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Angela Alberico, Vincenzo Ferone

Regularity properties of solutions of elliptic equations in $\mathbb{R}^2$ in limit cases

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Equazioni a derivate parziali. — *Regularity properties of solutions of elliptic equations in $\boldsymbol{R}^2$ in limit cases.* Nota di ANGELA ALBERICO e VINCENZO FERONE, presentata (*) dal Socio E. Magenes.

ABSTRACT. — In this paper the Dirichlet problem for a linear elliptic equation in an open, bounded subset of $\boldsymbol{R}^2$ is studied. Regularity properties of the solutions are proved, when the data are L^1-functions or Radon measures. In particular sharp assumptions which guarantee the continuity of solutions are given.

KEY WORDS: Elliptic equations; Lorentz spaces; Continuity properties.

RIASSUNTO. — *Proprietà di regolarità per soluzioni di equazioni ellittiche in $\boldsymbol{R}^2$ in casi limite.* In questa *Nota* si studia il problema di Dirichlet per un'equazione lineare ellittica in un insieme aperto, limitato di $\boldsymbol{R}^2$. Sono provate proprietà di regolarità per le soluzioni, quando i dati sono funzioni di L^1 oppure misure di Radon. In particolare sono date ipotesi ottimali che garantiscono la continuità delle soluzioni.

1. INTRODUCTION

Let Ω be an open bounded subset of $\boldsymbol{R}^n$, $n \geq 2$, and let $u \in H_0^1(\Omega)$ be solution of the problem

$$\begin{cases} -(a_{ij}(x)\, u_{x_j})_{x_i} + cu = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where a_{ij} are bounded functions satisfying:

$$a_{ij}(x)\,\xi_i\xi_j \geq |\xi|^2 \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \boldsymbol{R}^n\,, \tag{1.2}$$

and c satisfies the sign condition:

$$c(x) \geq 0\,. \tag{1.3}$$

In this paper we will give regularity properties of u in the case $n = 2$, under the hypothesis that f is in $L^1(\Omega)$ or is a Radon measure with bounded variation. We obtain results analogous to those valid when $n \geq 3$.

If $f \in L^p(\Omega)$, $p > n/2$, it is well known that u is Hölder-continuous. In the case $n \geq 3$ the simple continuity of u is guaranteed under the weaker assumption that f belongs to the Lorentz space $L(n/2, 1)$ (see [3, 13, 14]). In Section 2 we firstly give a similar result when $n = 2$. In particular we prove that if $f \in L \log L$ then u is bounded and continuous.

Using the above result and duality arguments, in Section 3 we prove that if f is a Radon measure with bounded variation, then, denoting by $|\Omega|$ the measure of Ω,

$$\int_\Omega e^{\beta|u|}\, dx \leq c\,|\Omega|\,, \tag{1.4}$$

for some $\beta > 0$ and c constant. Furthermore we give the optimal constants in (1.4) and

(*) Nella seduta del 3 ottobre 1995.

more precisely we show that if the total variation of f is less or equal to 1, then (1.4) holds for any $\beta < 4\pi$ and with $c = 4\pi/(4\pi - \beta)$. This result allows us to extend and to improve some properties of the solutions of the equation

$$-(a_{ij}(x)\, u_{x_j})_{x_i} = V(x)\, e^u ,$$

in two dimensions, found for example in [8, 10].

In Section 4 we study, still in the case $n = 2$, problem (1.1) when f belongs to intermediate spaces between L^1 and $L \log L$. We prove Moser-like estimates similar to those in [15, 4]. Under the hypothesis

$$\int_0^{|\Omega|} \left(\int_0^s f^*(t)\, dt \right)^p \frac{ds}{s} \leq 1 , \qquad 1 < p < \infty ,$$

where f^* is the decreasing rearrangement of f, we find β_p such that:

$$\int_\Omega e^{\beta_p |u|^{p'}} dx \leq c\,|\Omega| ,$$

where $c = c(p)$ is a constant. Such a value β_p is sharp.

Finally in Section 5 we give analogous estimates when $n \geq 3$ (see also [15]).

2. The case $f \in L \log L$

We briefly recall same notations which will be useful in the following. Let Ω be an open bounded subset of $\boldsymbol{R}^n$. If $\varphi\colon \Omega \to \boldsymbol{R}$ is a measurable function, we denote by $\mu_\varphi(t) = |\{x \in \Omega\colon |\varphi(x)| > t\}|$, the distribution function of φ and by $\varphi^*(s) = \sup\{t > 0\colon \mu_\varphi(t) > s\}$, the decreasing rearrangement of φ. We say that φ belongs to the space $L(\log L)^\alpha$, $0 < \alpha \leq 1$, if the quantity

$$\|\varphi\|_{L(\log L)^\alpha} = \int_0^{|\Omega|} \varphi^*(t) \left(\log \frac{|\Omega|}{t} \right)^\alpha dt$$

is finite (see for example [5]). When $\alpha = 1$ we will put $L \log L = L(\log L)^1$. We remark that, introducing the average function of φ^*:

$$\overline{\varphi}(s) = \frac{1}{s} \int_0^s \varphi^*(t)\, dt ,$$

we can write:

$$\|\varphi\|_{L \log L} = \int_0^{|\Omega|} \overline{\varphi}(s)\, ds .$$

Finally, we remind a result in [2] which allows us to estimate u by its gradient.

