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Some remarks on Set-theoretic Intersection Curves in $\mathbb{P}^3$

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Geometria algebrica. — *Some remarks on Set-theoretic Intersection Curves in* $\mathbb{P}^3$.
Nota (*) di Roberto Paoletti, presentata dal Corrisp. M. Cornalba.

Abstract. — Motivated by the notion of Seshadri-ampleness introduced in [11], we conjecture that the genus and the degree of a smooth set-theoretic intersection $C \subset \mathbb{P}^3$ should satisfy a certain inequality. The conjecture is verified for various classes of set-theoretic complete intersections.

Key words: Set-theoretic intersection; Seshadri-ampleness; Genus; Degree.

Riassunto. — *Alcune osservazioni sulle Curve Intersezioni Complete Insiemistiche in* $\mathbb{P}^3$. Con motivazione dalla nozione di Seshadri-ampiezza discussa in [11], si congettura che il genere e il grado di un'intersezione completa insiemistica liscia $C \subset \mathbb{P}^3$ soddisfino un'opportuna diseguaglianza. La congettura è verificata per varie classi di intersezioni complete insiemistiche.

1. Introduction

A notion of positivity for curves in a projective 3-fold, called Seshadri-ampleness, has been introduced in [11]. It is stronger than requiring that the normal bundle be ample and that the cohomological dimension of the complement should be one. In fact, it is satisfied by only «relatively few» curves in $\mathbb{P}^3$, inasmuch it implies an inequality involving the genus and the degree, that we write down explicitly below.

It is an open question whether any space curve is a set-theoretic intersection. While for surfaces in $\mathbb{P}^4$ the cohomological dimension of the complement suffices to separate set-theoretic complete intersection surfaces from most other surfaces, for $\mathbb{P}^3$ some finer positivity measure is required. We might ask whether in characteristic zero any set-theoretic complete intersection in $\mathbb{P}^3$ is Seshadri-ample. This would imply the following:

Conjecture 1.1 (char $(k) = 0$). *Let* $C \subset \mathbb{P}^3$ *be a smooth connected set-theoretic complete intersection of degree* d *and genus* g*; then* $g > (1/2)d(\sqrt{d} - 4) + 1$.

This is false in positive characteristic [7]. In this paper, we shall check the above inequality in some examples, and then prove it under a strong assumption on the singularities of the surface cutting out C. We shall also make a step towards proving it for any set-theoretic complete intersection with semistable normal bundle, by giving a lower bound on the Seshadri constant.

2. Example 1.1. It is easy to produce a large class of set-theoretic intersections for which the conjecture is true. Suppose that the ideal sheaf of C has a minimal resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-m)^r \to \mathcal{O}_{\mathbb{P}^3}(m-t)^r \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \to \mathcal{J}_C \to 0,$$

ove m, t, a are positive integers with $2m > t > ((2r-1)/r)m$ and $a = r(2m-t)$, and

(*) Pervenuta all'Accademia il 27 luglio 1995.

the induced morphism $\mathcal{O}_{\mathbb{P}^3}(-m)^r \to \mathcal{O}_{\mathbb{P}^3}(m-t)^r$ is represented by a symmetric matrix of homogeneous form (of positive degrees). Then C is projectively Cohen-Macauley and self-linked, by the theory of [12, 13]. By Proposition 2.3 of [11], C is ample, and therefore $g > (1/2)d(\sqrt{d}-4)+1$. This can be generalized to include the examples of Gallarati and Catanese [2], as we next show.

EXAMPLE 1.2. Following ideas of Barth [1], Rao has constructed examples of self-linked subschemes of codimension two in Grassmanians, which may be pulled-back to provide self-linked curves in $\mathbb{P}^3$ [13, Example 11]. Namely, to a general choice of a symmetric nondegenerate bilinear form Q on $\mathbb{C}^n$ and of a perturbation $Q' = Q + (s_i s_j)$ of Q, there is associated a reduced locally CM codimension two self-linked subscheme $Z \subset G(n-r, n)$ (the Grassmanian of $n-r$-dimensional subspaces of $\mathbb{C}^n$), having a resolution

$$(1) \qquad 0 \to \mathcal{S}(-2) \oplus \mathcal{Q}^*(-2) \to \bigoplus_{i=1}^{n+1} \mathcal{O}_G(-2) \to \mathcal{O}_G \to \mathcal{O}_Z \to 0\,,$$

where $\mathcal{S}$ and $\mathcal{Q}$ are, respectively, the universal subbundle and the universal quotient bundle on $G(n-r, n)$. Briefly, Z arises as follows. Set $s = n-r$ and define $D = \{x \in G(s, n) \mid \operatorname{rk}(Q|_{S(x)}) \leq s-2\}$, $D' = \{x \in G(s,n) \mid \operatorname{rk}(Q'|_{S(x)}) \leq s-2\}$. Then D and D' are irreducible hypersurfaces, whose singular locus has codimension 3 in $G(s, n)$, and they touch along Z, that is $D \cap D' = 2Z$.

