Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Edoardo Vesentini

Complex geodesics and isometries in Cartan domains of type four

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **6** (1995), n.4, p. 251–257.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1995_9_6_4_251_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1995.

Rend. Mat. Acc. Lincei
s. 9, v. 6:251-257 (1995)

Geometria. — *Complex geodesics and isometries in Cartan domains of type four.* Nota (*) del Socio EDOARDO VESENTINI.

ABSTRACT. — Holomorphic maps of Cartan domains of type four preserving the supports of complex geodesics are characterized, providing, in particular, a new description of holomorphic isometries.

KEY WORDS: Cartan domain; Complex geodesic; Holomorphic isometry; Moebius transformation.

RIASSUNTO. — *Geodetiche complesse ed isometrie in domini di Cartan del quarto tipo.* Vengono caratterizzate le applicazioni olomorfe dei domini di Cartan del quarto tipo che conservano i supporti di geodetiche complesse, ottenendo una nuova caratterizzazione delle isometrie olomorfe.

Isometries between Riemannian manifolds are characterized by their behaviour on geodesics: namely by their mapping geodesics onto geodesics and affine parameters onto affine parameters. The natural question arises whether a similar characterization of isometries still holds when differentiable manifolds, geodesics for Riemannian metrics, and differentiable maps are replaced by complex manifolds, complex geodesics for invariant Finsler metrics and by holomorphic maps. In this context the situation is far from being as clear-cut as in the differentiable case. First of all, complex geodesics may or may not exist; when they do exist, no general existence and uniqueness statement seems to be available which might be compared to the classical one holding in the differentiable case. Examples, in which the complex manifold is the open unit ball in some normed space, endowed with the Carathéodory metric, show that the behaviour of complex geodesics is related to the fine structure of different parts of the boundary of the ball. It is worth noticing, at this point, that the amount of information about the complex geodesics depends on the geometry of the ball, and – not unexpectedly – the information is more complete when the ball is a bounded, homogeneous, symmetric domain.

Concentrating the attention on these domains, the problems that will be discussed here are centered on the following question: given two bounded, homogeneous, symmetric domains D and D', endowed with their Carathéodory metrics, and an injective holomorphic map $F: D \to D'$, does the fact that F transforms all complex geodesics on D into complex geodesics on D' imply that F is a holomorphic isometry?

This question has been already investigated – and the answer turns out to be affirmative – in the two cases in which D and D' are open unit balls in complex Hilbert spaces or in the Banach spaces of all continuous functions on two compact Hausdorff spaces.

In this latter case the requirements on the behaviour of F on all complex geodesics can be weakened considerably, and that seems to indicate a qualitative difference with

(*) Pervenuta all'Accademia il 4 agosto 1995.

what happens in the case of Riemannian manifolds. It turns out that the hypothesis whereby F maps any complex geodesic passing through a pre-assigned point of D and converging to some extreme point of the boundary of D, onto a complex geodesic of the same kind in D' suffices to insure that F is a holomorphic isometry.

In this paper, the investigation will be carried out in the case of another class of bounded, homogeneous, symmetric domains: the Cartan domains of type four. Moreover, pursuing a research initiated in [4], the holomorphic functions $F: D \to D'$ mapping supports of complex geodesics into supports of complex geodesics – which have been called projective holomorphic mappings – will be characterized.

1. Let $\mathcal{H}$ be a complex Hilbert space with $\dim_C \mathcal{H} > 1$, and let $\mathcal{E}$ be a closed subspace of the Banach space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. It will be assumed henceforth that $\mathcal{E}$ is a Cartan factor of type four [1], *i.e.* that $\mathcal{E}$ is self-adjoint ($\mathcal{E}^* = \mathcal{E}$) and that, for every $X \in \mathcal{E}$, X^2 is a scalar multiple of the identity $I \in \mathcal{L}(\mathcal{H})$.

