping». Furthermore, in concrete cases related to boundary value problems for differential operators the following facts occur:

(*i*) $X, X_1, \ldots, X_r, Y, Y_1, \ldots, Y_r$ are spaces of $\boldsymbol{R}^n$-valued functions (for some n). A norm $\|\cdot\|$ is assigned on the linear space of real valued functions that are the n components of the elements x of X, and one considers on X a norm defining the product topology. A norm on $X_1, \ldots, X_r, Y_1, \ldots, Y_r$ is chosen in a similar way.

(*ii*) X is invariant under composition with all affine mappings from $\boldsymbol{R}^n$ into itself.

(*iii*) If an element x of X takes its values in a one-dimensional, linear subspace of $\boldsymbol{R}^n$, then $\tau_1(x), \ldots, \tau_r(x)$ take their values in that subspace.

(*iv*) The affine representation $g \mapsto \varrho_g$ of G on X arises from an affine representation $g \mapsto \alpha_g$ of G on $\boldsymbol{R}^n$ in the following way: $\varrho_g(x) = \alpha_g \circ x$.

When this occurs it is easy to see that, for every $g \in G$, ϱ_g remains continuous (and hence a homeomorphism) also when the linear space X is equipped with the topology defined by each of the seminorms $x \mapsto \|\tau_j(x)\|_{Y_j}, j = 1, \dots, r$, and also with the topology induced on X by that of each X_j, $j = 1, \dots, r$; thus ϱ_g can be extended to an affine homeomorphism ϱ_g^j from X_j onto itself, and, for each $g \in G$ and $j = 1, \dots, r$, there is $c_j(g) \in \boldsymbol{R}$ such that

$$\|\tau_j(l_g(x))\|_{Y_j} \leq c_j(g) \|\tau_j(x)\|_{Y_j}, \qquad \forall x \in X.$$

Then, putting for each $(x^j)_{j=1,\dots,r} \in X_1 \times \dots \times X_r$

$$\tilde{l}_g((\tau_j(x^j))_{j=1,\dots,r}) = ((\tau_j \circ \varrho_g^j)(x^j))_{j=1,\dots,r},$$

it is easy to verify that $\tilde{l}_g$ is a one-to-one, continuous, linear mapping (and hence a linear homeomorphism) from $Y_1 \times \dots \times Y_r$ onto itself such that $\tilde{l}_g \circ \tau = \tau \circ l_g$. Of course, $\tilde{l}_g$ is the linear part of the affine mapping $\tilde{\varrho}_g$ from $Y_1 \times \dots \times Y_r$ into itself defined by putting

$$\tilde{\varrho}_g((\tau_j(x^j))_{j=1,\dots,r}) = ((\tau_j \circ \varrho_g^j)(x^j))_{j=1,\dots,r}$$

for all $(x^j)_{j=1,\dots,r} \in X_1 \times \dots \times X_r$.

In conclusion: *if the conditions* (*i*), (*ii*), (*iii*) *are satisfied, then for every representation* $g \mapsto \varrho_g$ *of* G *on* X *of the type described in* (*iv*) *there is an affine representation* $g \mapsto \tilde{\varrho}_g$ *of* G *on* Y *which satisfies* (2.1).

6. Examples from Neumann's problems

In this section we present some examples of concrete operators A satisfying the symmetry assumptions (2.1), (2.2) and (2.3); they arise from the study of Neumann's boundary problems of the divergence type, in particular from the treatment of the «traction problem» in finite elastostatics and in linearized elastostatics.

To this end, let us denote by $\boldsymbol{M}_{m \times n}$ the set of real $m \times n$ matrices, by $\boldsymbol{M}_n$ the set of real $n \times n$ matrices, by $\boldsymbol{M}_n^+$ the set of $Z \in \boldsymbol{M}_n$ such that $\det Z > 0$, by I the unit element of the ring $\boldsymbol{M}_n$, by Sym_n the set of symmetric elements of $\boldsymbol{M}_n$, by Skew_n the set of skew symmetric elements of $\boldsymbol{M}_n$, and by $\boldsymbol{O}_n^+$ the set of elements Z of $\boldsymbol{M}_n^+$ such that $Z^T = Z^{-1}$, where Z^T is the transpose of the matrix Z.