Let now $\mathcal{E}$ be a nontrivial globally generated rank-r vector bundle on $\mathbb{P}^3$, $n \leq h^0(\mathbb{P}^3, \mathcal{E})$ and $V \subset H^0(\mathbb{P}^3, \mathcal{E})$ a general n-dimensional linear subspace generating $\mathcal{E}$. Then V determines a morphism $\gamma\colon \mathbb{P}^3 \to G(n-r, V) \cong G(n-r, n)$, and $C = \gamma^{-1}(Z) \subset \mathbb{P}^3$ will be a reduced locally CM self-linked curve. Set $E = \gamma^*(\mathcal{S}(-2) \oplus \mathcal{Q}^*(-2))$ and $F = \bigoplus_{i=1}^{n+1} \det(\mathcal{E})^{-2}$; then by (1) C has a resolution

$$(2) \qquad 0 \to E \to F \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0\,.$$

Let us suppose that C is irreducible and non-singular (this is the case of the examples of Catanese). Then $\deg(C) = c_2(F-E)$ and $\deg(N) = c_3(F-E)$, where N is the normal bundle to C. We have the isomorphisms $\gamma^*(\mathcal{Q}) \cong \mathcal{E}$, $\gamma^*(\mathcal{S}) \cong M_{\mathcal{E}}$, where $M_{\mathcal{E}} = \operatorname{Ker}(V \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{E})$. Thus

$$(3) \qquad c_t(F-E) = c_t(\det \mathcal{E}^{-2}) \cdot c_t(\mathcal{E} \otimes \det \mathcal{E}^{-2}) \cdot c_t(\mathcal{E}^* \otimes \det \mathcal{E}^{-2})^{-1}\,.$$

A rather long direct computation then gives $c_2(F-E) = 2c_1^2$ and $c_3(F-E) = 2c_3 + 6c_1^3 - 2c_1c_2$, where $c_i = c_i(\mathcal{E})$. Set $x_i = c_i(F-E) \cdot H^{3-i}$; the inequality in Conjecture 1.1 is equivalent to the other $x_3 > x_2\sqrt{x_2}$. The latter is in turn equivalent to $c_3 + (3-\sqrt{2})c_1^3 - c_1c_2 > 0$; the left-hand side can be rewritten $(2-\sqrt{2})c_1^3 + (c_3 + c_1^3 - 2c_1c_2) + c_1c_2$. This expression is non-negative, because $\mathcal{E}$ is globally generated [4, ch. 12]; [5, esp. the remark on page 57]. If it vanishes, then $c_1 = 0$ and $\mathcal{E}$ is trivial.

EXAMPLE 1.3. Let us consider the following explicit case. Let $d > 0$ be a fixed integer, and for $i, j > 0$ let $a_{ij} = a_{ji}$ be a homogeneous form on $\mathbb{P}^3$ of degree $d = s$. For each h consider the simmetric matrix $A_h = [a_{ij}]_{1 \leq i,j \leq h}$, and let $F_h := \operatorname{div}(\det(A_h)) \subset$

$\subset \mathbb{P}^3$. Then $\deg(F_h) = hs$, and if the a_{ij}'s are general, then the singular locus of F_h consists of $t_h = s^3h(h^2 - 1)/3$ nodes [2, § 1]. Then, by Theorem 2.2 and Lemma 2.3 of [2], F_h and F_{h+1} are tangent along a smooth curve C_h, which contains all the nodes of both F_h and F_{h+1}. Furthermore, we have $g \geqslant 1 + s^2h(h+1)(s(7h+8) - 8)/3$, $d = s^2h(h+1)/2$; it easily follows that $g > (1/2)d(\sqrt{d} - 4) + 1$.