The uniform norm in $\mathcal{E}$ will be denoted by $|||\ |||$, and $\|\ \|$ will stand for the Hilbert space norm in $\mathcal{E}$, defined by the inner product $(\ |\ )$:

$$2(X|Y)I = XY^* + Y^*X \qquad (X, Y \in \mathcal{E}).$$

The open unit ball D for the norm $|||\ |||$ in $\mathcal{E}$ (Cartan domain of type four) is defined, in terms of the norm $\|\ \|$, by

$$D = \left\{Z \in \mathcal{E}: \|Z\|^2 < \frac{1}{2}(1 + |(Z|Z^*)|^2) < 1\right\},$$

and the boundary ∂D of D is given by

$$\partial D = \{Z \in \mathcal{E}: |||Z||| = 1\} = \left\{Z \in \mathcal{E}: \|Z\|^2 = \frac{1}{2}(1 + |(Z|Z^*)|^2) \leqslant 1\right\}.$$

Setting, for $r > 0$, $B(r) = \{Z \in \mathcal{E}: \|Z\| < r\}$ and $\partial_0 D = \overline{B(1)} \cap \partial D$, then

$$\begin{aligned}\partial_0 D = \{Z \in \mathcal{E}: \|Z\| = |(Z|Z^*)| = 1\} &= \\ = \{Z \in \mathcal{E}: \|Z\| = 1, Z^* = e^{i\theta} Z \text{ for some } \theta \in \mathbf{R}\} &= \\ = \{e^{i\theta} X: \theta \in \mathbf{R}, X \text{ self-adjoint unitary operator contained in } \mathcal{E}\}&.\end{aligned}$$

The set $\partial_0 D$, which – borrowing the terminology from the finite-dimensional case – in this paper will be called the *distinguished boundary* of D, consists of all complex (= real) extreme points of the closure, cl(D), of D. If $\dim_C \mathcal{H} < \infty$, $\partial_0 D$ is the Shilov boundary of cl(D).

The adjunction operator $Z \mapsto Z^*$ defines a conjugation in $\mathcal{E}$, whose real elements are the bounded linear self-adjoint operators acting on $\mathcal{H}$.

Given a second Hilbert space $\mathcal{H}'$ with $\dim_C \mathcal{H}' > 1$, $\mathcal{E}' \subset \mathcal{L}(\mathcal{H}')$ will, in the following, be a Cartan factor of type four, and D', $\partial D'$, $\partial_0 D'$ will be the Cartan domain of type four associated to $\mathcal{E}'$, its boundary and its distinguished boundary; $|||\ |||$ and $\|\ \|$ will stand for the uniform norm and for the Hilbert space norm in both $\mathcal{E}$ and $\mathcal{E}'$; L^* will denote the adjoint of a bounded linear operator L in $\mathcal{E}$ or in $\mathcal{E}'$.

For $A \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ (the Banach space of all bounded linear operators from $\mathcal{E}$ to $\mathcal{E}'$), the conjugate $\overline{A} \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ is defined by

$$\overline{A}(Z) = A(Z^*)^*$$

for all $Z \in \mathcal{E}$. Hence, if $X \in \mathcal{E}$ is real, then $\overline{A}(X) = A(X)^*$.

Let $A \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ be such that

$$A(\partial_0 D) \subset \partial_0 D' . \tag{1}$$

That is equivalent to requiring that, if there exists $\xi(Z) \in \boldsymbol{R}$ such that $Z^* = e^{i\xi(Z)} Z$, then there is $\eta(Z) \in \boldsymbol{R}$ for which

$$A(Z)^* = e^{i\eta(Z)} A(Z) .$$

Thus, if X and Y are real,

$$\overline{A}(X) = A(X)^* = e^{i\eta(X)} A(X) , \qquad \overline{A}(Y) = A(Y)^* = e^{i\eta(Y)} A(Y) .$$

Since for $t \in \boldsymbol{R}$, also $X + tY$ is real, then

$$e^{i\eta(X)} A(X) + t\, e^{i\eta(Y)} A(Y) = e^{i\eta(X + tY)} (A(X) + tA(Y)) .$$

Hence, if $A(X)$ and $A(Y)$ are linearly independent, then $e^{i\eta(X+tY)} = e^{i\eta(X)} = e^{i\eta(Y)}$ for all $t \in \boldsymbol{R}$.

LEMMA 1. *If $A \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ is injective, there exists $\eta \in \boldsymbol{R}$ such that $A(Z)^* = e^{i\eta} A(Z^*)$ for all $Z \in \mathcal{E}$.*

Indeed, the continuous linear operators $\overline{A}$ and $e^{i\eta} A$ coincide on the real linear subspace $\operatorname{Re} \mathcal{E}$ consisting of all self-adjoint linear operators contained in $\mathcal{E}$, and therefore coincide on $\mathcal{E}$.

