Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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On «power-logarithmic» solutions of the Dirichlet problem for elliptic systems in $K_d \times \mathbb{R}^{n-d}$, where K_d is a d-dimensional cone

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **7** (1996), n.1, p. 17–30.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1996_9_7_1_17_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1996.

Rend. Mat. Acc. Lincei
s. 9, v. 7:17-30 (1996)

Analisi matematica. — *On «power-logarithmic» solutions of the Dirichlet problem for elliptic systems in $K_d \times R^{n-d}$, where K_d is a d-dimensional cone.* Nota di VLADIMIR A. KOZLOV e VLADIMIR G. MAZ'YA, presentata (*) dal Socio G. Fichera.

ABSTRACT. — A description of all «power-logarithmic» solutions to the homogeneous Dirichlet problem for strongly elliptic systems in a n-dimensional cone $K = K_d \times R^{n-d}$ is given, where K_d is an arbitrary open cone in R^d and $n > d > 1$.

KEY WORDS: Elliptic systems; Boundary singularities; Asymptotics of solutions.

RIASSUNTO. — *Sulle soluzioni «power-logarithmic» del problema di Dirichlet per sistemi ellittici in $K_d \times R^{n-d}$, dove K_d è un cono d-dimensionale.* Viene data una descrizione di tutte le soluzioni «power-logarithmic» del problema omogeneo di Dirichlet per un sistema fortemente ellittico in un cono n-dimensionale $K = K_d \times R^{n-d}$, dove K_d è un qualsiasi cono aperto in R^d e $n > d > 1$.

INTRODUCTION

«Power-logarithmic» solutions play an important role in the theory of elliptic boundary value problems in domains with piecewise smooth boundaries (see [1-3]). With the help of these special solutions one can describe asymptotic behavior of arbitrary solutions of boundary value problems near singularities of the boundary. In this article we consider the Dirichlet problem for strongly elliptic systems in a n-dimensional cone, which is invariant with respect to shifts along certain directions, *i.e.* in the cone $K = K_d \times R^{n-d}$, where K_d is an arbitrary open cone in R^d and $n > d > 1$. In particular, for $d = 2$, it is the case of a dihedral angle.

We are interested in the solutions of the homogeneous Dirichlet problem which have the form

$$|x|^{\lambda} \sum_{0 \leq k \leq \kappa} \frac{1}{(\kappa - k)!} (\log |x|)^{\kappa - k} u_k(x/|x|),$$

where $x \in K$ and u_k are vector-valued functions with finite Dirichlet integral in a domain Ω which is the intersection of the cone K and the $(n-1)$-dimensional unit sphere.

The main result is a description of such solutions in terms of similar solutions for the cone K_d (Theorems 1 and 2).

As an example we consider the Laplace operator and obtain all positive homogeneous solutions for it (here solutions with logarithmic terms are absent).

(*) Nella seduta del 10 febbraio 1996.

1. Formulation of the problem

We represent the space R^n, $n \geq 3$, as the Cartesian product $R^d \times R^{n-d}$, $1 < d < n$, and use the notation $x = (y, z)$, $y = (y_1, \dots, y_d)$, $z = (z_1, \dots, z_{n-d})$. We introduce the spherical coordinates (r, ω), (ϱ, ϕ) and (σ, θ) in the spaces R^n, R^d and R^{n-d}, where $r = |x|$, $\varrho = |y|$, $\sigma = |z|$ and $\omega \in S^{n-1}$, $\phi \in S^{d-1}$, $\theta \in S^{n-d-1}$.

Consider the open d-dimensional cone $K_d = \{y \in R^d : \varrho > 0, \phi \in \Omega_d\}$, where Ω_d is a domain on the sphere S^{d-1}, $\overline{\Omega}_d \neq S^{d-1}$.

Let K be the n-dimensional cone $\{x \in R^n : r > 0, \omega \in \Omega\}$ which can be represented as the product $K_d \times R^{n-d}$. In this case $\omega \in \Omega$ if and only if

$$\omega = (\phi \cos \tau, \theta \sin \tau), \qquad \text{where} \quad \tau \in (0, \pi/2), \quad \phi \in \Omega_d, \quad \theta \in S^{n-d-1}.$$

Consider the differential operator

$$\mathcal{a}(D_x) = \sum_{|\alpha| = 2m} A_\alpha D_x^\alpha, \tag{1.1}$$

where $D_x = i^{-1}$ grad and A_α are constant $l \times l$ matrices. This operator is assumed to be strongly elliptic which means that for any $\xi \in R^n$, $f \in \mathbb{C}^l$ the following inequality is valid $\operatorname{Re}(\mathcal{a}(\xi) f, f) \geq c_0 |\xi|^{2m} |f|^2$, $c_0 > 0$, where $(\, , \,)$ and $|\cdot|$ are the scalar product and the norm in $\mathbb{C}^l$.

We shall seek the vector-valued function U in the space $(\overset{\circ}{H}{}^m_{\mathrm{loc}}(K, 0))^l = \{U : \eta U \in (\overset{\circ}{H}{}^m(K))^l$ for all $\eta \in C_0^\infty(R^n \setminus \{0\})\}$ satisfying the system

$$\mathcal{a}(D_x) U = 0 \quad \text{on } K. \tag{1.2}$$

Our aim is to describe all the solutions of this Dirichlet problem which have the form

$$U(x) = r^\lambda \sum_{0 \leq k \leq \kappa} \frac{1}{(\kappa - k)!} (\log r)^{\kappa - k} u_k(\omega), \tag{1.3}$$

where u_k are vector-valued functions from $(\overset{\circ}{H}{}^m(\Omega))^l$.

Remark 1.1. It is easy to see that the distribution (1.3) belongs to the class $(\overset{\circ}{H}{}^m_{\mathrm{loc}}(K, 0))^l$ if and only if $u_k \in (\overset{\circ}{H}{}^m(\Omega))^l$.

Let $A(\lambda)$ denote the differential operator on S^{n-1} defined by the equality

$$A(\lambda) u = r^{2m - \lambda} \mathcal{a}(D_x)(r^\lambda u).$$

By $\mathcal{L} = \mathcal{L}(\lambda)$, $\lambda \in \mathbb{C}$, we mean the polynomial operator pencil

$$\mathcal{L}(\lambda) : (\overset{\circ}{H}{}^m(\Omega))^l \to (H^{-m}(\Omega))^l, \tag{1.4}$$

defined by $\mathcal{L}(\lambda) u = A(\lambda) u$.

The following assertion can be checked directly.