EXAMPLE 1.4. In his study of canonical surfaces in $\mathbb{P}^3$, Ciliberto has constructed smooth self-linked curves of genus $g = (n-7)(n-8)(2n-9)/6$ and degree $d = (n-5)(n-6)/2$, for $n = 7, \ldots, 10$ [3]; these examples also fall in the range described by the conjecture.

EXAMPLE 1.5. This example is taken from [8]. Let k be a field of characteristic zero, and let $\mathcal{S} = \{(d, g) \mid \exists C \subset \mathbb{P}^3$ smooth set-theoretic complete intersection of a cone with some other surface, having degree d and genus $g\}$. Then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = \{(ml, (1/2)ml(m + l - 4) + 1) \mid m, l \in N\}$, and $\mathcal{S}_2 = \{(ml + 1, (1/2)ml(m + l - 4) + m) \mid m, l \in N, 2 \leqslant l \leqslant m + 2\}$. It is then easy to check that $g > (1/2)d(\sqrt{d} - 4) + 1$ for all such pairs.

EXAMPLE 1.6. Let $\mathcal{E}$ be a rank-2 vector bundle on $\mathbb{P}^4$, with $s \in H^0(\mathbb{P}^4, \mathcal{E})$ such that $S = Z(s)$ (the zero-locus of s) is non-singular and connected. Let $\pi\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be a general projection with center $P \in \mathbb{P}^4$, $f = \pi|_S$. Then we can assume that $f(S)$ has only ordinary singularities. Hence, f is birational, generically 2-1 over the double curve $C \subset f(S)$, and 3-1 on a finite set points P_i of C; C is smooth away from the P_i's, the inverse images of the P_i's are nodes of the double point curve $D = f^{-1}(C) \subset S$, and D is otherwise smooth; the other singularities of $f(S)$ are pinch points, at whose inverse images $D \to C$ is simply ramified. The pinch points are the images of the tangents to S going through the center P of the projection. By Theorem 15 of [13], $C \subset \mathbb{P}^3$ is self-linked, and therefore a set-theoretic complete intersection. Let K_S denote the canonical bundle of S, $n = \deg(S)$ and $c_2(S) = c_2(T_S)$. Then the number t of triple points is given by $6t = n(n^2 - 12n + 44) + 4K_S^2 - 2c_2(S) - 3H \cdot K_S(n - 8)$ [9, p. 59]. In our case, $n = c_2(\mathcal{E})$, $K_S \cong \mathcal{O}_S(c_1(\mathcal{E}) - 5)$, and $c_t(T_S) = (1 + tH)^5 \cdot c_t(\mathcal{E})^{-1}$. Hence, $6t = c_2\{c_2^2 + 2c_1^2 - 6c_1 + 5c_2 - 3c_1c_2 + 4\}$, where $c_i = c_i(\mathcal{E})$ (viewed as integers). Let us assume that c_1 and c_2 are such that $t = 0$; this happens for example if $(c_1, c_2) = (4, 4), (5, 6), (6, 8)$. Then, both $C \subset f(S)$ and $D \subset S$ are nonsingular.

By the double point formula [4, § 9.3] we find that $D \equiv (c_2 + 1 - c_1)H|_S$, and in particular that D is an ample divisor on S; thus it is connected, and being non-singular it is irreducible. Hence the same holds for C. Since $g = f|_D \colon D \to C$ has degree 2, $\deg(D) = 2\deg(C)$; set $d = \deg(C)$, and denote by g_D and g_C, respectively, the genera of D and C. By Riemann-Hurwitz, we have $2g_D - 2 = 2(2g_C - 2) + \deg(R)$, where R is the ramification divisor of g, and therefore $\deg(R)$ is simply the number of pinch points of $f(S)$. Clearly $\deg(D) = c_2(c_2 + 1 - c_1)$, and so $\deg(C) = (1/2)c_2(c_2 + 1 - c_1)$; adjunction gives $2g_D - 2 = c_2(c_2 - 4)(c_2 + 1 - c_1)$.