Setting $C = e^{i\eta/2} A \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$, then $C(Z)^* = e^{-i\eta/2} A(Z)^* = e^{i\eta/2} A(Z^*) = C(Z^*)$ for all $Z \in \mathcal{E}$; that is to say, C is a real operator. For $X \in \mathcal{E} \setminus \{0\}$ self-adjoint, consider the unitary operator $U = (1/\|X\|) X$ acting on $\mathcal{H}$. Then $C(U)^* = C(U^*) = C(U)$, and, in view of (1), $\|C(U)\| = \|U\| = 1$, *i.e.* $C(X)^* = C(X)$ and $\|C(X)\| = \|X\|$ for all real $X \in \mathcal{E}$.

If $X, Y \in \mathcal{E}$ are real, the equalities $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2 + 2(X \mid Y)$ and $\|C(X + Y)\|^2 = \|C(X)\|^2 + \|C(Y)\|^2 + 2(C(X) \mid C(Y))$ imply then: $(C(X) \mid C(Y)) = (X \mid Y)$, *i.e.*

$$(C^{\odot} C(X) \mid Y) = (X \mid Y) , \tag{2}$$

where $C^{\odot} \in \mathcal{L}(\mathcal{E}', \mathcal{E})$ is the adjoint of C with respect to the Hilbert space structures of $\mathcal{E}$ and $\mathcal{E}'$. Since, for any $T \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$, $\overline{T}^{\odot} = \overline{T^{\odot}}$, then $C^{\odot} C$ is a real operator, which, by (2), is the identity on $\operatorname{Re} \mathcal{E}$ and therefore on $\mathcal{E}$. Hence the following proposition holds.

PROPOSITION 1. *If $A \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ is injective and satisfies* (1), *there is a constant $\alpha \in \boldsymbol{R}$ such that $e^{i\alpha} A$ is a real linear isometry for the norm $\| \, \|$.*

As a consequence of Theorem I of [3] (1), that yields

THEOREM 1. *Let the operator* $A \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ *be injective. Then* A *is a linear isometry for the uniform norm* $\|\!|\ \|\!|$ *if, and only if,* A *maps* $\partial_0 D$ *into* $\partial_0 D'$.

Now, let $F: D \to D'$ be a holomorphic map for which $F(0)$, and such that, for every $X \in \partial_0 D$, either there is $\zeta \in \Delta \backslash \{0\}$ (where Δ is the open unit disc of $\boldsymbol{C}$) for which

$$(1/\zeta) F(\zeta X) \in \partial_0 D', \tag{3}$$

or

$$\lim_{\zeta \to 0} (1/\zeta) F(\zeta X) \in \partial_0 D'. \tag{4}$$

Since $\partial_0 D$ and $\partial_0 D'$ are the sets of all complex extreme points of $\mathrm{cl}(D)$ and $\mathrm{cl}(D')$, by the Schwarz lemma [1],

$$F(\zeta Z) = \zeta dF(0) Z \tag{5}$$

for all $Z \in D$ and all $\zeta \in \boldsymbol{C}$ with $|\zeta| < 1/ \|\!| Z \|\!|$. Being $dF(0)X \in \partial_0 D'$ for all $X \in \partial_0 D$, Theorem 1 implies that $dF(0)$ is a linear isometry for $\|\!|\ \|\!|$. Thus (cf. [3] and bibliographical references therein), the following proposition holds

PROPOSITION 2. *If the holomorphic map* $F: D \to D'$ *is such that:* $F(0) = 0$, $dF(0)$ *is injective and, for every* $X \in \partial_0 D$, *either* (3) *or* (4) *holds, then F is an isometry for the Carathéodory metrics in D and in D'.*

COROLLARY 1. *If* $F(0) = 0$, *if* $dF(0)$ *is injective and if* $dF(0)X$ *is a unitary self-adjoint operator for every self-adjoint unitary operator* $X \in \mathcal{E}$, *then F is an isometry for the Carathéodory metrics in D and in D'.*

2. These latter results will now be stated also in terms of complex geodesics. If D is the open unit ball in a complex Banach space $\mathcal{E}$, a complex geodesic in D is a holomorphic map $\phi: \Delta \to D$ which is an isometry for the Poincaré metric on Δ and the Carathéodory metric on D [2]. The set $\phi(\Delta)$, which is closed in D and is called the support of ϕ, determines the complex geodesic up to a Moebius transformation of Δ, in the sense that, if the support of a complex geodesic $\psi: \Delta \to D$ coincides with that of ϕ, then there is a Moebius transformation λ of Δ onto Δ such that $\psi = \phi \circ \lambda$; and viceversa.

