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Extension of distributions and representation by derivatives of continuous functions.

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Analisi funzionale. — *Extension of distributions and representation by derivatives of continuous functions.* Nota di Jérôme Lemoine e Jacques Simon, presentata (*) dal Socio E. Magenes.

Abstract. — It is proved that any Banach valued distribution on a bounded set can be extended to all of $\boldsymbol{R}^d$ if and only if it is a derivative of a uniformly continuous function. A similar result is given for distributions on an unbounded set. An example shows that this does not extend to Frechet valued distributions. This relies on the fact that a Banach valued distribution is locally a derivative of a uniformly continuous function. For sake of completeness, a global representation of a Banach valued distribution by derivatives of functions with compact supports is given.

Key words: Distributions; Extension; Vector-values; Representation.

Riassunto. — *Estensione di distribuzione e rappresentazione per mezzo di derivate continue*. Si dimostra che ogni distribuzione a valori in uno spazio di Banach, in un aperto limitato di $\boldsymbol{R}^d$ può essere estesa a tutto $\boldsymbol{R}^d$ se e solo se è una derivata di una funzione uniformemente continua. Un risultato simile è dato anche per distribuzioni in un insieme non limitato. Un esempio dimostra che questa proprietà non si può estendere alle distribuzioni a valori in spazi di Fréchet. La dimostrazione dipende dal fatto che una distribuzione a valori in un spazio di Banach è localmente una derivata di una funzione uniformemente continua. Per completezza è data anche una rappresentazione globale di una distribuzione a valori di uno spazio di Banach per mezzo di derivate di una funzione con supporto compatto.

1. Introduction

In this work we prove that any Banach valued distribution on a bounded set can be extended to all of $\boldsymbol{R}^d$ if and only if it is a derivative of a uniformly continuous function (Theorem 3). That is $f \in \mathcal{D}'(\Omega; E)$, E Banach space, has an extension in $\mathcal{D}'(\boldsymbol{R}^d; E)$ if and only if it can be written as $f = \partial^\beta g$.

More generally, a distribution in any open set has an extension if and only if, on all bounded subset, it is a derivative of an uniformly continuous function (Generalization 4).

This relies on the fact that any Banach valued distribution f is locally a derivative of a uniformly continuous function. That is, in each compact subset of Ω, $f = \partial^\beta g$ (Theorem 1). The function g is given explicitely, using an elementary solution λ of $(\partial_1 \partial_2 \ldots \partial_d)^n \lambda = \delta_0$, and it depends «continuously» on f.

An example shows that these results do not extend to Frechet valued distributions, and *a fortiori* to locally convex spaces (Theorem 5).

For sake of completeness, in a second part (Sections 6 and 7) we give a global representation of any Banach valued distribution by an infinite but locally finite sum of derivatives of uniformly continuous functions with compact supports. That is $f = \sum_{i \in N} \partial^{\beta_i} g_i$ with g_i which cancel in any $\omega \subset\subset \Omega$ from a finite order i_ω (Theorem 11).

(*) Nella seduta del 18 novembre 1995.

This relies on a global representation of any distribution with compact support by a finite sum (Theorem 7), using a parametrix.

For real distributions on all of $\boldsymbol{R}^d$, this global representation was yet obtained by L. Schwartz [3]. On other hand he gave in [2] a representation by a finite sum which holds for more general spaces than Banach ones (quasi-complete dual of Frechet spaces), but which is only local.

2. Review

Let E be a Banach space and Ω an open subset of $\boldsymbol{R}^d$. We denote by $\mathcal{D}'(\Omega; E)$ the set of all continuous linear maps from $\mathcal{D}(\Omega)$ into E. The space $\mathcal{D}(\Omega)$ is endowed with the inductive limit topology of $\mathcal{D}_K(\Omega)$ (space of functions of $\mathcal{D}(\Omega)$ with support in K) for all compact K included in Ω. Each $\mathcal{D}_K(\Omega)$ is a Frechet space for the following semi-norms (which are increasing with m)

$$|||\varphi|||_m = \sup_{0 \leqslant |\beta| \leqslant m, x \in \Omega} |\partial^\beta \varphi(x)| .$$

A subset of $\mathcal{D}(\Omega)$ is open if its intersection with any $\mathcal{D}_K(\Omega)$ is open.