Bearing Neumann's problems for second order differential operators in mind, we make the following two choices of the spaces X, X_j, Y_j considered in the previous section:

$$\begin{cases} X = W^{k+2,p}(\Omega, \boldsymbol{R}^m), \\ X_1 = Y_1 = W^{k,p}(\Omega, \boldsymbol{R}^m), \\ X_2 = W^{k+1,p}(\Omega, \boldsymbol{R}^m), \\ Y_2 = W^{k+1-1/p,p}(\partial\Omega, \boldsymbol{R}^m), \end{cases} \qquad \begin{cases} X = C^{k+2,\lambda}(\overline{\Omega}, \boldsymbol{R}^m), \\ X_1 = Y_1 = C^{k,\lambda}(\overline{\Omega}, \boldsymbol{R}^m), \\ X_2 = C^{k+1,\lambda}(\overline{\Omega}, \boldsymbol{R}^m), \\ Y_2 = C^{k+1,\lambda}(\partial\Omega, \boldsymbol{R}^m), \end{cases}$$

where Ω is a sufficiently smooth, bounded, open subset of $\boldsymbol{R}^n$, and $1 < p \in \boldsymbol{R}$, $\lambda \in$ $\in]0, 1[$, and $p(k+1) > n$. They are spaces of $\boldsymbol{R}^m$-valued functions; their definitions and properties can by found, for instance, in Valent [2]. For both choices of these spaces, we take as $\tau_1 : X_1 \to Y_1$ and $\tau_2 : X_2 \to Y_2$ the functions defined by putting, for any $x_1 \in$ $\in X_1$ and $x_2 \in X_2$,

$$\tau_1(x_1) = x_1 , \qquad \tau_2(x_2) = x_2 \,|_{\partial\Omega} ,$$

and we take as a completion of $Y_1 \times Y_2$ the pair $((H, \beta_H), \varphi_H)$, where H is the product

$$L^2(\Omega, \boldsymbol{R}^n) \times L^2(\partial\Omega, \boldsymbol{R}^n),$$

φ_H is the identity function from $Y_1 \times Y_2$ into H, and β_H is the inner product on H defined by

$$\beta_H((\bar{y}_1, \bar{y}_2), (y_1, y_2)) = \int_\Omega \bar{y}_1(t)\cdot y_1(t)\,dt + \int_{\partial\Omega} \bar{y}_2(t)\cdot y_2(t)\,d\sigma ,$$

with $\cdot$ denoting the inner product on $\boldsymbol{R}^n$. Thus, by (3.2), we have

$$\beta_X(x_1, x_2) = \int_\Omega x_1(t)\cdot x_2(t)\,dt + \int_{\partial\Omega} x_1(t)\cdot x_2(t)\,d\sigma , \qquad \forall x_1,\ x_2 \in X .$$

Moreover, we take as U the set of orientation-preserving diffeomorphisms of $\overline{\Omega}$ onto a subset of $\boldsymbol{R}^n$ belonging to X. We observe that, since X is continuously embedded in $C^1(\overline{\Omega}, \boldsymbol{R}^n)$, U is open in X (cfr. [1, Ch. 2, Th. 1.4]).

Here, we deal with a (Neumann) operator $A \colon U \to Y_1 \times Y_2$ of the form

(6.1) $$A(x) = (-\operatorname{div} S(x) , S(x)\,|_{\partial\Omega}\, \nu) ,$$

where ν is the outward, unit normal to $\partial\Omega$ and $S(x)$ is the mapping from $\overline{\Omega}$ into $\boldsymbol{M}_{m\times n}$ obtained from a given smooth function $s \colon \overline{\Omega} \times \boldsymbol{M}_n \to \boldsymbol{M}_{m\times n}$ by setting

(6.2) $$S(x)(t) = s(t, \partial x(t)) , \qquad \forall t \in \overline{\Omega} ,$$

with ∂x the gradient of x. (In Valent [2, Chapter II], it is proved that, actually, A maps X into $Y_1 \times Y_2$ for both previous choices of the spaces X, Y_1, Y_2, provided s and Ω are sufficiently smooth).

Example 1. We take as G the group of translations of $\boldsymbol{R}^n$ and define ϱ_g, $\tilde{\varrho}_g$ for any $g \in G$ and τ by putting

(6.3) $$\varrho_g(x) = g \circ x , \qquad \tilde{\varrho}_g(y_1, y_2) = (g \circ y_1, g \circ y_2) , \qquad \tau(x) = (x, x\,|_{\partial\Omega})$$

for all $x \in X$ and $(y_1, y_2) \in Y_1 \times Y_2$. Then conditions (2.1) and (2.2) are evidently satisfied. Also condition (2.3) is satisfied, by virtue of the divergence theorem; indeed, since $T_e G$ is the set of constant functions from $\boldsymbol{R}^n$ into itself and $R_v(x) = v$, $\forall x \in X$ and $\forall v \in T_e G$, condition (2.3) becomes

$$-\int_\Omega \operatorname{div} S(x) + \int_{\partial\Omega} S(x)\,\nu = 0 , \qquad \forall x \in U .$$