Denoting by $\|\!|\ \|\!|$ the norm in $\mathcal{E}$, for any $V \in \mathcal{E} \backslash \{0\}$, the map $\zeta \mapsto (\zeta / \|\!| V \|\!|) V$ of Δ into D, is a complex geodesic whose support contains 0 and whose tangent vector at 0 is collinear to V.

For $Z \in D$ and $V \in \mathcal{E} \backslash \{0\}$, let $\Gamma(Z; V)$ be the set of all complex geodesics ϕ such that $Z \in \phi(\Delta)$ and that the tangent vector at Z is collinear to V. If $\phi \in \Gamma(0; V)$, replacing ϕ by $\phi \circ \lambda$, for a suitable choice of the Moebius transformation λ of Δ onto Δ, one

(1) The theorem was established in [3] when $\mathcal{E} = \mathcal{E}'$. However, the proof carries over, with only minor formal changes, to the case in which $\mathcal{E} \neq \mathcal{E}'$.

can assume that $\phi(0) = 0$. By the maximum principle $|||\phi'(0)||| = 1$. Moreover, if $(1/|||V|||)V$ is a complex extreme point of $\mathrm{cl}(D)$, then, by the strong maximum principle, ϕ is given by $\phi(\zeta) = \zeta\phi'(0)$. In particular, the support of ϕ is $\{\zeta V: |\zeta|\,|||V||| < 1\}$, and is thus uniquely determined by V. Viceversa, if all $\phi \in \Gamma(0; V)$ have the same support, then V is collinear to a complex extreme point of $\mathrm{cl}(D)$. That proves

LEMMA 2. *For all $V \in \mathcal{E}\backslash\{0\}$, $\Gamma(0; V) \neq \emptyset$. All elements of $\Gamma(0; V)$ have the same support if, and only if, $(1/|||V|||)V$ is a complex extreme point of D.*

Going back to the case in which $\mathcal{E}$ is a Cartan factor of type four, Corollary 1 and Lemma 2 yield

LEMMA 3. *For $V \in \mathcal{E}\backslash\{0\}$, all complex geodesics contained in $\Gamma(0; V)$ have the same support if, and only if, V is self-adjoint.*

Since the group $\mathrm{Aut}\, D$ of all holomorphic automorphisms of D acts transitively, then $\Gamma(Z; V) \neq \emptyset$ for all $Z \in D$ and all $V \in \mathcal{E}\backslash\{0\}$.

For $Z \in D$ and $V \in \partial D\backslash\{0\}$, let $\Xi(Z; V)$ be the set of all complex geodesics ϕ for which there is a holomorphic map ϑ of a neighbourhood of $\mathrm{cl}(\Delta)$ into $\mathcal{E}$, whose restriction to Δ is ϕ, and such that $Z \in \phi(\Delta)$, $V \in \vartheta(\partial\Delta)$.

If $V \in \partial_0 D$, then $\Gamma(0; V) = \Xi(0; V)$. Since every element $G \in \mathrm{Aut}\, D$ is the restriction to D of a bi-holomorphic map $\widehat{G}$ of a neighbourhood of $\mathrm{cl}\,(D)$ onto a neighbourhood of $\mathrm{cl}\,(D)$, which leaves $\partial_0 D$ invariant, then

$$\Xi(Z; V) = \widehat{G}(\Xi(0; \widehat{G}^{-1}(V)))$$

for all $Z \in D$, and all $V \in \partial_0 D$, where G is any holomorphic automorphism of D mapping 0 to Z.