Proposition 1.1. *The vector-valued function* (1.3) *satisfies the system* (1.2) *if and only if*

$$\sum_{0 \leq j \leq s} \frac{\mathcal{L}^{(j)}(\lambda)}{j!} u_{s-j} = 0, \qquad s = 0, 1, \dots, \kappa,$$

where $\mathcal{L}^{(j)}(\lambda)$ *is the derivative of order j with respect to* λ. *In other words, the exponent* λ *in* (1.3) *is an eigenvalue of the pencil* $\mathcal{L}$, u_0 *is its eigenvector and* $u_1, \dots, u_\kappa$ *are generalized eigenvectors.*

REMARK 1.2. Proposition 1.1 implies that the dimension of the space of solutions of the form (1.3) coincides with the algebraic multiplicity of the eigenvalue λ. The dimensions of spaces of solutions (1.3) for a fixed κ are uniquely determined by the geometrical and partial multiplicities of λ. For $\kappa = 0$ the corresponding dimension coincides with the geometrical multiplicity of the eigenvalue λ.

The next assertion is generally known and easily verified.

PROPOSITION 1.2. (i) *The spectrum of the operator pencil* $\mathcal{L}$ *consists of eigenvalues with finite algebraic multiplicities having only the limit point at infinity.*

(ii) *The line* $\operatorname{Re}\lambda = m - n/2$ *contains no eigenvalues of the operator pencil.*

Consider the formally adjoint operator of $\mathfrak{A}(D_x)$:

$$\mathfrak{A}^*(D_x) = \sum_{|\alpha| = 2m} A_\alpha^* D_x^\alpha .$$

Let $A^*(\lambda)$ be the differential operator on the unit sphere defined by the equality

$$A^*(\lambda)u = r^{2m-\lambda}\mathfrak{A}^*(D_x)(r^\lambda u).$$

The operator pencil

$$\mathcal{L}^*(\lambda) = (\overset{\circ}{H}{}^m(\Omega))^l \to (H^{-m}(\Omega))^l \tag{1.5}$$

is defined by $\mathcal{L}^*(\lambda)u = A^*(\lambda)u$. Proposition 1.1, Remark 1.2 and Proposition 1.2 are also valid for this operator pencil. Moreover, the operator pencils $\mathcal{L}$ and $\mathcal{L}^*$ are connected by $(\mathcal{L}(\lambda))^* = \mathcal{L}(2m - n - \bar{\lambda})$ (see [2]), which leads to the following assertion.

PROPOSITION 1.3. *The number* λ *is an eigenvalue of the operator pencil* $\mathcal{L}$ *if and only if* $2m - n - \bar{\lambda}$ *is an eigenvalue of the operator pencil* $\mathcal{L}^*$. *The algebraic, geometrical and partial multiplicities of both eigenvalues coincide.*

2. SOLUTION OF (1.3) FOR $\operatorname{Re}\lambda > m - n/2$

PROPOSITION 2.1. *If* U *is a solution of the system* (1.2) *of the form* (1.3) *where* $\lambda \in \mathbb{C}$ *and* $u_k \in (\overset{\circ}{H}{}^m(\Omega))^l$, *then the vector-valued functions* $D_z^\gamma U$ *have the same properties for an arbitrary multi-index* γ.

PROOF. Applying the local energy estimate to the derivative $\partial_{z_j} U_h$, where U_h is a mollification of U in z with radius h, and passing to the limit as $h \to 0$, we obtain $\partial_{z_j} U \in (\overset{\circ}{H}{}^m_{\text{loc}}(K, 0))^l$. It is also clear that $\partial_{z_j} U$ is a solution of (1.2) and has the form (1.3). Using Remark 1.1 we arrive at the desired result for $|\gamma| = 1$. It remains to apply the induction in $|\gamma|$. ■

Below we need the set of solutions of the system

$$(2.1) \qquad \mathcal{A}(D_y, 0) = 0 \quad \text{on } K_d ,$$

which have the form

$$(2.2) \qquad U(y) = \varrho^l \sum_{0 \leq k \leq \kappa} \frac{1}{(\kappa - k)!} (\log \varrho)^{\kappa - k} u_k(\phi),$$

where u_k are vector-valued functions from $(\overset{\circ}{H}{}^m(\Omega_d))^l$.

Similarly to Section 1 we associate the operator pencil

$$\mathcal{L}_d(\lambda) : (\overset{\circ}{H}{}^m(\Omega_d))^l \to (H^{-m}(\Omega_d))^l$$

with the equation (2.1). Propositions 1.1 and 1.2 are valid (with obvious changes) for this operator pencil. In particular, the line $\operatorname{Re}\lambda = m - d/2$ contains no eigenvalues of the operator pencil $\mathcal{L}_d$.

Let $\{\mu_j\}_{j \in Z}$ be a sequence of eigenvalue of $\mathcal{L}_d$ numerated with regard to their algebraic multiplicity, and let the eigenvalues, lying in the half-plane $\operatorname{Re}\mu > m - d/2$, have non-negative indices while the remaining eigenvalues have negative indices. We can assume that each eigenvalue μ_j generates one solution of (2.1), which has the form $v_j(y) = \varrho^{\mu_j} Q_j(\phi, \log \varrho)$, where Q is a polynomial in the second argument with coefficients in $(\overset{\circ}{H}{}^m(\Omega_d))^l$. If $\mu_j = \mu_{j+1} = \ldots = \mu_{j+N-1}$, where N is the algebraic multiplicity, then the polynomials $Q_j, \ldots, Q_{j+N-1}$ are linear independent.

PROPOSITION 2.2. *Let α be a $(n-d)$-dimensional multi-index and let j be a non-negative integer. To each pair (j, α) there corresponds a solution of the system* (1.2), *having the form*

$$(2.3) \qquad V_{j\alpha}(x) = \sum_{\beta \leq \alpha} z^\beta \varrho^{\mu_j + |\alpha - \beta|} Q_{j\beta}(\phi, \log \varrho),$$

where $Q_{j\beta}$ are polynomials in $\log \varrho$ with coefficients in $(\overset{\circ}{H}{}^m(\Omega_d))^l$ and $Q_{j\alpha} = Q_j$.

PROOF. For the sake of brevity let the coefficient of z^β in the right-hand side of (2.3) be denoted by $\Psi_\beta(y)$. The equality $\mathcal{A}(D_x) V_{j\alpha} = 0$ is equivalent to the system of equations

$$(2.4) \quad \mathcal{A}(D_y, 0)\Psi_\gamma = - \sum_{\gamma < \beta \leq \alpha} \frac{(-i)^{|\beta - \gamma|} \beta!}{(\beta - \gamma)!} \mathcal{A}^{(\beta - \gamma)}(D_y, 0) \Psi_\beta(y) \quad \text{on } K_d ,$$

where γ is an arbitrary $(n-d)$-dimensional multi-index satisfying $\gamma \leq \alpha$ and $\mathcal{A}^{(\delta)}(\eta, \xi) = (\partial_\xi^\delta A)(\eta, \xi)$. Suppose that all Ψ_β are constructed for $\beta > \gamma$. Then Ψ_γ can be determined by (2.4) using Proposition 7.1 [3].