Example 2. Let $m = n$. Then mapping A defined by ((6.1), (6.2)) is the (n-dimensional version of the) *finite elastostatics operator*. In the physical context, Ω represents a reference configuration of an elastic body and function x represents a deformation of the body, while function s defines the elastic response in the sense that $s(t, \partial x(t))$ is the first Piola-Kirchhoff stress at point t when the body is deformed by x. In accordance with the principle of material frame-indifference, and the balance of angular momentum we suppose that

(6.4) $$s(t, RZ) = Rs(t, Z), \qquad \forall (t, Z, R) \in \Omega \times \boldsymbol{M}_n^+ \times \boldsymbol{O}_n^+ ,$$

(6.5) $$s(t, Z) Z^T = \mathrm{Sym}_n , \qquad \forall (t, Z) \in \Omega \times \boldsymbol{M}_n^+ .$$

In this example we take as G the group of isometries of $\boldsymbol{R}^n$ (*i.e.*, functions from $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ of the type $y \mapsto c + Ry$, with $c \in \boldsymbol{R}$ and $R \in \boldsymbol{O}_n^+$), and define τ and ϱ_g, $\tilde{\varrho}_g$ for each isometry g of $\boldsymbol{R}^n$ as in (6.3).

It is evident that (2.1) holds and that (2.2) follows from (6.4) (combined with (6.1) and (6.2)). We now show that (6.5) (combined with (6.1) and (6.2)) implies symmetry (2.3). Since $T_e G$ is the set of (affine) functions $v: \boldsymbol{R}^n \to \boldsymbol{R}^n$ of the type $v(t) = c + Wt$, $(t \in \boldsymbol{R}^n)$, with $c \in \boldsymbol{R}^n$ and $W \in \mathrm{Skew}_n$, and $R_v(x) = v \circ x$, $\forall v \in T_e G$ and $\forall x \in X$, condition (2.3)′ (equivalent to (2.3)) says that

$$- \int_{\Omega} ((\operatorname{div} S(x))(t)) \cdot (c + Wx(t))\, dt + \int_{\partial\Omega} (S(x)(t)\, \nu(t)) \cdot (c + Wx(t))\, d\sigma = 0$$

for all $c \in \boldsymbol{R}^n$, $W \in \mathrm{Skew}_n$ and $x \in X$, where $\cdot$ denotes the inner product of $\boldsymbol{R}^n$. In view of (6.2) and the divergence theorem, this condition becomes

$$W \int_{\Omega} \partial x(t)\, s(t, \partial x(t))^T\, dt = 0 , \qquad \forall W \in \mathrm{Skew}_n \quad \text{and } \forall x \in X ,$$

namely

$$\int_{\Omega} \partial x(t)\, s(t, \partial x(t))^T\, dt \in \mathrm{Sym}_n , \qquad \forall x \in X .$$

Then, in order to conclude our proof, it suffices to observe that the last condition is satisfied if (6.5) holds.

Remark. Suppose that there is a C^1-functions $w: \overline{\Omega} \times \boldsymbol{M}_n \to \boldsymbol{R}$ such that

(6.6) $$s(t, Z) = \partial_z w(t, Z), \qquad \forall (t, Z) \in \overline{\Omega} \times \boldsymbol{M}_n^+ ,$$

and set

(6.7) $$f_x(x) = \int_{\Omega} w(t, \partial x(t))\, dt$$

for $x \in U$. Consider on X the inner product β_X, and observe that the Gâteaux-differential $f_x'(\bar{x})$ of f_x at $\bar{x}$ is (the continuous, linear form on (X, β_X)) defined by

$$f'(\bar{x})(x) = \int_{\Omega} w(t, \partial x(t))\, \partial x(t)\, dt , \qquad (x \in X),$$

namely, in view of the divergence theorem, by

$$f'(\bar{x})(x) = -\int_{\Omega} ((\operatorname{div} S(\bar{x}))(t)) \cdot x(t)\, dt + \int_{\partial\Omega} (S(\bar{x})(t)\, \nu(t)) \cdot x(t)\, d\sigma .$$