A distribution is therefore a linear map of $\mathcal{D}(\Omega)$ into E which is continuous on each $\mathcal{D}_K(\Omega)$, that is such that, for each compact $K \subset \Omega$, there exists $m \in \mathbf{N}$ and $b \in \boldsymbol{R}$ such that: $\forall \varphi \in \mathcal{D}_K(\Omega)$,

$$\|\langle f, \varphi \rangle\|_E \leqslant b |||\varphi|||_m . \tag{1}$$

Remark. The topology of $\mathcal{D}(\Omega)$ is generated, cf. [4], by the (filtering) family of following seminorms: for each function $q \in \mathcal{C}(\Omega)$, a seminorm is defined by

$$\|\varphi\|_{\mathcal{D}(\Omega); q} = \sup_{x \in \Omega} \sup_{0 \leqslant |\beta| \leqslant |q(x)|} |q(x) \, \partial^\beta \varphi(x)| .$$

A linear map f from $\mathcal{D}(\Omega)$ into E is therefore continuous if there exists $q \in \mathcal{C}(\Omega)$ and $c \in \boldsymbol{R}$ such that: $\forall \varphi \in \mathcal{D}(\Omega)$,

$$\|\langle f, \varphi \rangle\|_E \leqslant c \|\varphi\|_{\mathcal{D}(\Omega); q} . \tag{2}$$

This property characterizes distributions. Indeed, (2) implies (1) since, if support $\varphi \subset K$, $\|\varphi\|_{\mathcal{D}(\Omega); q} \leqslant m |||\varphi|||_m$ for $m = \sup_{x \in K} |q(x)|$; conversely (1) implies (2), cf. [4]. ■

3. Local representation

A distribution with values in a Banach space is locally the derivative of a continuous function, according to the following result.

Theorem 1. *Let $f \in \mathcal{D}'(\Omega; E)$. For all $\omega \subset\subset \Omega$, there exists $g \in \mathcal{C}_u(\Omega; E)$ and $\beta \in \mathbf{N}^d$ such that*

$$f = \partial^\beta g \quad \text{in } \omega . \quad ■$$

We denote by $\mathcal{C}_u(\Omega; E)$ the space of uniformly continuous functions. For real valued distributions on $\boldsymbol{R}^d$, this property has been established by L. Schwartz [3, Theorem XXI, p. 82], by using Hann-Banach theorem.

PROOF. Let K be a compact set such that $\omega \subset\subset K \subset\subset \Omega$, and m satisfying (1). Let $\lambda \in \mathcal{C}^m(\boldsymbol{R}^d)$ be defined by

$$\lambda(x) = ((m+1)!)^{-d}(x_1 x_2 \ldots x_d)^{m+1} \text{ if all of } x_i \text{ are } \geqslant 0, \qquad \lambda(x) = 0 \text{ else}.$$

It satisfies $(\partial_1 \partial_2 \ldots \partial_d)^{m+2}\lambda = \delta_0$. Indeed $(\partial_1 \partial_2 \ldots \partial_d)^m \lambda = \mu$ in classical sense where $\mu(x) = x_1 x_2 \ldots x_d$ if all of x_i are $\geqslant 0$, $\mu(x) = 0$ else, and $(\partial_1 \partial_2 \ldots \partial_d)^2 \mu = \delta_0$ in distribution sense.

Let α and ζ be two «localization» functions such that, denoting $r = \sup_{x \in K} |x|$,

$$\alpha \in \mathcal{D}(\boldsymbol{R}^d), \qquad \alpha(x) = 1 \text{ if } x \in \omega, \qquad \alpha(x) = 0 \text{ if } x \notin K,$$

$$\zeta \in \mathcal{D}(\boldsymbol{R}^d), \qquad \zeta(x) = 1 \text{ if } |x| \leqslant 2r, \qquad \zeta(x) = 0 \text{ if } |x| \geqslant 2r+1.$$

We define $g \in \mathcal{D}'(\boldsymbol{R}^d; E)$ by $g = \alpha(\widetilde{\alpha f} * (\zeta\lambda))$, where $\tilde{}$ is the extension by 0 to $\boldsymbol{R}^d$. This definition is lawful since ζ, and thus $\zeta\lambda$, have a compact support, which allows to define the convolution. The announced properties – and some others – are satisfied according to the following lemma. ■

LEMMA 2. *One has* $g \in \mathcal{C}_u(\boldsymbol{R}^d; E)$, support $g \subset K$, $(\partial_1 \ldots \partial_d)^{m+2} g = f$ *in* ω *and there exists* $c_{m,\omega,K}$ *such that*

$$\sup_{x \in \boldsymbol{R}^d} \|g(x)\|_E \leqslant b c_{m,\omega,K}. \qquad ■$$

This result shows that, in addition to Theorem 1, one can choose g with compact support depending «continuously» on f (through b) and $|\beta| = (m+2)d$ where m is the order of f in ω, defined by (1).

PROOF. *Support.* According to the properties of the support of a product, support $g \subset$ support $\alpha \subset K$.