REMARK 2.1. If among the numbers $\mu_j + 1, \ldots, \mu_j + |\alpha|$ there are no eigenvalues of the operator pencil $\mathcal{L}_d(\mu)$ then $V_{j\alpha}$ is uniquely defined. Moreover, the degrees of the polynomials $Q_{j\beta}$, $\beta \leq \alpha$, do not exceed the degree of $Q_{j\alpha}$.

REMARK 2.2. Suppose that the collection $\mu_j + 1, \ldots, \mu_j + |\alpha|$ contains s different eigenvalues with the maximal partial multiplicities $\kappa_1, \ldots, \kappa_s$. Then the degree of the polynomial $Q_{j\beta}$ in (2.3) does not exceed $\kappa_0 + \kappa_1 + \ldots + \kappa_s$, where κ_0 is the degree of the polynomial Q_j. Moreover, $V_{j\alpha}$ is unique up to a linear combination of solutions $V_{j'\alpha'}$, $\alpha' < \alpha$ and $\mu_j' + |\alpha'| = \mu_j + |\alpha|$. Thus, to each pair (j, α) there corresponds a solution of the form (2.3). Since the coefficients $\varrho^{\mu_j} Q_j$ of z^α in (2.3) are linear independent, it follows that the same is true for $V_{j\alpha}$.

PROPOSITION 2.3. *Let the vector-valued function*

$$U(x) = \sum_{|\alpha| \leq N} z^\alpha \varrho^{\lambda - |\alpha|} Q_\alpha(\phi, \log \varrho), \tag{2.5}$$

where Q_α is a polynomial in $\log \varrho$ *with coefficients in* $(\overset{\circ}{H}{}^m(\Omega_d))^l$ *and* $\operatorname{Re}\lambda - N > m - d/2$, *be a solution of the homogeneous Dirichlet problem for* (1.2). *Then $\lambda = \mu_s + k$ for some s, $k \geq 0$ and*

$$U(x) = \sum_{\mu_j + |\alpha| = \mu_s + k} c_{j\alpha} V_{j\alpha}(x). \tag{2.6}$$

PROOF. It is clear that the coefficient of z^α with $|\alpha| = N$ in (2.5) satisfies $\mathfrak{A}(D_y, 0)(\varrho^{\lambda - N} Q_\alpha) = 0$ on K_d. Hence and from the inequality $\operatorname{Re}\lambda - N > m - d/2$ it follows that $\lambda - N = \mu_j$, $j \geq 0$, and

$$\varrho^{\lambda - N} Q_\alpha(\phi, \log \varrho) = \sum_{\{j: \mu_j = \lambda - N\}} c_{j\alpha} V_{j\alpha}(y), \qquad c_{j\alpha} = \text{const}.$$

Therefore the difference

$$U(x) - \sum_{\{j: \mu_j = \lambda - N\}} c_{j\alpha} V_{j\alpha}(x)$$

is a solution of the homogeneous Dirichlet problem (1.2) and can be represented in the form (2.6) with $N - 1$ instead of N. Subsequently reducing the order of the multi-index α we arrive at (2.6). ■

THEOREM 1. *The vector-valued function U of the form* (1.3), *where* $\operatorname{Re}\lambda > m - n/2$ *and* $u_k \in (\overset{\circ}{H}{}^m(\Omega))^l$, *is a solution of the system* (1.2) *if and only if $\lambda = \mu_s + q$ for some non-negative integer s and q, and U is a combination of the vector-valued functions $V_{j\alpha}$ with $\mu_j + |\alpha| = \mu_s + q$.*

PROOF. By Proposition 2.1 $D_z^\gamma U$ is a solution of the homogeneous Dirichlet problem for the system (1.2), which can be represented as (1.3) with coefficients in $(\overset{\circ}{H}{}^m(\Omega))^l$. The role of γ is played by $\lambda - |\gamma|$. Hence it follows for an arbitrary multi-index γ that

$$D_y^\alpha D_z^\gamma U(x) = r^{\lambda - |\alpha| - |\gamma|} \sum_{k=0}^{K} (\log r)^k \phi_k(\omega), \tag{2.7}$$

where $\phi_k \in (\overset{\circ}{H}{}^{m - |\alpha|}(\Omega))^l$ and $|\alpha| \leq m$.

Let $F(y, \zeta)$ be the Fourier transform with respect to the second variable of the vector-valued function $U(y, z)$ and let $F(y, \cdot) \in (S'(R^{n-d}))^l$ for all $y \in K_d$. Since for $\zeta \neq 0$ and $M = 0, 1, \ldots,$ we have

$$F(y, \zeta) = |\zeta|^{-2M} \int_{R^{n-d}} e^{i(\zeta, z)} (-\Delta_z)^M u(y, z)\, dz, \tag{2.8}$$

it follows that the function $\zeta \to F(\cdot, \zeta)$ belongs to the class $C^\infty(R^{n-d}\backslash\{0\}; (\overset{\circ}{H}{}^m_{\text{loc}}(K_d))^l)$. For all $\zeta \neq 0$ the system

$$\mathfrak{A}(D_y, \zeta) F(y, \zeta) = 0 \tag{2.9}$$

is satisfied on K_d. From (2.8) and (2.7) it follows

$$\int_{\Omega_d} |D_y^\alpha F(|y|\,\theta, \zeta)|^2 d\theta \leqslant c_N(\zeta)|y|^{-N}, \tag{2.10}$$

where $|\alpha| \leqslant m$ and $N = 0, 1, \ldots$.

We show that the function $F(\cdot, \zeta)$ belongs to the space $(\overset{\circ}{H}{}^m(K_d))^l$ for any $\zeta \neq 0$. Let $\chi \in C_0^\infty(R^{n-d})$, $\chi = 0$ outside the unit ball and $\chi = 1$ for $|z| \leqslant 1/2$. By $F_0(y, \zeta)$ and $F_\infty(y, \zeta)$ we denote the Fourier transform in z of the functions χU and $(1-\chi)U$, respectively. For $|\alpha| \leqslant m$ we have

$$\int_{C_d} dy \int_{R^{n-d}} |D_y^\alpha F_0|^2 d\zeta = c \int_{C_d} dy \int_{R^{n-d}} |D_y^\alpha(\chi U)|^2 dz, \tag{2.11}$$

where $C_d = \{y \in K_d : |y| < 1\}$. The condition $\operatorname{Re}\lambda > m - n/2$ and (2.7) imply that the right-hand side in (2.11) is finite.