Then

$$f_x'(\bar{x}) = j_H(A(\bar{x})) \circ \tau , \qquad \forall \bar{x} \in U , \tag{6.8}$$

where j_H is the canonical isomorphism of H onto its dual defined by putting

$$j_H(\bar{y}_1, \bar{y}_2)(y_1, y_2) = \beta_H((\bar{y}_1, \bar{y}_2), (y_1, y_2))$$

for all $(\bar{y}_1, \bar{y}_2)$, $(y_1, y_2) \in H$. Note that, in the particular case we are discussing, (6.8) coincides with (3.6), because here $\varphi_H(A(\bar{x})) = A(\bar{x})$ and $\psi_H = \tau$. Thus, when (6.6) holds, *the function* f_x *defined by* (6.7) *is a potential for* A *with respect to* τ *and the inner product* β_H *on* $Y_1 \times Y_2$, (see Sect. 3). Consequently, in view of Theorem 4.1, symmetry (2.2) implies (2.3) when (6.6) holds. On the other hand, this is in agreement with the fact that *symmetry* (6.5) *is a consequence of* (6.4) *when* (6.6) *holds*. In order to see that (6.4) implies (6.5) provided (6.6) holds, we observe that, for any fixed $t \in \Omega$ and $Z \in \boldsymbol{M}_n$, from (6.4) it follows that

$$s(t, (\exp W) Z) = (\exp W) s(t, Z), \qquad \forall W \in \operatorname{Skew}_n ,$$

which easily gives $\partial_z s(t, Z) ZW = Ws(t, Z)$, $\forall W \in \operatorname{Skew}_n$, and hence

$$\partial_z s(t, Z) ZWZ^T = Ws(t, Z) Z^T , \qquad \forall W \in \operatorname{Skew}_n .$$

Now, if (6.6) holds, this implies $Ws(t, Z) Z^T = 0$, $\forall w \in \operatorname{Skew}_n$, namely (6.5), because $\partial_z s(t, Z) \in \operatorname{Sym}_n$ by (6.6), while $ZWZ^T \in \operatorname{Skew}_n$.

EXAMPLE 3. Let $m = n$, and

$$S(x) = \left(\sum_{h,k=1}^{n} s_{ijhk}\, \partial_k x_h \right)_{i,j=1,\dots n} , \tag{6.9}$$

where x_h is the h-th component of the $\boldsymbol{R}^n$-valued function x, and the s_{ijhk} are given real-valued functions defined on $\bar{\Omega}$ such that

$$s_{ijhk} = s_{hkij} , \tag{6.10}$$

$$s_{ijhk} = s_{jihk} . \tag{6.11}$$

In this case, the mapping A defined by ((6.1), (6.2), (6.9)) is the (n-dimensional version of the) *linear elastostatics operator*. Functions s_{ijhk}, $(i, j, h, k = 1, \dots, n)$, having the properties (6.10) and (6.11) can be obtained from the $\boldsymbol{M}_n$-valued function $s(= (s_{ij})_{i,j=1,\dots,n})$ in Example 2 by putting $s_{ijhk}(t) = \partial_{Z_{hk}} s_{ij}(t, I)$; indeed, symmetries (6.10), (6.11) follow from the symmetries (2.2) and (2.3) of s. Here we take as G the tangent space at the identity function from $\boldsymbol{R}^n$ into itself to the manifold of isometries of $\boldsymbol{R}^n$; thus G is the (additive) group of (affine) functions $g: \boldsymbol{R}^n \to \boldsymbol{R}^n$ of the type $g(t) = c + Wt$, $(t \in \boldsymbol{R}^n)$, with $c \in \boldsymbol{R}^n$ and $W \in \operatorname{Skew}_n$. Furthermore, we consider the (affine) representations $g \mapsto \varrho_g$ of G on X and $g \mapsto \tilde{\varrho}_g$ of G on $Y_1 \times Y_2$ defined by putting

$$\varrho_g(x) = x + g|_{\bar{\Omega}} , \qquad \tilde{\varrho}_g(y_1, y_2) = (y_1 + g|_{\bar{\Omega}}, y_2 + g|_{\partial\Omega}) .$$