An identity. In ω, $g = \widetilde{\alpha f} * (\zeta\lambda)$ thus

$$(\partial_1 \ldots \partial_d)^{m+2} g = \widetilde{\alpha f} * (\partial_1 \ldots \partial_d)^{m+2}(\zeta\lambda) \quad \text{in } \omega. \tag{3}$$

In $\{x : |x| < 2r\}$ one has $\zeta = 1$ thus, $(\partial_1 \ldots \partial_d)^{m+2}(\zeta\lambda) = (\partial_1 \ldots \partial_d)^{m+2}\lambda = \delta_0$. It follows that, in all of Ω, $(\partial_1 \ldots \partial_d)^{m+2}(\zeta\lambda) = \delta_0 + \theta$ where support $\theta \subset \{x : |x| \geqslant 2r\}$. Hence (3) yields $(\partial_1 \ldots \partial_d)^{m+2} g = \widetilde{\alpha f} * \delta_0 + \widetilde{\alpha f} * \theta$ in ω.

On the other hand, support $(\widetilde{\alpha f} * \theta) \subset$ support $\widetilde{\alpha f}$ + support $\theta \subset \{x : |x| \leqslant r\}$ + + $\{x : |x| \geqslant 2r\} \subset \{x : |x| \geqslant r\}$. The last set has an empty intersection with ω therefore, in the above equation, it remains

$$(\partial_1 \ldots \partial_d)^{m+2} g = \widetilde{\alpha f} * \delta_0 = \widetilde{\alpha f} = f \quad \text{in } \omega.$$

Properties of regularized functions g_n. In order to get the continuity and to bound the norm of g by passing to the limit, uniform properties on regularized functions g_n are proved now. Given a mollifier ϱ_n, that is $\varrho_n \in \mathcal{D}(\boldsymbol{R}^d)$ such that $\varrho_n(x) = 0$ if $|x| \geqslant 1/n$ and $\int \varrho_n = 1$, a function $\lambda_n \in \mathcal{C}^\infty(\boldsymbol{R}^d)$ is defined by $\lambda_n = \lambda * \varrho_n$ and a function $g_n \in \mathcal{C}_u^\infty(\Omega; E)$ is defined by

$$g_n = \alpha(\widetilde{\alpha f} * (\zeta\lambda_n)) . \tag{4}$$

One has $\widetilde{\alpha f} * (\zeta\lambda_n) \in \mathcal{C}^\infty(\boldsymbol{R}^d; E)$ and, setting $\check{h}(y) = h(-y)$,

$$(\widetilde{\alpha f} * (\zeta\lambda_n))(x) = \langle \widetilde{\alpha f}, \tau_{-x}(\zeta\lambda_n)^\vee \rangle_{\boldsymbol{R}^d} = \langle f, \alpha\tau_{-x}(\zeta\lambda_n)^\vee \rangle_\Omega .$$

Together with (1) this yields: $\forall x \in \Omega$,

$$\|(\widetilde{\alpha f} * (\zeta\lambda_n))(x)\|_E \leqslant b |||\alpha\tau_{-x}(\zeta\lambda_n)^\vee|||_m . \tag{5}$$

The Leibnitz's formula gives, since ζ and thus $\zeta\lambda_n$ have their supports in $\Lambda =$ $= \{z : |z| < 2r + 1\}$,

$$|||\alpha\tau_{-x}(\zeta\lambda_n)^\vee|||_{m,\Omega} \leqslant c_m |||\alpha|||_{m,\Omega} |||(\zeta\lambda_n)^\vee|||_{m,\Omega - x} =$$
$$= c_m |||\alpha|||_{m,\Omega} |||\zeta\lambda_n|||_{m,\Lambda} \leqslant (c_m)^2 |||\alpha|||_{m,\Omega} |||\zeta|||_{m,\Lambda} |||\lambda_n|||_{m,\Lambda} .$$

Multiplying (5) by $|\alpha(x)|$ and taking the upper bounds, we therefore obtain

$$\sup_{x \in \Omega} \|(g_n)(x)\|_E \leqslant b c_{m,\alpha,\zeta} |||\lambda_n|||_{m,\Lambda} . \tag{6}$$

Passing to the limit. Let us prove that $(g_n)_{n \in N}$ is a Cauchy sequence. Replacing in (4) λ_n by $\lambda_n - \lambda_{n'}$, we obtain $g_n - g_{n'}$ instead of g_n. The above calculus is still lawful after this substitution, therefore (6) yields

$$\sup_{x \in \Omega} \|(g_n - g_{n'})(x)\|_E \leqslant b c_{m,\alpha,\zeta} |||\lambda_n - \lambda_{n'}|||_{m,\Lambda} . \tag{7}$$

Since $\lambda \in \mathcal{C}^m(\boldsymbol{R}^d)$ and Λ is bounded, one has, when $n \to \infty$, $|||\lambda_n - \lambda|||_{m,\Lambda} \to 0$. Thus the inequality (7) implies that the g_n form a Cauchy sequence in $\mathcal{C}_u(\Omega; E)$. It converges to g in $\mathcal{D}'(\Omega; E)$ and therefore in $\mathcal{C}_u(\Omega; E)$. Passing to the limit in (6) we finally obtain

$$\sup_{x \in \Omega} \|g(x)\|_E \leqslant b c_{m,\alpha,\zeta} |||\lambda|||_{m,\Lambda} = b c_{m,\omega,K}$$

which completes the proofs of Lemma 2, and therefore of Theorem 1. ■

4. Existence of an extension

A distribution in a bounded open set has an extension if it is the derivative of a uniformly continuous function, according to the following result.