THEOREM 2.1. *If u is compactly supported in Ω, then*:

$$u^*(s) \leqslant (nC_n)^{-1/n} \|Du\|_n \left(\log \frac{|\Omega|}{s}\right)^{1-1/n}$$

where C_n is the measure of the unit sphere in $\boldsymbol{R}^n$.

We will consider the case $n = 2$. Using a well known result in [18] and Theorem 2.1, a solution $u \in H_0^1(\Omega)$ of (1.1) can be sharply estimated in terms of f.

THEOREM 2.2. *If $u \in H_0^1(\Omega)$ is the solution of* (1.1), *with $f \in L(\log L)^{1/2}$, then*:

$$u^*(s) \leqslant (4\pi)^{-1} \int_s^{|\Omega|} \bar{f}(t)\,dt, \qquad s \in]0, |\Omega|].$$

PROOF. Theorem 2.1 implies that if $f \in L(\log L)^{1/2}$ then f is in the dual of $H_0^1(\Omega)$. This means that one can use the arguments of the proof of Theorem 1 in [18]. ∎

An immediate consequence of this result is:

COROLLARY 2.3. *If $u \in H_0^1(\Omega)$ is the solution of* (1.1), *with $f \in L\log L$, then $u \in L^\infty(\Omega)$ and*:

$$\|u\|_\infty \leqslant (4\pi)^{-1} \|f\|_{L\log L}.$$

Using arguments similar to those in [3] and [13], by Theorem 2.2 one can obtain a continuity result for local solutions of (1.1), that is, for functions $u \in H^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega (a_{ij} u_{x_j} \varphi_{x_i} + cu\varphi) = \int_\Omega f\varphi, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{2.1}$$

THEOREM 2.4. *Let $u \in H^1_{\mathrm{loc}}(\Omega)$ satisfy* (2.1), *with $f \in L\log L$. Then u is continuous in Ω.*

PROOF. Let x_0 be in Ω and let us denote, for $\varrho > 0$, $B_\varrho(x_0) = \{x \in \boldsymbol{R}^2 : |x - x_0| < \varrho\}$. We take ϱ such that $B_{8\varrho}(x_0) \subset \Omega$, and we put $u = v + w$, where $v \in H_0^1(B_{8\varrho}(x_0))$ is the solution of the problem:

$$-(a_{ij}(x) v_{x_j})_{x_i} + cv = f, \qquad \text{in } B_{8\varrho}(x_0),$$

and w is the solution of the problem:

$$\begin{cases} -(a_{ij}(x) w_{x_j})_{x_i} + cw = 0 & \text{in } B_{8\varrho}(x_0), \\ w = u & \text{on } \partial B_{8\varrho}(x_0). \end{cases}$$

If $\overline{\omega}(w, \varrho)$ is the oscillation of $w(x)$ in $B_\varrho(x_0)$, then a constant $\eta < 1$ exists, such that (see [17, Lemma 7.3])

$$\overline{\omega}(w, \varrho) \leqslant \eta \overline{\omega}(w, 4\varrho). \tag{2.2}$$

Furthermore by Corollary 2.3,

$$\overline{\omega}(v,\varrho)\leqslant\overline{\omega}(v,4\varrho)\leqslant 2\sup_{B_{8\varrho}(x_0)}|v|\leqslant(2\pi)^{-1}\|f\|_{L\log L,B_{8\varrho}(x_0)}$$

where $\|f\|_{L\log L,B_{8\varrho}(x_0)}$ is the norm of f restricted to $B_{8\varrho}(x_0)$. Then, we have:

$$\overline{\omega}(v,\varrho)\leqslant F(\varrho), \tag{2.3}$$

where $F(\varrho)\to 0$ as $\varrho\to 0$. Then from (2.2) and (2.3) we get

$$\overline{\omega}(u,\varrho)\leqslant\eta\overline{\omega}(w,4\varrho)+F(\varrho)\leqslant\eta\overline{\omega}(u,4\varrho)+(\eta+1)F(\varrho)$$

and theorem follows. ■

If Ω is regular enough one can establish the same regularity up to the boundary of Ω. More precisely one can suppose, for example, that Ω is H_0^1-admissible (1) (see [17, Definition 6.2]). Indeed, in such a case inequality (2.2) holds also when $x_0\in\partial\Omega$ (see [17, Lemma 7.4]), where the oscillation is considered on $\Omega\cap B_\varrho(x_0)$. Then one can state the following:

THEOREM 2.5. *If Ω is H_0^1-admissible and $u\in H_0^1(\Omega)$ is a weak solution of* (1.1), *then u is continuous in $\overline{\Omega}$.*

REMARK 2.1. The results in Theorems 2.4 and 2.5 are sharp in the sense that if in (1.1) $f\in L\log L$ but $f\notin L^p$, $p>1$, then we can only say that the solution u of (1.1) is continuous but not Hölder-continuous. As an example let us consider the problem

$$\begin{cases}-\Delta u=f(|x|) & \text{in } B_R,\\ u=0 & \text{on } \partial B_R,\end{cases} \tag{2.4}$$

where $f(\varrho)=\pi^{-1}\varrho^{-2}(\log(1/(\pi\varrho^2)))^{-\beta}$, $\beta>2$, and B_R is a ball centered at the origin of radius R, such that the function $f^*(s)=1/(s(\log s^{-1})^\beta)$ is decreasing in $]0,|B_R|]$. It is easy to recognize that $f^*(s)$ is the decreasing rearrangement of f and that the solution of (2.4) can be written as:

$$u(x)=\frac{1}{4\pi(\beta-1)(\beta-2)}\left[\left(\log\frac{1}{|B_R|}\right)^{2-\beta}-\left(\log\frac{1}{\pi|x|^2}\right)^{2-\beta}\right].$$

It follows that u is simply continuous in B_R. On the other hand it is easy to verify that $f\in L\log L$ but $f\notin L^p$, $\forall p>1$.

REMARK 2.2. Theorems 2.4 and 2.5 are sharp also because, at least in the $L(\log L)^\alpha$ scale, the hypothesis $f\in L\log L$ cannot be weakened. If one considers again problem

(1) We recall that an open bounded set $\Omega\subset\mathbf{R}^n$ is H_0^1-admissible if there exist two positive constants α and ϱ_0 such that for $0<\varrho<\varrho_0$ and $x_0\in\partial\Omega$, the following inequality holds, for every $u\in C^1(\overline{\Omega\cap B_\varrho(x_0)})$ vanishing on $\partial\Omega$:

$$|u(x)|\leqslant\alpha\int_{\Omega\cap B_\varrho(x_0)}|u_x(t)|\,|x-t|^{1-n}\,dt.$$

(2.4) with $f(\varrho) = \pi^{-1}\varrho^{-2}(\log(1/(\pi\varrho^2)))^{-2}$ one can easily check that the solution

$$u(x) = \frac{1}{4\pi}\left[\log\left(\log\frac{1}{\pi|x|^2}\right) - \log\left(\log\frac{1}{|B_R|}\right)\right]$$

is not continuous at the origin. This time $f \notin L\log L$, but $f \in L(\log L)^\alpha$, $\forall \alpha$ such that $0 < \alpha < 1$.

3. The case f is a measure

In the following we suppose that $\Omega \subset \boldsymbol{R}^2$ is smooth (for example H_0^1-admissible) and that (1.2), (1.3) hold. If μ is a Radon measure with bounded variation, supported in Ω, we say (see [17]) that $u \in L^1(\Omega)$ is a weak solution of

$$-(a_{ij}u_{x_j})_{x_i} + cu = \mu\,, \tag{3.1}$$

vanishing on $\partial\Omega$, if

$$\int_\Omega u[-(a_{ji}\varphi_{x_j})_{x_i} + c\varphi]\,dx = \int_\Omega \varphi\,d\mu \tag{3.2}$$

for any $\varphi \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$ such that $[-(a_{ji}\varphi_{x_j})_{x_i} + c\varphi] \in C^0(\overline{\Omega})$. Using the results of Section 2 we can prove:

Theorem 3.1. *If μ is a Radon measure of bounded variation and*

$$\int_\Omega |d\mu| \leqslant 1\,,$$

equation (3.1) *admits a weak solution u vanishing on $\partial\Omega$, such that*

$$\int_\Omega e^{\beta|u(x)|}\,dx \leqslant \frac{4\pi}{4\pi - \beta}|\Omega|\,, \qquad \forall\beta < 4\pi\,. \tag{3.3}$$