We represent F_∞ as

$$F_\infty(y, \zeta) = |\zeta|^{-2M} \int_{R^{n-d}} e^{i(\zeta, z)}(-\Delta_z)^M [(1-\chi)U]\, dz,$$

where M is a sufficiently large integer. By Parseval's theorem we have

$$\int_{C_d} dy \int_{R^{n-d}} |\zeta|^{4M} |D_y^\alpha F_\infty|^2 d\zeta \leqslant c \sum_{|\gamma| + |\delta| = 2M} \int_{C_d} dy \int_{R^{n-d}} |D_z^\gamma(1-\chi)|^2 |D_z^\delta D_y^\alpha U|^2 dz.$$

The boundedness of the right-hand side follows from (2.7).

Thus $F(\cdot, \zeta) \in (\overset{\circ}{H}{}^m(K_d))^l$ for all $\zeta \neq 0$. Since the vector-valued function $F(\cdot, \zeta)$ is a solution of (2.9) with strongly elliptic operator $\mathfrak{A}(D_y, \zeta)$, it follows that $F(\cdot, \zeta)$ vanishes for $\zeta \neq 0$. Therefore,

$$U(y, z) = \sum_{|\alpha| \leqslant N} z^\alpha \phi_\alpha(y).$$

From this and (1.3) we find that $\phi_\alpha(y) = \varrho^{\lambda - |\alpha|} Q_\alpha(\phi, \log \varrho)$, where Q_α is a polynomial in the second argument with coefficients in $(\overset{\circ}{H}{}^m(\Omega_d))^l$. Since $U \in (\overset{\circ}{H}{}^m_{\text{loc}}(K))^l$, it follows that $\operatorname{Re}\lambda - N > m - d/2$. The reference to Proposition 2.3 completes the proof. ■

REMARK 2.3. From the above theorem and the linear independence of vector-valued functions $V_{j\alpha}$ mentioned in Remark 2.2 it follows that the algebraic multiplicity of the eigenvalue λ of the operator pencil $\mathfrak{L}$ with $\operatorname{Re}\lambda > m - n/2$ is equal to the number of pairs (j, α) such that $\lambda = \mu_j + |\alpha|$, $j \geqslant 0$.

The theorem just proved gives a description of all solutions of the form (1.3) to the system (1.2) for $\operatorname{Re}\lambda > m - n/2$. According to Proposition 1.2(ii) the line

$\operatorname{Re}\lambda = m - n/2$ does not contain solutions with the exponent λ. It remains to study the case $\operatorname{Re}\lambda < m - n/2$.

3. Solutions of the form (1.3) with $\operatorname{Re}\lambda < m - n/2$

Proposition 3.1. *Let $j < 0$. There exists a solution of the Dirichlet problem for the system*

$$\mathfrak{A}(D_y, \zeta) W = 0 \quad \text{on } K_d , \tag{3.1}$$

which has the form

$$W_j(y, \zeta) = \chi(\varrho\zeta) \sum_{|\alpha| \leq m - d/2 - \operatorname{Re}\mu_j} \zeta^\alpha \varrho^{\mu_j + |\alpha|} Q_{j\alpha}(\phi, \log \varrho) + R_j(y, \zeta). \tag{3.2}$$

Here $\chi \in C_0^\infty(R^{n-d})$, $\chi = 1$ in a neighbourhood of the origin, $Q_{j\alpha}$ are polynomials in $\log \varrho$ *with coefficients from $(\overset{\circ}{H}{}^m(\Omega_d))^l$, $Q_{j0} = Q_j$, the vector-valued function R_j belongs to the space $C^\infty(R^{n-d}\backslash\{0\}, \overset{\circ}{H}{}^m(K_d)^l)$ and can be expressed in the form*

$$R_j(y, \zeta) = |\zeta|^{-\mu_j} \sum_{0 \leq k \leq N_j} R_{jk}(y|\zeta|, \zeta/|\zeta|)(\log |\zeta|)^k , \tag{3.3}$$

where N_j is the largest degree of the polynomials $Q_{j\alpha}$, and the vector-valued functions $\theta \to R_{jk}(\cdot, \theta)$ belong to the space $C^\infty(S^{n-d-1}; (\overset{\circ}{H}{}^m(K_d))^l)$.

Proof. We put $\psi_{j\alpha}(y) = \varrho^{\mu_j + |\alpha|} Q_{j\alpha}(\phi, \log \varrho)$ and seek the vector-valued functions $\psi_{j\alpha}$ by the equality

$$\mathfrak{A}(D_j, \zeta)\Big(\sum_\alpha \zeta^\alpha \psi_{j\alpha}(y)\Big) = 0 \quad \text{on } K_d .$$

Making the coefficient of ζ^α equal to zero we obtain the equation

$$\mathfrak{A}(D_y, 0)\psi_{j\alpha} = -\sum_{\beta < \alpha} \frac{1}{(\alpha - \beta)!} \mathfrak{A}^{(\alpha - \beta)}(D_y, 0)\psi_{j\beta}(y) \quad \text{on } K_d . \tag{3.4}$$

Starting with the vector-valued function $\psi_{j0} = v_j$ one can find subsequently all $\psi_{j\alpha}$ from (3.4) using Proposition 7.1 [3]. Hence the remainder R_j in (3.2) satisfies the system

$$\mathfrak{A}(D_y, \zeta) R_j(y, \zeta) = F_j(y, \zeta) \quad \text{on } K_d . \tag{3.5}$$

The right-hand side admits the representation

$$F_j(y, \zeta) = \chi(\varrho\zeta) \sum_{m < |\alpha| + \operatorname{Re}\mu_j + d/2 \leq 3m} \zeta^\alpha \varrho^{\mu_j + |\alpha| - 2m} Q_{j\alpha}^{(1)}(\phi, \log \varrho) + \tag{3.6}$$
$$+ \sum_{|\alpha| < 3m - d/2 - \operatorname{Re}\mu_j} \chi_\alpha(\varrho\zeta)\zeta^\alpha \varrho^{\mu_j + |\alpha| - 2m} Q_{j\alpha}^{(2)}(\phi, \log \varrho),$$

where $\chi_\alpha \in C_0^\infty(R^{n-d}\backslash\{0\})$, $Q_{j\alpha}^{(1)}$, $Q_{j\alpha}^{(2)}$ are polynomials in $\log \varrho$ of degree not higher than the largest degree of the polynomials $Q_{j\alpha}$ and with coefficients from the class $(H^{-m}(\Omega_d))^l$.