Theorem 3. *Suppose that Ω is bounded. A distribution $f \in \mathcal{D}'(\Omega; E)$ has an extension in $\mathcal{D}'(\boldsymbol{R}^d; E)$ if and only if: there exists $g \in \mathcal{C}_u(\Omega; E)$ and $\beta \in \boldsymbol{N}^d$ such that*

$$f = \partial^\beta g \quad \text{in } \Omega . \quad ■$$

In an arbitrary open set, such a representation in each bounded part is sufficient. This is the goal of the following result.

GENERALIZATION 4. *A distribution* $f \in \mathcal{D}'(\Omega; E)$ *has an extension in* $\mathcal{D}'(\boldsymbol{R}^d; E)$ *if and only if: for any bounded open set* $\omega \subset \Omega$, *there exists* $g \in \mathcal{C}_u(\omega; E)$ *and* $\beta \in \mathbf{N}^d$ *such that*

$$f = \partial^\beta g \quad in \ \omega . \qquad \blacksquare$$

PROOF. The condition is necessary since, if there exists an extension, the Theorem 1 shows that it is of the form $\tilde{f} = \partial^\beta g$ in ω, with $g \in \mathcal{C}(\boldsymbol{R}^d; E)$, and therefore $f = \partial^\beta g|_\omega$ with $g|_\omega \in \mathcal{C}_u(\omega; E)$.

Let us prove that the condition is sufficient. If Ω is bounded, we can choose $\omega = \Omega$, and the corresponding function g has an extension $\tilde{g} \in \mathcal{C}(\boldsymbol{R}^d; E)$ (cf. for example [1, Theorem 5.2, p. 302]). Then $\partial^\beta \tilde{g}$ is an extension of f.

If Ω is not bounded, we consider a partition of unity on $\boldsymbol{R}^d$ by functions α_n such that $\alpha_n \in \mathcal{D}(\boldsymbol{R}^d)$, support $\alpha_n \subset \{x : n-1 < |x| < n+1\}$, $\sum_{n \geq 0} \alpha_n = 1$. In $\omega_n = \{x \in \Omega : |x| < n\}$ one can, by assumption, represent $f = \partial^{\beta_n} g_n$ with $g_n \in \mathcal{C}_u(\omega_n; E)$, and we choose an extension $\tilde{g}_n \in \mathcal{C}(\boldsymbol{R}^d; E)$.

One defines a distribution $\tilde{f} \in \mathcal{D}'(\boldsymbol{R}^d; E)$ by $\tilde{f} = \sum_n \alpha_n \partial^{\beta_n} \tilde{g}_n$ because this series is converging in $\mathcal{D}'(\Omega; E)$ since, in each compact set of $\boldsymbol{R}^d$, only a finite number of $\alpha_n \partial^{\beta_n} \tilde{g}_n$ are not zero. This distribution is an extension of g because the equality $\alpha_n \partial^{\beta_n} \tilde{g}_n = \alpha_n f$ holds in ω_n and in $\Omega \backslash$support α_n, and therefore in their collection Ω, so that summing with respect to n, $\tilde{f} = f$ in Ω. $\blacksquare$

5. NON-REPRESENTATION OF DISTRIBUTIONS WITH VALUES IN A FRECHET SPACE

Suppose now that E is a Frechet space. That is a space endowed with a sequence of seminorms $\| \, \|_{E, \nu}$ which turn E into a Hausdorff complete space. The space $\mathcal{D}'(\Omega; E)$ is again the set of linear continuous maps from $\mathcal{D}(\Omega)$ into E.

The above results do not extend to the distributions with values in a Frechet space according to the following result.

THEOREM 5. *One can choose open sets* $\omega \subset\subset \Omega \subset\subset \boldsymbol{R}^d$, *a Frechet space E and a distribution* $f \in \mathcal{D}'(\Omega; E)$ *with compact support which, in* ω, *is not the derivative of a continuous function. That is*

$$\forall \beta \in \mathbf{N}^d , \qquad \forall g \in \mathcal{C}(\Omega; E), \qquad f|_\omega \neq (\partial^\beta g)|_\omega .$$

More generally, f is not the sum of derivatives of continuous functions. That is,

$$\forall I \text{ finite}, \quad \forall \beta_i \in \mathbf{N}^d , \quad \forall g_i \in \mathcal{C}(\Omega; E), \quad f|_\omega \neq \sum_{i \in I} (\partial^{\beta_i} g_i)|_\omega . \qquad \blacksquare$$

REMARK. The results of preceding sections may be extended to the distributions which have the following additional property: $\forall K \subset\subset \Omega$, there exists $m \in \mathbf{N}$ and $b \in \boldsymbol{R}$ such that, $\forall \nu \in \boldsymbol{N}$, $\forall \varphi \in \mathcal{D}(\Omega)$ such that support $\varphi \subset K$, $\|\langle f, \varphi \rangle\|_{E; \nu} \leq b \||\varphi|\|_m$ (in general m and b depend on ν, that is on the semi-norm of E).