In view of (5) and of Corollary 1, the holomorphic isometries for the Carathéodory metrics on D and D' can be characterized in terms of $\Xi(Z; V)$ and of the analogous sets $\Xi'(Z'; V')$ defined on D'. More specifically, (5) and Corollary 1 yield

THEOREM 2. *Let F be a holomorphic map of the Cartan domain of type four $D \subset \mathcal{E}$ into the Cartan domain of type four $D' \subset \mathcal{E}'$. Then, F is an isometry for the Carathéodory metrics of D and D' if, and only if, there is some $Z \in D$ such that $dF(Z)$ is injective and, for every $\phi \in \Xi(Z; V)$ with $V \in \partial_0 D$, $F \circ \phi \in \Xi(\phi(Z); V')$ for some $V' \in \partial_0 D'$.*

3. The hypotheses of Corollary 1 on the holomorphic map $F: D \to D'$ will now be weakened, assuming only that $F(0) = 0$, that $dF(0)$ is injective, and that, if $X \in \mathcal{E}$ is a self-adjoint unitary operator, then $F(tX) \in \mathcal{E}'$ is a self-adjoint operator for all t contained in a neighbourhood of 0 in $[0, 1)$. This latter condition can be rephrased by requiring that there exist $\varepsilon \in (0, 1]$ and $\gamma(t, X) \in \boldsymbol{R}$, such that

$$F(tX)^* = e^{i\gamma(t, X)} F(tX) \tag{6}$$

for all $t \in [0, \varepsilon)$.

Let

$$F(Z) = P_1(Z) + P_2(Z) + \dots \quad (Z \in D)$$

be the power series expansion of F in D, where $P_n : \mathcal{E} \to \mathcal{E}'$ is a continuous homogeneous polynomial of degree $n = 1, 2, \dots$.

The above condition amounts to requiring that, whenever $X^* = X \in \mathcal{E}$ and $\|X\| = 1$, then

$$P_n(X)^* = e^{i\gamma(t,X)} P_n(X)$$

for all $n = 1, 2, \dots$ and all $t \in [0, \varepsilon)$. Hence $\gamma(t, X)$ is independent of t.

Let $P_1 = dF(0) \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ be injective. Then, by Lemma 1, $\gamma(X)$ is independent of X, and therefore there is $\gamma \in \boldsymbol{R}$ such that

$$P_n(X)^* = e^{i\gamma} P_n(X)$$

for $n = 1, 2, \dots$ and for all self-adjoint operators $X \in \mathcal{E}$. The holomorphic function $Z \mapsto P_n(Z)^* - e^{i\gamma} P_n(Z^*)$ vanishes on $\mathrm{Re}\,\mathcal{E}$, and therefore vanishes identically on $\mathcal{E}$. Hence

$$F(Z)^* = e^{i\gamma} F(Z^*)$$

for all $Z \in D$.

If $W' \in \mathcal{E}'$ is orthogonal to $F(X)$ for all self-adjoint operators $X \in D$, then the scalar-valued holomorphic function $Z \mapsto (F(Z) \mid W')$ vanishes on $D \cap \mathrm{Re}\,\mathcal{E}$, and therefore vanishes on D, *i.e.* $W' \perp F(Z)$ for all $Z \in D$. Thus, if $W' \in \mathcal{E}' \setminus \{0\}$ and $W' \perp dF(0)X$ for all self-adjoint operators $X \in \mathcal{E}$ implies that $W' \perp F(X)$ for all self-adjoint operators $X \in D$, then, for every $Z \in D$, there is $f(Z) \in \boldsymbol{C}$ such that

$$F(Z) = f(Z)\, dF(0) Z . \tag{7}$$

Since $dF(0)$ has been assumed to be injective, for any $Z_0 \in D \setminus \{0\}$ there is a continuous linear form λ' on $\mathcal{E}'$ such that $\lambda'(dF(0)Z_0) \neq 0$. There is a neighbourhood U of Z_0 in D such that $\lambda'(dF(0)Z) \neq 0$ for all $Z \in U$. For $Z \in U$, $f(Z) = \lambda'(F(Z))/\lambda'(dF(0)Z)$, showing that f is a scalar-valued holomorphic function on D. Hence the following lemma holds.