Then (2.1) holds with τ defined by $\tau(x) = (x, x|_{\partial\Omega})$, $(x \in X)$. Here $e = 0$, and $R_v(x) = v$ for all $x \in X$ and all $v \in T_e G = G$; moreover $\tilde{l}_v(y_1, y_2) = (y_1, y_2)$ for all $v \in T_e G$ and all $(y_1, y_2) \in Y_1 \times Y_2$. Thus (2.2) becomes $A(x + g|_{\bar{\Omega}}) = A(x)$, $\forall g \in G$ and $\forall x \in X$, namely $A(g|_{\bar{\Omega}}) = 0$, $\forall g \in G$; this condition is satisfied, for, in view of ((6.10), (6.11)), we have $s_{ijhk} = s_{ijkh}$ and hence $S(g|_{\bar{\Omega}}) = 0$, $\forall g \in G$. As regards condition (2.3), we observe that here (2.3)′, which is equivalent to (2.3), becomes

$$-\int_{\Omega} ((\operatorname{div} S(x))(t)) \cdot (c + Wt)\, dt + \int_{\partial\Omega} (S(x)(t)\, \nu(t)) \cdot (c + Wt)\, d\sigma = 0$$

for all $x \in X$, $c \in \boldsymbol{R}^n$ and $W \in \mathrm{Skew}_n$ (where $\cdot$ denotes the inner product on $\boldsymbol{R}^n$), and we note that, by the divergence theorem, this condition means that, for all $x \in X$, the matrix

$$\left(\sum_{h,k=1}^{h} \int_{\Omega} s_{ijhk}(t)\, \partial_k x_h(t)\, dt \right)_{i,j=1,\dots,n}$$

is symmetric. Thus (2.3)′ is a consequence of (6.11).

7. Concluding remarks

In this article our intention is to present and analyze the framework within which next papers of the present author devoted to the local analysis of solutions of perturbation problems with affine symmetries could find their natural context. In such papers we shall consider an equation of the form $F(x, \varepsilon) = 0$ with F a mapping from $X \times \boldsymbol{R}$ into Y such that $F(\cdot, 0)$ has the symmetries described and discussed here. As a first step we shall deal with a mapping F affine in ε, and hence with an equation of the type

$$A(x) + \varepsilon B(x) = 0\,, \tag{7.1}$$

where A and B are given mappings from X into Y, and A has the symmetries considered in sect. 2.

We remark that, when A is the *finite elastostatics operator* defined in Example 2 in the previous section the meaning of B is that of a *loading operator*. An interesting example of (loading) operator $B = (B_1, B_2)$ is the following:

$$\begin{cases} B_1(x)(t) = b_1(t, x(t), \partial x(t))\,, & t \in \Omega\,, \\ B_2(x)(t) = b_2(t, x(t), (\operatorname{cof} \partial x(t))\, \nu(t))\,, & t \in \partial\Omega\,, \end{cases} \tag{7.2}$$

where $\operatorname{cof} \partial x(t)$ is the matrix of cofactors of the matrix $\partial x(t)$ and

$$b_1 : \Omega \times \boldsymbol{R}^n \times \boldsymbol{M}_n \to \boldsymbol{R}^n\,, \qquad b_2 : \partial\Omega \times \boldsymbol{R}^n \times \boldsymbol{R}_n \to \boldsymbol{R}^n$$

are given functions. Observe that $(\operatorname{cof} \partial x(t))\, \nu(t)$ is an element of $\boldsymbol{R}^n$ parallel to the normal to the boundary of $x(\Omega)$ at $x(t)$, and that example (7.2) includes the simple but significant case when B describes the load which acts on a heavy elastic body submerged in a quiet, homogeneous, heavy liquid; in this case

$$\begin{cases} B_1(x)(t) = \mu_1(t)u\,, & t \in \Omega\,, \\ B_2(x)(t) = -\mu_2\big((x(t)\cdot u)\operatorname{cof}\partial x(t)\big)\,\nu(t)\,, & t \in \partial\Omega\,, \end{cases}$$

where u is a fixed element or $\boldsymbol{R}^3$ with $|u| = 1$, μ_1 is a real-valued positive function defined on Ω, μ_2 is a positive constant, and $\cdot$ denotes the inner product on $\boldsymbol{R}^3$.

Only the case of *dead loadings* (*i.e.* the case when B is a constant operator) has been completely studied from the point of view of the local existence and bifurcation analysis (see [1, and references therein]).

In a subsequent paper we succeed in associating to any abstract (perturbation) operator B, at any $(x_0, g_0) \in X \times G$, certain linear subspaces of $T_e G$ that serve to discriminate situations of essential singularity from those in which the singularity is apparent. Moreover, a local existence theorem is proved for the equation (7.1) when A possesses the symmetries considered in this article. Thus a wide variety of those perturbation problems with symmetries where boundary operators appear will be treated in a unitary way. In particular, such theorem applies to the perturbation problems arising, in finite elastostatics, when one deals with loadings which depend on the unknown deformation x in a general manner (*live loadings*), as in example (7.2).

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