This property is equivalent, cf. [4], to the following: there exists $q \in \mathcal{C}(\Omega)$ and $c \in \boldsymbol{R}$ such that, $\forall \varphi \in \mathcal{D}(\Omega)$, $\forall \nu \in \boldsymbol{N}$, $\|\langle f, \varphi \rangle\|_{E; \nu} \leq c \|\varphi\|_{\mathcal{D}(\Omega); q}$. $\blacksquare$

In order to prove Theorem 5, we will use the following seminorms on $\mathcal{D}'(\Omega; E)$ defined, for all $\varphi \in \mathcal{D}(\Omega)$ and $\nu \in \mathbf{N}$, by:

$$\|f\|_{\mathcal{D}'(\Omega; E); \varphi, \nu} = \|\langle f, \varphi \rangle\|_{E; \nu} \tag{8}$$

and we will use the following property proved in [1].

LEMMA 6. *The space $\mathcal{D}'(\Omega; E)$ endowed with those seminorms is Hausdorff and sequentially complete.* ■

PROOF OF THEOREM 5. Choose $\Omega = \mathbf{R}$, $\omega =]-1, 1[$ and $E = \mathcal{C}(\mathbf{R})$ endowed with the following seminorms, indexed by $\nu \in \mathbf{N}$, $\|h\|_{\mathcal{C}(\mathbf{R}); \nu} = \sup_{|z| < \nu} |h(z)|$.

Definition of f. Let $h \in \mathcal{C}(\mathbf{R})$ satisfy support $h = [0, 1/2]$, $\sup_{z \in \mathbf{R}} |h(z)| = 1$, $h(1/4) = 1$. Define $f \in \mathcal{D}'(\mathbf{R}; \mathcal{C}(\mathbf{R}))$ by: $\forall \varphi \in \mathcal{D}(\mathbf{R})$, $\forall z \in \mathbf{R}$,

$$\langle f, \varphi \rangle(z) = \sum_{n \geqslant 1} \partial^n \varphi(0) h(z - n).$$

This definition is lawful since the series defining f is converging in $\mathcal{D}'(\mathbf{R}; \mathcal{C}(\mathbf{R}))$. To check it, define $f_N \in \mathcal{D}'(\mathbf{R}; \mathcal{C}(\mathbf{R}))$ by $\langle f_N, \varphi \rangle(z) = \sum_{1 \leqslant n \leqslant N} \partial^n \varphi(0) h(z - n)$, and fix ν. For $N' \geqslant N \geqslant \nu$,

$$\|\langle f_{N'} - f_N, \varphi \rangle\|_{\mathcal{C}(\mathbf{R}); \nu} = \sup_{|z| < \nu} \left| \sum_{N+1 \leqslant n \leqslant N'} \partial^n \varphi(0) h(z - n) \right| = 0$$

since $n \geqslant |z| + 1$ implies $h(z - n) = 0$.

Hence $(f_N)_{N \in \mathbf{N}}$ is a Cauchy sequence in $\mathcal{D}'(\mathbf{R}; \mathcal{C}(\mathbf{R}))$ for the seminorms defined by (8). By Lemma 6, this sequence has a limit which is, by definition, the distribution f.

Support of f. Since support $f_N = \{0\}$, passing to the limit gives support $f = \{0\}$.

Non-representability by only one derivative. Suppose that there exist $g \in \mathcal{C}(\mathbf{R}; \mathcal{C}(\mathbf{R}))$ and $\beta \in \mathbf{N}$ such that

$$f = \partial^\beta g \quad \text{in }]-1, 1[. \tag{9}$$

It would satisfy $\langle f, \varphi \rangle(z) = (-1)^\beta \int_{-1}^{1} g(x, z) \partial^\beta \varphi(x)\, dx$ for all $\varphi \in \mathcal{D}(]-1, 1[)$ and $z \in \mathbf{R}$, thus, $\forall \nu$,

$$\|\langle f, \varphi \rangle\|_{\mathcal{C}(\mathbf{R}); \nu} \leqslant c_\nu \sup_{|x| < 1} |\partial^\beta \varphi(x)| \tag{10}$$

where $c_\nu = \sup_{|x| < 1,\, |z| < \nu} |g(x, z)|$.