LEMMA 4. *Let $F(0) = 0$ and let $dF(0)$ be injective. If, for all $W' \in \mathcal{E}' \setminus \{0\}$ such that $W' \perp dF(0)X$ for all self-adjoint operators $X \in \mathcal{E}$ one has that $W' \perp F(X)$ for all self-adjoint operators $X \in D$, then there is a holomorphic function $f : D \to \boldsymbol{C}$ such that $f(0) = 0$ and that* (7) *holds for all $Z \in D$.*

If $f(Z) = 1 + p_1(Z) + p_2(Z) + \dots$ is the power series expansion of f in D, where $p_n : \mathcal{E} \to \boldsymbol{C}$ is a continuous homogeneous polynomial of degree $n = 1, 2, \dots$, then $P_n(Z) = p_{n-1}(Z)\, dF(0)Z$, and

$$\overline{p_{n-1}(Z)}\, (dF(0)Z)^* = e^{i\gamma} p_{n-1}(Z^*)\, dF(0)(Z^*),$$

whence $p_{n-1}(Z) = \overline{p_{n-1}(Z^*)}$, and therefore $f(Z) = \overline{f(Z^*)}$ for all $Z \in D$. In particular, $f(X) \in \boldsymbol{R}$ for all self-adjoint operators $X \in D$.

THEOREM 3. *Let F be a holomorphic map of D into D' satisfying the following conditions:*

$F(0) = 0$ *and* $dF(0)$ *is injective;*

for every $X \in \partial_0 D$, *there is a neighbourhood* $U(X)$ *of* 0 *in* $[0, 1)$, *and, for all* $t \in U(X)$, *there exists* $\gamma(t, X) \in \boldsymbol{R}$ *such that* (6) *holds for all* $t \in [0, \varepsilon)$;

if $W' \perp dF(0)X$ *for all* $X \in \partial_0 D$, *then* $W' \perp F(tX)$ *for all* $X \in \partial_0 D$ *and all* $t \in U(X)$;

for any $X \in \partial_0 D$ *there is a scalar Moebius transformation of* Δ *into* $\boldsymbol{C}$ $\zeta \mapsto \alpha(X)\,\zeta/(1 - \beta(X)\,\zeta)$ *with* $\alpha(X)$, $\beta(X) \in \boldsymbol{C}$, *such that*

$$F(tX) = (\alpha(X)\,t/(1 - \beta(X)\,t))\,dF(0)X \quad \textit{for all } t \in U(X).$$

Then there is a continuous linear form μ *on* $\mathcal{E}$, *with norm* $\|\mu\| < 1$, *such that*

$$F(Z) = (1/(1 - \mu(Z)))\,dF(0)Z \quad \textit{for all } Z \in D.$$

PROOF. For $t \in U(X)$, $\alpha(X)t(1 + \beta(X)t + (\beta(X)t)^2 + \ldots) = t(1 + tp_1(X) + t^2 p_2(X) + \ldots)$. Hence $\alpha(X) = 1$, $p_n(X) = \beta(X)^n$ for $n = 1, 2, \ldots$, and therefore $p_n(Z) = (p_1(Z))^n$ for all $Z \in \mathcal{E}$.

Setting $\mu = p_1$, all is left to prove is that $\|p_1\| < 1$.

By the maximum principle and the fact that p_1 is real, $\|p_1\| = \sup\{|p_1(Z)| : Z \in D\} = \sup\{|p_1(X)| : X \in \partial_0 D\}$. If $\|p_1\| \geq 1$, for every $\sigma \in (0, 1)$ there is $X \in \partial_0 D$ such that $p_1(X) \geq 1 - \sigma$. Since $1 + (1 - \sigma)p_1(X) + ((1 - \sigma)p_1(X))^2 + \ldots \geq 1/(1 - (1 - \sigma)^2)$, $\|F((1 - \sigma)X)\|$ tends to infinity as σ tends to zero, contradicting the inclusion $F(D) \subset D'$.

REFERENCES

[1] L. A. HARRIS, *Bounded symmetric homogeneous domains in infinite dimensional spaces*. Lecture Notes in Mathematics, 634, Springer-Verlag, Berlin 1974, 13-40.

[2] E. VESENTINI, *Complex geodesics*. Compositio Math., 44, 1981, 375-394.

[3] E. VESENTINI, *Holomorphic isometries of Cartan domains of type four*. Rend. Mat. Acc. Lincei, s. 9, 3, 1992, 287-294.

[4] E. VESENTINI, *Holomorphic projective mappings*. Rend. Ist. Mat. Università di Trieste, 26, 1994, 297-308.

Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 PISA