From (3.6) we obtain $F_j \in C^\infty(R^{n-d}; (H^{-m}(K_d))^l)$ and

$$F_j(y, \zeta) = |\zeta|^{2m-\mu_j} \sum_{0 \leq k \leq N} F_{jk}(y|\zeta|, \zeta/|\zeta|)(\log|\zeta|)^k, \tag{3.7}$$

where N_j is the largest degree of the polynomials $Q_{j\alpha}$, $|\alpha| \leq m - d/2 - \mu_j$, and the vector-valued function $\theta \to F_{jk}(\cdot, \theta)$ belongs to the class $C^\infty(S^{n-d-1}; (H^{-m}(K_d))^l)$.

By the strong ellipticity of the operator $A(D_x)$ we find that the mapping

$$(A(D_y, \zeta))^{-1} : (H^{-m}(K_d))^l \to (\mathring{H}^m(K_d))^l$$

is bounded together with all its derivatives uniformly with respect to ζ, $|\zeta| = 1$. Therefore the system (3.5) has a unique solution in

$$C^\infty(R^{m-d} \backslash \{0\}; (\mathring{H}^m(K_d))^l).$$

By (3.7) this solution can be expressed in the form (3.3), where

$$R_{jk}(\cdot, \theta) = (\mathfrak{A}(D_y, \theta))^{-1} F_{jk}(\cdot, \theta).$$

The proof is complete. ■

Remark 3.1. If among the numbers $\mu_j + 1, \dots, \mu_j + [m - d/2 - \mu_j]$ there are no points of the spectrum of the operator pencil $\mathfrak{L}_d(\mu)$, then the vector-valued function W_j is uniquely defined. Moreover, the degrees of the polynomials $Q_{j\alpha}$ and the number N_j do not exceed the degree of the polynomial Q_j.

Remark 3.2. Suppose that the collection $\mu_j + 1, \dots, \mu_j + [m - d/2 - \mu_j]$ contains s different eigenvalues with maximal partial multiplicities $\kappa_1, \dots, \kappa_s$. Then the degrees of the polynomials $Q_{j\alpha}$ and the number N_j do not exceed $\kappa_0 + \kappa_1 + \dots + \kappa_s$, where κ_0 is the degree of the polynomial Q_j.

The vector-valued function W_j is unique up to a linear combination of solutions $\zeta^\alpha W_l$, $0 \leq |\alpha| = \mu_l - \mu_j$. Henceforth we assume that the solutions $W_j, j < 0$, which have the properties mentioned in Proposition 3.1, are fixed.

Remark 3.3. Since the functions $\varrho^{\mu_j} Q_j$ are linear independent, it follows that the functions $\zeta^\alpha W_j$, $j < 0$, $|\alpha| \geq 0$, are also linear independent.

The following two lemmas will be used in the study of the inverse Fourier transform of the vector-valued function R_j with respect to the variable ζ.

Lemma 3.1. *Let R_j be introduced in Proposition 3.1. Then*

$$D_\zeta^\delta R_j(y, \zeta) = |\zeta|^{-\mu_j - |\gamma|} \sum_{0 \leq k \leq N_j} R_{j\gamma k}(y|\zeta|, \zeta/|\zeta|)(\log|\zeta|)^k,$$

where the vector-valued functions $\theta \to R_{j\gamma k}(\cdot, \theta)$ belong to the class $C^\infty(S^{n-d-1}; (\mathring{H}^m(K_d))^l)$ and γ is an arbitrary multi-index.

PROOF. Formula (3.6) implies

$$(3.8)\qquad D_\zeta^\gamma F_j(y,\zeta) = |\zeta|^{2m-\mu_j-|\gamma|} \sum_{0\leq k\leq N_j} F_{j\gamma k}(y|\zeta|, \zeta/|\zeta|)(\log|\zeta|)^k,$$

where $\theta \to F_{j\gamma k}(\cdot,\theta)$ is a vector-valued function from $C^\infty(S^{n-d-1}; (H^{-m}(\Omega_d))^l)$. Using the obvious equality

$$\mathcal{A}(D_y,\zeta)(D_\zeta^\delta R_j) = D_\zeta^\delta F_j - \sum_{\beta<\gamma} \frac{1}{(\gamma-\beta)!}\mathcal{A}^{(\gamma-\beta)}(D_y,\zeta)(D_\zeta^\beta R_j)$$

together with (3.8), we complete the proof by induction in $|\gamma|$.

LEMMA 3.2. *For any positive integer N the sums*

$$\sum_{|\alpha|\leq m} \int_{K_d} (1+\varrho^2)^N |D_y^\alpha R_{jk}(y,\theta)|^2 dy, \qquad k=0,\dots,N_j,$$

are bounded uniformly with respect to θ.

PROOF. It is clear that

$$\mathcal{A}(D_y,\theta)(1+\varrho^2)^{1/2}R_{jk} = (1+\varrho^2)^{1/2}F_{jk} - [\mathcal{A}(D_y,\theta),(1+\varrho^2)^{1/2}]R_{jk}.$$

From (3.6) and (3.7) it follows that the vector-valued function $(1+\varrho^2)^{N/2}F_{jk}$ belongs to $C^\infty(S^{n-d-1}; (H^m(K_d))^l)$ for all N. Using the boundedness of the operator

$$[\mathcal{A}(D_y,\theta),(1+\varrho^2)^{1/2}]: (\mathring{H}^m(K_d))^l \to (H^{-m}(K_d))^l$$

and the induction in N we obtain the boundedness of the norm of $(1+\varrho^2)^{n/2}R_{jk}$ in $(\mathring{H}^m(K_d))^l$. The lemma is proved.

Let $\hat{f}$ denote the inverse Fourier transform of the distribution $f=f(\zeta)$, *i.e.*

$$\hat{f}(z) = \frac{1}{(2\pi)^{n-d}} \int_{R^{n-d}} e^{-i(z,\zeta)} f(\zeta)\,d\zeta.$$

Then by (3.2) we have

$$(3.9)\quad (\hat{W}_j - \hat{R}_j)(y,z) = \varrho^{\mu_j+a-n} \sum_{|\alpha|\leq m-d/2-\operatorname{Re}\mu_j} (D^\alpha\hat{\chi})(\varrho^{-1}z)\,Q_{j\alpha}(\phi,\log\varrho).$$

Since $\hat{\chi}\in S(R^{n-d})$, it follows that for $|x|\leq b<\infty$ all components of derivatives in x of the vector-valued function (3.9) of order not higher than m do not exceed

$$(3.10)\qquad r^{-\mu}\varrho^{M-m+\mu_j+d-n}(1+|\log\varrho|)^{N_j}q(\phi),$$

where M is an arbitrary positive number and q is a positive function in $L_2(\Omega_d)$. It is readily verified that (3.10) is a square summable function on the set $\{x\in K: 0<a\leq|x|\leq b<\infty\}$. Therefore the function (3.9) belongs to the space $(\mathring{H}^m_{\text{loc}}(K))^l$.