Let us go back to the definition of f. It gives $\langle f, \varphi \rangle(\beta + 1 + 1/4) = \partial^{\beta + 1} \varphi(0)$ (since $h(\beta + 1 + 1/4 - n) = 0$ if $n \neq \beta + 1$ and $h(1/4) = 1$) thus $\|\langle f, \varphi \rangle\|_{\mathcal{C}(\mathbf{R}); \beta + 2} \geqslant |\partial^{\beta + 1} \varphi(0)|$. Together with (10), this would give, for all $\varphi \in \mathcal{D}(]-1, 1[)$,

$$|\partial^{\beta + 1} \varphi(0)| \leqslant c_{\beta + 2} \sup_{|x| < 1} |\partial^\beta \varphi(x)|.$$

This, and therefore (9), are not true, since for any given c, one can choose φ such that $\sup_{|x| < 1} |\partial^\beta \varphi(x)| \leqslant 1$ and $|\partial^{\beta + 1} \varphi(0)| > c$.

Non-representability by a sum of derivatives. It cannot exist a finite set $I \subset \boldsymbol{N}$, $g_i \in \mathcal{C}(\boldsymbol{R}, \mathcal{C}(\boldsymbol{R}))$ and $\beta_i \in \boldsymbol{N}$ such that $f = \sum_{i \in I} \partial^{\beta_i} g_i$ in $]-1, 1[$. Indeed, otherwise, (9) should be satisfied with $\beta = \max_i \{\beta_i\}$ and $g = \sum_i G_i$, where G_i is any anti-derivative of order $\beta - \beta_i$ of g_i (that is $\partial^{\beta - \beta_i} G_i = g_i$). ■

6. Representation of a distribution with compact support by derivatives of continuous functions with compact support

For sake of completeness, we will give in next section a global representation of any Banach valued distribution by an infinite sum of derivatives of continuous functions.

In order to prove it, we begin by the following representation of distributions with compact support, which uses the parametrix method of L. Schwartz.

Theorem 7. *Let $f \in \mathcal{D}'(\boldsymbol{R}^d; E)$ with compact support. There exists a finite number I of functions $g_i \in \mathcal{C}_u(\boldsymbol{R}^d; E)$ and $\beta_i \in \boldsymbol{N}^d$ such that*

$$f = \sum_{i \leqslant I} \partial^{\beta_i} g_i \quad \text{in } \boldsymbol{R}^d. \tag{11}$$

The g_i can be choosen depending (linearly) on f, and with support in an arbitrary neighbourhood of the support of f. ■

For the real valued distributions, these results have been established by L. Schwartz [3, Theorem XXVI, p. 91].

Remark. Theorem 7 extends to the case where E is a quasi-complete dual of Frechet space. Indeed L. Schwartz proved in [2] that a distribution with compact support is of finite order [2, Corollary 2, p. 85], and that a distribution of finite order satisfies (11) [2, Corollary 2, p. 90]. ■

We will use a parametrix of order $m + d + 1$, where m is the order of f. More precisely, we represent Dirac mass as following.

Lemma 8. *For all $r > 0$ and $m \in \boldsymbol{N}$, one has*

$$\delta_0 = \eta + \sum_{|\beta| = m+d+1} \partial^\beta \gamma_\beta$$

where $\eta \in \mathcal{D}(\boldsymbol{R}^d)$, $\gamma_\beta \in \mathcal{C}_u^m(\boldsymbol{R}^d)$, and η and γ_β have their support in $B = \{x \in \boldsymbol{R}^d : |x| \leqslant r\}$. ■

Proof. *The case where $m + d + 1$ is even.* Consider the elementary solution of $\delta_0 = \Delta^{(m+d+1)/2} X$ in $\boldsymbol{R}^d$ that is $X(x) = c|x|^{m+1}$ if d is odd $X(x) = c|x|^{m+1} \log|x|$ if d is even. This equation can be written $\delta_0 = \sum_{|\beta| = m+d+1} c_\beta \partial^\beta X$.

Let θ be a localization function such that $\theta \in \mathcal{C}^\infty(\boldsymbol{R}^d)$, $\theta(x) = 0$ if $x \geqslant r$, $\theta(x) = 1$ if $|x| \leqslant r/2$. Multiplying the two members of the equation by θ it comes, $\delta_0 = \theta\delta_0 = \sum_{|\beta| = m+d+1} c_\beta \theta \partial^\beta X$.

According to the dual Leibnitz's formula, $\theta \partial^\beta X = \partial^\beta(\theta X) + \eta_\beta$ with

$$\eta_\beta = \sum_{0 \leq \sigma < \beta} (-1)^{|\beta - \sigma|} \binom{\sigma}{\beta} \partial^\sigma (X \partial^{\beta - \sigma} \theta).$$

This function is $\mathcal{C}^\infty$ (it cancels in a neighbourhood of 0), thus we get the announced decomposition with $\eta = \sum_\beta c_\beta \eta_\beta$ and $\gamma_\beta = c_\beta \theta X$.