To compute $\deg(R)$, we argue as follows. Let $V = \mathbb{C}^4$, and view $\mathbb{P}^4 = \mathbb{P}(V)$ as the projective space of lines in V. The Gauss map $\gamma_S\colon S \to G(2, \mathbb{P}^4) = G(3, V)$ is given by $\gamma_S(x) = [\overline{T_x S}]$, where $\overline{T_x S} \subset \mathbb{P}^4$ is the projective tangent space. The pull-back of the tautological sequence on $G(3, V)$ is the exact sequence $0 \to E(-1) \to \to V \otimes \mathcal{O}_S \to N(-1) \to 0$, where $N = N_{S/\mathbb{P}^4}$, and E also sits in the exact sequence $0 \to \to \mathcal{O}_S \to E \to T_S \to 0$. The fiber of $E(-1)$ at $x \in S$ gets identified with the cone over $\overline{T_x S}$. Let $l \subset V$ be the line corresponding to the center P of the given projection π. Then we have a composition $\phi\colon E(-1) \to V \otimes \mathcal{O}_S \to (V/l) \otimes \mathcal{O}_S$. Set $F = E(-1)$; then ϕ drops rank at $x \in S$ if and only if $V \supset F(x) \supset l$, *i.e.* if and only if $P \in \overline{T_x S}$, which is equivalent to saying that x lies over a pinch point. Hence, $\deg(R) = \deg(X_2(\phi))$, where $X_2(\phi) = = \{x \in S \mid \mathrm{rank}(\phi(x)) \leq 2\}$. Since $c(F)^{-1} = c(N(-1))$, and $N = \mathcal{E}|_S$, Porteous' formula [4, § 14.4] yields

$$\deg(R) = c_2(\mathcal{E})^2 - c_1(\mathcal{E})c_2(\mathcal{E}) + c_2(\mathcal{E}).$$

The inequality $g > (1/2)d(\sqrt{d} + 4) + 1$ is then equivalent to the other $c_2^2 - 5c_2 + + c_1c_2 + 2 \geq 0$; the pairs (c_1, c_2) for wich this fails may be excluded by use of Scharzenberger's condition $c_2(c_2 + 1 - 3c_1 - 2c_1^2) \equiv 0 (\mathrm{mod}\ 12)$ [10, ch. I].

Example 1.7. Let $\mathcal{E}$ be the Horrocks-Mumford bundle on $\mathbb{P}^4$, and fix $x \in \mathbb{P}^4$ a general point. Barth has shown that the family of jumping lines of $\mathcal{E}$ through x is parametrized by a nonsingular curve $R_x \subset \mathbb{P}^3$, having degree 8 and genus 5, which is the curve of contact of two Kummer surfaces, both having all of their nodes on R_x [1]. Clearly the conjecture is satisfied in this case. The curve R_x is not projectively normal.

Example 1.8. Let C be a smooth irreducible curve of genus g and degree d, which is the set-theoretic complete intersection of two integral hypersurfaces V and W, and let $\langle V, W\rangle$ be as in Proposition 2.2.

Proposition 2.1. *Suppose that for a general $S \in \langle V, W\rangle$ the exceptional divisor in the blow up of S along C irreducible. Then* $g > (1/2)d(\sqrt{d} - 4) + 1$.

To see this, as in Proposition 2.2 below we let s be the multiplicity of a general $S \in \langle V, W\rangle$ along C. Let $H \subset P_C$ be the inverse image of a general hyperplane; then $\widetilde{S}_C \in \in |aH - sE_C|$ moves with no fixed components on H, and therefore $(aH - sE_C)^2 \cdot H \geq \geq 0$. This inequality is equivalent to $s/a \leq 1/\sqrt{d}$.

On the other hand, for a general $S \in \langle V, W\rangle$, as we have seen, one has $\widetilde{S}_C = \phi^* \widetilde{S}_G$, and thus $\widetilde{S}_C \setminus E_C \cong \widetilde{S}_G \setminus E_G$. However, for S general the line bundle $\mathcal{O}_{\widetilde{S}_G}(E_G)$ is ample (for such is $\mathcal{O}_{\widetilde{V}_G}(E_G)$) and so the latter open set is affine. Now we only need to prove the following lemma, which is basically a specialization of a result of Goodman [6]:

Claim 2.1. *Let S be an irreducible projective surface, and $E \subset S$ an irreducible Cartier divisor, such that $U = S \setminus E$ is affine. Then* $E^2 > 0$.

Proof. Let f be a nonconstant function on U. We may regard f as a rational function on S, with polar divisor supported on E. Let $(f)_\infty$ and $(f)_0$ be the polar and the zero divisors of f. Then E^2 is a positive multiple of $(f)_\infty^2 = (f)_0 \cdot (f)_\infty \geq 0$.