Since $\varrho = r\cos\tau$, $z/\varrho = \theta\tan\tau$, (3.9) can be transformed in the following way

$$(r\cos\tau)^{\mu_j+d-n} \sum_{|\alpha|\leq m-d/2-\mathrm{Re}\,\mu_j} (D^\alpha\hat{\chi})(\theta\tan\tau)\, Q_{j\alpha}(\phi, \log(r\cos\tau)) =$$

$$= (r\cos\tau)^{\mu_j+d-n} \sum_{\nu=0}^{N_j} \frac{(\log r)^\nu}{\nu!} \sum_{|\alpha|\leq m-d/2-\mathrm{Re}\,\mu_j} (D^\alpha\hat{\chi})(\theta\tan\tau)\, \frac{Q_{j\alpha}^{(\nu)}(\phi, \log\cos\tau)}{\nu!},$$

where $Q_{j\alpha}^{(\nu)}(\phi, t) = \partial_t^\nu Q_{j\alpha}(\phi, t)$. Thus, the function (3.9) has the form (1.3) with coefficients from $(\overset{\circ}{H}{}^m(\Omega))^l$.

By Lemma 3.2 we have

$$(3.11)\qquad \sum_{|\alpha|\leq m} \int_{K_d} \frac{|D_y^\alpha R_j(y, \zeta)|^2}{|y|^{2(m-|\alpha|)}}\, dy \leq c\,|\zeta|^{-2\mathrm{Re}\,\mu_j - d + 2m}(1 + |\log|\zeta||)^{2N_j} \times$$

$$\times \sum_{0\leq k\leq N_j} \max_{\theta\in S^{n-d-1}} \|R_{jk}(\cdot, \theta)\|^2_{(\overset{\circ}{H}{}^m(K_d))^l}.$$

Since $-2\,\mathrm{Re}\,\mu_j - d + 2m > d - n$, it follows that the inverse Fourier transform $\hat{R}_j(y, \cdot) \in (S'(R^{n-d}))^l$ is defined for almost all $y \in K_d$. Moreover, by (3.11) the function R_j can be considered as an element of the space $S'(R^{n-d}; (\overset{\circ}{L}{}_2^m(K_d))^l)$ where $\overset{\circ}{L}{}_2^m(K_d)$ is the completion of $C_0^\infty(K_d)$ with respect to the Dirichlet integral of order m. The first term in the right-hand side of (3.2) belongs to the space $(C^\infty \cap S')(R^{n-d}; (\overset{\circ}{H}{}^m_{\mathrm{loc}}(K_d))^l)$, hence $W_j \in S'(R^{n-d}; (\overset{\circ}{H}{}^m_{\mathrm{loc}}(K_d))^l)$. Therefore $\hat{W}_j \in S'(R^{n-d}; (\overset{\circ}{H}{}^m_{\mathrm{loc}}(K_d))^l)$.

We show that $\mathcal{A}(D_x)\hat{W}_j = 0$ on K in the sense of the distribution theory. For this purpose it suffices to verify that $\mathcal{A}(D_y, \zeta) W_j = 0$ on K in the space $S'(R^{n-d}; (H_{\mathrm{loc}}^{-m}(K_d, 0))^l)$. Indeed, by (3.11) the function W_j belongs to $L_{\infty,\mathrm{loc}}(R^{n-d}; (\overset{\circ}{H}{}^m_{\mathrm{loc}}(D_k, 0))^l)$ and hence $\mathcal{A}(D_y, \zeta) W_j \in L_{\infty,\mathrm{loc}}(R^{n-d}; (\overset{\circ}{H}{}^m_{\mathrm{loc}}(K_d, 0))^l)$. Thus, the equation $\mathcal{A}(D_y, \zeta) W_j = 0$ obtained earlier for $\zeta \neq 0$ can be extended to all ζ.

PROPOSITION 3.2. *The vector-valued function* $\hat{W}_j$ *has the form* (1.3) *where* $\lambda = \mu_j + d - n$, $\kappa \leq N_j$, $u_k \in (\overset{\circ}{H}{}^m(\Omega))^l$ *and* $\hat{W}_j$ *satisfies the system* (1.2) *on* K.

PROOF. We have shown in the proof of Lemma 3.2 that $\hat{W}_j - \hat{R}_j$ has the form (1.3), where $\lambda = \mu_j + d - n$, $\kappa \leq N_j$, $u_k \in (\overset{\circ}{H}{}^m(\Omega))^l$, and that $\hat{W}_j$ satisfies (1.2). In order to complete the proof it suffices to use the following assertion.

LEMMA 3.3. *The vector-valued function* $\hat{R}_j$ *belongs to* $(\overset{\circ}{H}{}^m_{\mathrm{loc}}(K))^l$ *and the representation* (1.3) *is valid for* $\hat{R}_j$ *with* $\kappa \leq N_j$.

PROOF. Let $R_j = R_j^{(1)} + R_j^{(2)}$, where

$$R_j^{(1)}(y, \zeta) = \chi(\zeta) R_j(y, \zeta), \qquad R_j^{(2)}(y, \zeta) = (1 - \chi(\zeta)) R_j(y, \zeta).$$

Using Parseval's theorem we find

$$(3.12)\qquad \int_{K_{d,b}} \frac{dy}{|y|^{2(m-|\alpha|)}} \int_{R^{n-d}} |D_y^\alpha D_z^\beta \widehat{R}_j^{(1)}(y,z)|^2 dz =$$

$$= c \int_{R^{n-d}} |\zeta^\beta \chi(\zeta)|^2 \int_{K_{d,b}} |D_y^\alpha R_j(y,\zeta)|^2 \frac{dy}{|y|^{2(m-|\alpha|)}},$$

where $K_{d,b} = \{y \in K_d : |y| < b\}$, $b = \text{const}$ and $|\alpha| + |\beta| \leqslant m$. From (3.11) it follows that the right-hand side in (3.12) does not exceed

$$c \int_{R^{n-d}} |\chi(\zeta)|^2 |\zeta|^{-2\mathrm{Re}\mu_j - d + 2m + 2|\beta|} (1 + |\log|\zeta||)^{2N_j} d\zeta < \infty .$$

Now let us estimate the integral

$$(3.13)\qquad \int_{K_{d,b}} \int_{R^{n-d}} (|y|^{4N} + |z|^{4N}) |D_y^\alpha D_z^\beta \widehat{R}_j^{(2)}(y,z)|^2 dy\, dz$$

for $|\alpha| + |\beta| \leqslant m$ and for a sufficiently large N. By Parseval's theorem this integral is equal to

$$(3.14)\qquad c \int_{K_{d,b}} \int_{R^{n-d}} |(-\Delta_\zeta)^N (\zeta^\beta (1 - \chi(\zeta))) D_y^\alpha R_j(y,\zeta)|^2 dy\, d\zeta +$$

$$+ c \int_{K_{d,b}} \int_{R^{n-d}} \big| |y|^{2N} \zeta^\beta (1 - \chi(\zeta)) D_y^\alpha R_j(y,\zeta) \big|^2 dy\, d\zeta .$$