The case where $m + d + 1$ is odd. The above decomposition relative to the even number $(m+1) + d + 1$ yields, denoting $Y = c|x|^{m+2}$ or $c|x|^{m+2} \log |x|$ whether the parity of d, and $\beta + e_i = (\beta_1, \ldots, \beta_i + 1, \ldots, \beta_d)$,

$$\delta_0 = \eta_{m+1} + \sum_{|\beta| = m + d + 1} \sum_{1 \leq i \leq d} \partial^\beta \partial_i (c_{\beta + e_i} \theta Y).$$

Hence the announced decomposition holds with

$$\eta = \eta_{m+1} + \sum_{|\beta| = m + d + 1} \sum_{1 \leq i \leq d} c_{\beta + e_i} Y \partial_i \theta, \qquad \gamma_\beta = \theta \sum_{1 \leq i \leq d} c_{\beta + e_i} \partial_i Y. \qquad \blacksquare$$

Proof of Theorem 7. Let us first prove that f has a finite order in all of $\boldsymbol{R}^d$. Let K be a compact set such that support $f \subset\subset K$. According to (1), there exists m such that, $\forall \varphi \in \mathcal{D}(\boldsymbol{R}^d)$, support $\varphi \subset K$,

$$\|\langle f, \varphi \rangle\|_E \leq b_K \, |||\varphi|||_m .$$

Let $\alpha \in \mathcal{D}(\boldsymbol{R}^d)$ be such that $\alpha = 1$ in a neighbourhood of support f, support $\alpha \subset K$. Assume now that $\varphi \in \mathcal{D}(\boldsymbol{R}^d)$. Then $\langle f, \varphi \rangle = \langle f, \alpha\varphi \rangle$ thus

$$\|\langle f, \varphi \rangle\|_E \leq b_K \, |||\varphi|||_m \leq b_K c_m \, |||\alpha|||_m \, |||\varphi|||_m \leq b \, |||\varphi|||_m$$

which proves that the order of f is m in all of $\boldsymbol{R}^d$.

Decomposing

$$f = f * \delta_0 = f * \eta + \sum_{|\beta| = m + d + 1} f * \partial^\beta \gamma_\beta = f * \eta + \sum_{|\beta| = m + d + 1} \partial^\beta (f * \gamma_\beta)$$

the announced properties in the Theorem 7 – and some others – are given by the following lemma. $\blacksquare$

Lemma 9. *One has $f * \eta \in \mathcal{C}_u^\infty(\boldsymbol{R}^d; E)$, $f * \gamma_\beta \in \mathcal{C}_u(\boldsymbol{R}^d; E)$, $f * \eta$ and $f * \gamma_\beta$ have their support in* support $f + B$, *and*

$$\sup_{x \in \boldsymbol{R}^d} \|(f * \eta)(x)\|_E \leq b c_{m,d,r}, \qquad \sup_{x \in \boldsymbol{R}^d} \|(f * \gamma_\beta)(x)\|_E \leq b c_{m,d,r}. \qquad \blacksquare$$

Proof. The imbeddings on supports result from the fact that support $f * g \subset$ $\subset$ support $f +$ support g.

The regularity of $f * \eta$ is classical, since η is regular. In order to bound its norm, we observe that $(f * \eta)(x) = \langle f, \tau_{-x} \eta \rangle$ hence $\|(f * \eta)(x)\|_E \leq b \, |||\eta|||_m$.

To establish the continuity and to bound the norm, we are going to regularize γ_β, which will allow some similar calculus to those done on $f * \eta$, then we will pass to the limit. Given a mollifier ϱ_n, a function $\gamma_\beta^n \in C_u^\infty(\boldsymbol{R}^d)$ is defined by $\gamma_\beta^n = \gamma_\beta * \varrho_n$.

One has $f * \gamma_\beta^n \in \mathcal{C}_u^\infty(\boldsymbol{R}^d; E)$ and, $\forall x$, $(f * \gamma_\beta^n)(x) = \langle f, \tau_{-x} \check{\gamma}_\beta^n \rangle$ thus

$$\|(f * \gamma_\beta^n)(x)\|_E \leq b \||\tau_{-x} \check{\gamma}_\beta^n \||_m = b \||\gamma_\beta^n\||_m$$

hence

$$\sup_{x \in \boldsymbol{R}^d} \|(f * \gamma_\beta^n)(x)\|_E \leq b \||\gamma_\beta^n\||_m . \tag{12}$$

Let us prove that the $f * \gamma_\beta^n$ form a Cauchy sequence. Replacing in these calculus γ_β^n by $\gamma_\beta^n - \gamma_\beta^{n'}$ we bound likewise,

$$\sup_{x \in \boldsymbol{R}^d} \|(f * \gamma_\beta^n - f * \gamma_\beta^{n'})(x)\|_E \leq b \||\gamma_\beta^n - \gamma_\beta^{n'}\||_m .$$