If equality held, U would contain the complete curve supporting $(f)_0$, absurd. ∎

By the claim, we have $E^2 \cdot (aH - sE) > 0$, which can be rewritten $(s/a)\deg(N) > d$. Thus we conclude $\deg(N) > d\sqrt{d}$, and the proposition easily follows. ∎

Example 1.8 should serve as a toy model for a general proof of the conjecture, which in this case follows from the simoultaneous inequalities $(aH - sE)\cdot H^2 > 0$ and $\deg(N) > d\sqrt{d}$. The following should then be a step towards a generalization:

PROPOSITION 2.2 (char $(k) = 0$). *Let $C \subset \mathbb{P}^3$ be a smooth and irreducible curve, with semistable normal bundle. Assume that C is the set-theoretic complete intersection of two integral surfaces V and W, of degree $a \geq b$ respectively. Let $\langle V, W\rangle$ be the linear series of all surfaces with equations $\lambda F_V + G\cdot F_W$, where F_V, F_W are the defining equations of V and W, $\lambda \in \mathbb{C}$ and G ranges over all polynomials of degree $a - b$. Let s be the multiplicity along C of a general $S \in \langle V, W\rangle$. Then $\varepsilon(C) \geq s/a$.*

PROOF. Let $G = V \cap W$ be the scheme-theoretic complete intersection of V and W, and denote by P_C and P_G the blow-ups of $\mathbb{P}^3$ along C and G, respectively, and by E_C and E_G the corresponding exceptional divisors. The inclusion of ideals $\mathfrak{I}_G \subset \mathfrak{I}_C$ induces a rational map $\phi\colon P_C \to P_G$. Since P_C is smooth, the singular locus Σ of ϕ has dimension ≤ 1. ϕ induces the identifications $P_C\backslash E_C \cong P_G\backslash E_G$, and then $\Sigma \subset E_C$. For a general $S \in \langle V, W\rangle$, it is easy to see that $\widetilde{S}_G \in |aH - E_G|$. On the other hand, by definition, the proper transform of S in P_C is $\widetilde{S}_C \equiv aH - sE_C$.

LEMMA 2.1. *For $S \in \langle V, W\rangle$ general, $\widetilde{S}_C \equiv \phi^* \widetilde{S}_G$.*

PROOF. There is an isomorphism $\langle V, W\rangle \cong |aH - E_G|$, and on the other hand the latter linear series is base point free. Hence, $\overline{\phi(E_C\backslash\Sigma)} \not\subset \widetilde{S}_G$ for a general S. ∎

Since the normal bundle N is semistable, the smooth surface $E_C \cong \mathbb{P}N$ does not contain any curve of negative self-intersection. Hence, the proof of the Proposition is reduced to the following lemma:

LEMMA 2.2. *Let X, Y be projective threefolds, with X smooth, and $\phi\colon X \dashrightarrow Y$ a rational map. Let $\Sigma \subset X$ be the singular locus of ϕ, and suppose that there exists a smooth surface $S \subset X$, containing Σ and such that for each 1-dimensional component Σ_i of Σ one has $\Sigma_i \cdot_S \Sigma_i \geq 0$. Let L be a globally generated line bundle on Y. Then $\phi^* L$ is nef.*

In fact, the lemma implies that $aH - sE_C$ is nef, and therefore that $\varepsilon(C) \geq$ $\geq s/a$.

PROOF OF LEMMA 2.2. Let $D \subset X$ be an irreducible curve. If $D \not\subset \Sigma$, then there is $Z \in$ $\in |L|$ such that $Z \not\supset \phi(D\backslash\Sigma)$. Then $\phi^* Z \not\supset D$, and thus $\phi^* L \cdot D \geq 0$. Now suppose that D is an irreducible component of Σ. Let $\widetilde{\phi}$ be the maximal extension of $\phi|_S$. Since the singular locus of $\tilde{\phi}$ is at most 0-dimensional, $\tilde{\phi}^* L$ is nef. In fact, there exists $Z \in |L|$ such that $\tilde{\phi}^* Z \not\supset \Sigma_i$, for all i. Hence $\tilde{\phi}^*(Z) \cdot D \geq 0$. Now, $\tilde{\phi} = \phi|_S$ away from Σ, and so $\phi^* Z \cap S = \tilde{\phi}^* Z + T$, where T is an effective curve in S supported on Σ. The statement of the lemma then follows by the hypothesis. ∎

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