By Lemma 3.2 the first integral on the right does not exceed

$$(3.15)\qquad c \int_{|\zeta| > a} |\zeta|^{2\mu_j - 4N + 2|\alpha| + 2|\beta| - d} (1 + |\log|\zeta||)^{2N_j} \times$$

$$\times \int_{K_d} \sum_{0 \leqslant k \leqslant N_j} \sum_{|\gamma| \leqslant 2N} |D_y^\alpha R_{jk\gamma}(y, \zeta/|\zeta|)|^2 dy\, d\zeta .$$

Since the inner integral is bounded and since N is large, it follows that the value (3.15) is finite.

The second integral in the right-hand side of (3.14) is not greater than

$$c \int_{|\zeta| > a} |\zeta|^{-2\mu_j - 4N + 2|\alpha| + 2|\beta| - d} (1 + |\log|\zeta||)^{2N_j} \times$$

$$\times \int_{K_d} |y|^{4N} \sum_{0 \leqslant k < N_j} |D_y^\alpha R_{jk}(y, \zeta/|\zeta|)|^2 dy\, d\zeta .$$

This value is also finite because the inner integral is uniformly bounded with respect to $\zeta/|\zeta|$ by Lemma 3.2.

Since the integrals (3.12) and (3.13) are bounded, it follows that the integral

$$\int_{x\in K,\, a<|x|<b} \sum_{|\alpha|\le m} |D_x^\alpha \widehat{R}_j(x)|^2 dx$$

converges.

We verify the zero Dirichlet conditions for $\widehat{R}_j$ on $\partial K \backslash \{0\}$. Put

$$R_{j\varepsilon}(y,\zeta) = (1-\chi(\zeta/\varepsilon))\chi(\varepsilon\zeta)R_j(y,\zeta),$$

where ε is a small positive number. By Lemma 3.1

$$R_{j\varepsilon} \in S(R^{n-d}; (\mathring{H}^m(K_d))^l).$$

Therefore $\widehat{R}_{j\varepsilon} \in S(R^{n-d}; (\mathring{H}^m(K_d))^l)$. Moreover, $\widehat{R}_{j\varepsilon} \in (\mathring{H}^m(K))^l$. Replacing R_j by $R_j - R_{j\varepsilon}$ in the above argument we obtain that the integrals

$$\int_{K_{d,b}} \frac{dy}{|y|^{2(m-|\alpha|)}} \int_{R^{n-d}} |D_y^\alpha D_z^\beta (\widehat{R}_j^{(1)} - \widehat{R}_{j,\varepsilon}^{(1)})|^2 dz,$$

$$\int_{K_{d,b}} \int_{R^{n-d}} (|y|^{4N} + |z|^{4N}) |D_y^\alpha D_z^\beta (\widehat{R}_j^{(2)} - \widehat{R}_{j,\varepsilon}^{(1)})|^2 dy\, dz,$$

tend to zero as $\varepsilon \to 0$. Consequently, $\widehat{R}_j \in (\mathring{H}^m_{\text{loc}}(K))^l$.

By (3.3) we have

$$\widehat{R}_j(x) = \frac{r^{\mu_j+d-n}}{(2\pi)^{n-d}} \int_{R^{n-d}} e^{-i((z/r),\zeta)} |\zeta|^{-\mu_j} \sum_{0\le k\le N_j} R_j\left(\frac{y}{r}|\zeta|, \frac{\zeta}{|\zeta|}\right)\left(\log\frac{|\zeta|}{r}\right)^k d\zeta,$$

which is equivalent to (1.3) for $d = N_j$. The lemma and Proposition 3.2 are proved.

The following assertion is an immediate corollary of Proposition 3.2, Remark 3.3 and Proposition 2.1.

PROPOSITION 3.3. *For any multi-index γ the vector-valued function $D_z^\gamma \widehat{W}_j$ has the form* (1.3), *where $\lambda = \mu_j + d - n - |\gamma|$, $\kappa \le N_j$, $u_k \in (\mathring{H}^m(\Omega))^l$, and satisfies the system* (1.2). *The vector-valued functions $D_z \widehat{W}_j$, $j<0$, $|\gamma|>0$ are linear independent.*

Now we can give a description of all solutions of the form (1.3) for $\operatorname{Re}\lambda < m - n/2$.

THEOREM 2. *The vector-valued function U of the form* (1.3), *where* $\operatorname{Re}\lambda < m - n/2$ *and $u_k \in (\mathring{H}^m(\Omega))^l$, is a solution of the system* (1.2) *if and only if $\lambda = \mu_s + d - n - k$ for some integers $k \ge 0$, $s<0$. Moreover U is a linear combination of the vector-valued functions $D_z^\gamma \widehat{W}_j$, $\lambda = \mu_j + d - n - |\gamma|$.*

PROOF. Let $\mathcal{L}^*(\lambda)$ be the operator pencil (1.5) and let $\mathcal{L}_d^*(\lambda)$ be a similar operator pencil constructed with regard to the operator $A^*(D_y, 0)$ in K_d. By Proposition 1.3 the spectrum of $\mathcal{L}_d^*(\lambda)$ consists of the eigenvalues $2m - d - \overline{\mu}_j$, $j = 0, \pm 1, \dots$, numerated with account taken of their algebraic multiplicity. This and Theorem 1 imply that the spectrum of $\mathcal{L}^*(\lambda)$ consists of the eigenvalues $\lambda^*_{j,\gamma} = 2m - d - \overline{\mu}_j + |\gamma|$ in the half-plane $\operatorname{Re}\lambda > m - n/2$ (here $j < 0$ and γ is an arbitrary $(n-d)$-dimensional multi-index). Applying Proposition 1.3 again we obtain that the spectrum of the operator pencil $\mathcal{L}(\lambda)$ consists of the eigenvalues $2m - n - \overline{\lambda^*_{j,\gamma}} = \mu_j + d - n - |\gamma|$ numerated with account taken of their algebraic multiplicity and placed in the half-plane $\operatorname{Re}\lambda < m - n/2$. Therefore, according to Remark 2.2, the number of linear independent solutions of the system (1.2), which have the form (1.3) with $\operatorname{Re}\lambda < m - n/2$ and $u_k \in (\overset{\circ}{H}{}^m(\Omega))^l$, is equal to the number of the representations of λ as $\mu_j + d - n - |\gamma|$ with $j < 0$, $|\gamma| \geqslant 0$. It was shown in Proposition 3.3 that the solutions $D_z^\gamma \widehat{W}_j$ are linear independent. Consequently, all solutions of the system (1.2), which have the form (1.3), are generated by linear combinations of $D_z^\gamma \widehat{W}_j$.