Since $\gamma_\beta \in \mathcal{C}^m(\boldsymbol{R}^d)$ and has a compact support, $\||\gamma_\beta^n - \gamma_\beta\||_m \to 0$ thus this inequality implies that $(f * \gamma_\beta^n)_{n \in N}$ is a Cauchy sequence in $\mathcal{C}_u(\boldsymbol{R}^d; E)$. It converges to $f * \gamma_\beta$ in $\mathcal{D}'(\boldsymbol{R}^d; E)$ and therefore in $\mathcal{C}_u(\boldsymbol{R}^d; E)$. Passing to the limit in (12), we obtain

$$\sup_{x \in \boldsymbol{R}^d} \|(f * \gamma_\beta)(x)\|_E \leq b \||\gamma_\beta\||_m$$

which completes the proof of Lemma 9, and therefore the proof of Theorem 7. ■

Remark. A distribution with compact support can be represented by only one derivative, that is $f = \partial^\beta g$ in all $\boldsymbol{R}^d$, but g has not necessary a compact support.

Such a representation is obtained by using the local representation of Theorem 1 for the extension $\tilde{f}$ by 0, and for the compact $K = \overline{\Omega}$. One checks that g cannot always been choosen with compact support by considering $\Omega = \boldsymbol{R}$ and $f = \delta_0$; none of its primitives has a compact support. ■

Corollary 10. *Let Ω be an open subset of $\boldsymbol{R}^d$ and $f \in \mathcal{D}'(\Omega; E)$, such that the support of f is compact and enclosed in Ω.*

There exists a finite number I of $g_i \in \mathcal{C}_u(\Omega; E)$ and $\beta_i \in \mathbf{N}^d$ such that

$$f = \sum_{1 \leq i \leq I} \partial^{\beta_i} g_i .$$

For all $K \supset\supset$ support f, *one can choose g_i such that* support $g_i \subset K$. ■

Proof. It suffices to apply Theorem 7 to the extension by 0 of f to all of $\boldsymbol{R}^d$. ■

7. Global representation of a distribution

A distribution (again with values in a Banach space) can be globally represented by an infinite converging sum, which is locally finite, of derivatives of uniformly continuous functions, in the following way.

Theorem 11. *Let $f \in \mathcal{D}'(\Omega; E)$. There exists $g_i \in \mathcal{C}_u(\Omega; E)$ and $\beta_i \in \mathbf{N}^d$ such that*

$$f = \sum_{i \in N} \partial^{\beta_i} g_i \quad \textit{in } \mathcal{D}'(\Omega; E),$$

support g_i *is compact and included in* Ω *and, for each* $\omega \subset\subset \Omega$, *all the* g_i *cancel in* ω *from a finite order* i_ω. *Therefore*

$$f = \sum_{i \leqslant i_\omega} \partial^{\beta_i} g_i \quad \text{in } \omega . \quad \blacksquare$$

For real valued distributions, this property has been established by L. Schwartz [3, Theorem XXX, p. 96].

PROOF. We decompose Ω in a collection of open sets ω_n which converge to the boundary in the following way: $\forall n \in \mathbf{N}$,

$$\Omega_n = \{x : |x| < n, \overline{\text{ball}}(x, 1/n) \subset \Omega\} (\Omega_0 = \emptyset), \qquad \omega_n = \Omega_{n+2} \setminus \overline{\Omega}_n .$$

To the covering of Ω by the ω_n we associate a partition of unity $\{\alpha_n\}_{n \in \mathbf{N}}$. It satisfies $\alpha_n \in \mathcal{D}(\omega_n)$, $\sum_{n \geqslant 0} \alpha_n = 1$ in Ω. We decompose $f = \sum_{n \geqslant 0} f_n$ where $f_n = \alpha_n f$. One has support $f_n \subset\subset \omega_n \subset\subset \Omega$ which allows to decompose f_n with the Corollary 10. This latter gives the existence of a finite number I_n of $g_{i,n} \in \mathcal{C}(\Omega; E)$ and of $\beta_{i,n} \in \mathbf{N}^d$ such that $f_n = \sum_{i \leqslant I_n} \partial^{\beta_{i,n}} g_{i,n}$ and support $g_{i,n} \subset \omega_n$. Finally,

$$f = \sum_{n \geqslant 0} \sum_{i \leqslant I_n} \partial^{\beta_{i,n}} g_{i,n}$$

and renumbering the couples (i, n), we obtain the announced properties. $\blacksquare$

REMARK. The representation formula of Theorem 11 extends to the case where E is a quasi-complete dual of Frechet space.

Indeed Theorem 7 extends to this case, as seen in a former remark, and we can conclude by the above proof. $\blacksquare$

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