4. The Laplace operator in $K_d \times R^{n-d}$

In the case of the Laplace operator in $K_d \times R^{n-d}$ Theorems 1 and 2 admit a more explicit interpretation. Here $\mathfrak{A}(D_x) = \Delta_x$, $\mathfrak{A}(D_y, 0) = \Delta_y$. $\mathcal{L}_d(\mu) = \delta + \mu(\mu + d - 2)$, where δ is the Beltrami operator on S^{d-1}. Let $\{\mu_j\}_{j \geqslant 0}$ be the sequence of positive eigenvalues of the operator pencil $\mathcal{L}_d(\mu)$ numerated with account taken of their multiplicities and let $\{\Psi_j\}_{j \geqslant 0}$ be the sequence of real eigenfunctions. The number $2 - d - \mu_j$ is also an eigenvalue and Ψ_j is the corresponding eigenfunction.

In a similar way we denote by $\{\lambda_j\}$ a sequence of eigenvalues of the operator pencil $\mathcal{L}(\lambda)$ generated by the operator Δ_x in the cone K and by $\{u_j\}_{j \geqslant 0}$ a sequence of corresponding eigenfunctions. The same eigenfunction corresponds both to the eigenvalues λ_j and $2 - n - \lambda_j$. In particular, the eigenfunction $(\cos\tau)^{\mu_j}\Psi_j(\phi)$ generates two solutions (1.3): $\varrho^{\mu_j}\Psi_j(\phi)$ and $v_j(x) = r^{2-n-2\mu_j}\varrho^{\mu_j}\Psi_j(\phi)$.

In order to find all eigenfunctions of the operator pencil $\mathcal{L}(\lambda)$ and consequently all solutions of the form (1.3) it suffices to do it for the eigenvalues $2 - n - \lambda_j$.

First of all we show that one can use v_j as the functions $\widehat{W}_j$ introduced in the preceding section. We have

$$(4.1)\qquad \int_{R^{n-d}} e^{i(z,\zeta)} v_j(y,z)\,dz = \varrho^{\mu_j}\Psi_j(\phi) \int_{R^{n-d}} \frac{e^{i(z,\zeta)}\,dz}{(\varrho^2 + |z|^2)^{\mu_j + 1 + n/2}} =$$

$$= \frac{2^{2-d/2-\mu_j}\pi^{(n-d)/2}}{\Gamma(\mu_j - 1 + n/2)}\, \varrho^{2-d-\mu_j}\Psi_j(\phi)(\varrho|\zeta|)^{\mu_j - 1 + d/2} K_{1-\mu_j-d/2}(\varrho|\zeta|),$$

where K_ν is the modified Bessel function.

The function $\varrho^{1-d/2}K_{1-\mu_j-d/2}(\varrho)\,\Psi_j(\phi)$ satisfies the zero Dirichlet condition on $\partial K_d \backslash \{0\}$ and is a solution of the equation $\Delta u - u = 0$ on K_d. Since for $\varrho \to 0$ we have $K_{1-\mu_j-d/2}(\varrho) = \varrho^{1-\mu_j-d/2}P_k(\varrho^2) + O(\varrho^\varepsilon)$, $\varepsilon > 0$, where P_k is a polynomial of degree k, $2k \leqslant \mu_j - 1 + d/2$, then the right-hand side of (4.1) admits the representation (3.2) and hence it can play the role of W_j. This and Proposition (1.3) imply that the linear combinations of the functions $D_z^\gamma v_j$ exhaust the set of solutions of the form (1.3) for $\operatorname{Re}\lambda < 1 - n/2$.

We introduce the polynomials $Q_{\gamma,\kappa}(z)$ of degree $|\gamma|$ by the equality

$$Q_{\gamma,\kappa}(z) = (|z|^2+1)^{|\gamma|+\kappa/2} D_z^\gamma (|z|^2+1)^{-\kappa/2}$$

and represent them as the sum of the homogeneous polynomials

$$Q_{\gamma,\kappa} = \sum_{0\leq s\leq |\gamma|/2} Q_{\gamma,\kappa}^{(s)}, \ \deg Q_{\gamma,\kappa}^{(s)} = |\gamma| - 2s\,.$$

Then

$$r^{\kappa+2|\gamma|} D_z^\gamma r^{-k} = \sum_{0\leq s\leq |\gamma|/2} \varrho^{2s} Q_{\gamma,\kappa}^{(s)}(z)$$

and therefore the eigenvalue $2-n-\mu_j-k$, $k=0,1,\dots$, has eigenfunctions

$$u_{\nu,\gamma}(\omega) = (\cos\tau)^{\mu_\nu}\Psi_\nu(\phi) \sum_{0\leq s\leq |\gamma|/2} (\cos\tau)^{2s} Q_{\gamma,\kappa_\nu}^{(s)}(\theta\sin\tau)\,,$$

where $\kappa_\nu = n-2+2\mu_\nu$ and $\mu_\nu + |\gamma| = \mu_j + k$. By Theorem 2 there are no other linear independent eigenfunctions. The same eigenfunctions are generated by the eigenvalue μ_j+k.

Thus we have proved the following assertion.

THEOREM 3. *The function* $U \in \overset{\circ}{H}{}^m_{\mathrm{loc}}(K,0)$ *of the form* (1.3) *is a solution of the equation* $\Delta_x U = 0$ *in* K *if and only if*

(i) $\lambda = \mu_j + k$, $j,k = 0,1,\dots$, *and* U *is a linear combination of the functions*

$$\varrho^{\mu_\nu}\Psi_\nu(\phi) \sum_{0\leq s\leq |\gamma|/2} \varrho^{2s} Q_{\gamma,\kappa_\nu}^{(s)}(z)\,,$$

where $\mu_\nu + |\gamma| = \mu_j + k$;

(ii) $\lambda = 2-n-\mu_j-k$, $j,k=0,1,\dots$, *and* U *is a linear combination of the functions*

$$\varrho^{\mu_\nu}\Psi_\nu(\phi) D_z^\gamma (r^{2-n-2\mu_\nu}) = \varrho^{\mu_\nu}\Psi_\nu(\varphi) r^{2-n-2\mu_\nu} \sum_{0\leq s\leq |\gamma|/2} \varrho^{2s} Q_{\gamma,\kappa_\nu}^{(s)}(z)\,,$$

where $\mu_\nu + |\gamma| = \mu_j + k$.

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