Proof. By definition, if $u \in L^1(\Omega)$ is a weak solution of (3.1), vanishing on $\partial\Omega$, then (3.2) holds. Then, for any ψ continuous on $\overline{\Omega}$, we have:

$$\int_\Omega u\psi\,dx = \int_\Omega G(\psi)\,d\mu\,,$$

where $\varphi = G(\psi)$ denotes the solution of the problem:

$$\begin{cases} -(a_{ji}\varphi_{x_j})_{x_i} + c\varphi = \psi & \text{in } \Omega\,, \\ \varphi = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By Theorem 2.2 we get:

$$\left|\int_\Omega u\psi\,dx\right| \leqslant \frac{\|\psi\|_{L\log L}}{4\pi}\int|d\mu| \leqslant \frac{1}{4\pi}\|\psi\|_{L\log L}\,. \tag{3.4}$$

Taking into account the fact that $C^0(\overline{\Omega})$ is dense in $L\log L$, (3.4) implies that the linear functional $A(\psi) = \int_\Omega u\psi\,dx$ is continuous on $L\log L$. This means that u belongs to $L_{\exp}$, *i.e.*

$$\int_\Omega e^{\lambda |u|}\, dx < \infty ,$$

for a certain $\lambda > 0$. Furthermore (3.4) gives, for any $q \in \mathbf{N}$,

$$|A(\psi)| \leqslant (4\pi)^{-1} \int_0^{|\Omega|} \psi^*(t) \log(|\Omega|/t)\, dt \leqslant$$

$$\leqslant (4\pi)^{-1} \|\psi\|_{q/(q-1)} \left(\int_0^{|\Omega|} [\log(|\Omega|/t)]^q\, dt \right)^{1/q} = (4\pi)^{-1} (|\Omega| q!)^{1/q} \|\psi\|_{q/(q-1)} .$$

Then

$$\|u\|_q \leqslant (|\Omega| q!)^{1/q}/(4\pi).$$

It follows:

$$\int_\Omega e^{\beta |u|}\, dx = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \|u\|_k^k \leqslant |\Omega| \sum_{k=0}^{\infty} \left(\frac{\beta}{4\pi}\right)^k . \qquad \blacksquare$$

REMARK 3.1. Theorem 3.1 improves analogous results in [8] (where the case $a_{ij} = \delta_{ij}$ and $\mu \in L^1(\Omega)$ is considered). For example in [8] the following inequality is found:

$$\int_\Omega e^{\beta |u(x)|}\, dx \leqslant \frac{4\pi^2}{4\pi - \beta} (\operatorname{diam} \Omega)^2 , \qquad \forall \beta < 4\pi .$$

Furthermore (3.3) is sharp because it holds as an equality when μ is a Dirac mass concentrated at the origin and Ω is a ball centered at the origin.

In general, under the hypotheses of Theorem 3.1, the integral

$$\int_\Omega e^{\beta |u(x)|}\, dx \tag{3.5}$$

does not need to be finite if $\beta \geqslant 4\pi$. It is enough to consider the case μ is a Dirac mass concentrated at the origin and Ω is a ball centered at the origin. On the other hand if μ is an L^1-function, proceeding as in the proof of Corollary 1 in [8], one easily obtains:

COROLLARY 3.2. *Let u be solution of* (3.1), *vanishing on $\partial\Omega$, with $\mu \in L^1(\Omega)$. Then for every $\beta > 0$ we have $e^{\beta|u|} \in L^1(\Omega)$.*

SKETCH OF THE PROOF. Let us split $\mu \in L^1(\Omega)$ as $\mu = \mu_1 + \mu_2$, with $\|\mu_1\|_1 < \varepsilon$, $\varepsilon > 0$, and $\mu_2 \in L^\infty(\Omega)$. Obviously $u = u_1 + u_2$, where u_1 and u_2 are the solutions vanishing on $\partial\Omega$ of (3.1), with μ substituted by μ_1 and μ_2, respectively. The function u_2 is bounded, while, applying Theorem 3.1 to the function u_1, we have:

$$\int_\Omega e^{|u_1(x)|/\varepsilon}\,dx < +\infty\,.$$

Because of the arbitrariness of ε the assert follows. ■

REMARK 3.2. As already observed in [8], in the case $\mu \in L^1(\Omega)$ it is not possible to bound the integral (3.5) independently of μ, when $\beta \geq 4\pi$. In this respect Theorem 3.1 extends the result in [8] and improves an analogous result in [10] where only the existence of $\overline{\beta} > 0$ such that

$$\int_\Omega e^{\overline{\beta}|u(x)|}\,dx \leq c(\operatorname{diam}\Omega)^2\,,$$

with a suitable constant c, is proved.

Now we state some corollaries of Theorem 3.1. They can be proved essentially as in [8], where the case $a_{ij} = \delta_{ij}$ is considered. In particular we can recover all the results contained in Section 3 of [10] with the difference that in certain cases we are able to make sharp hypotheses. For the sake of brevity we only consider such cases.

COROLLARY 3.3. *Assume* $(u_m) \subset L^1(\Omega)$ *is a sequence of solutions of*

$$-(a_{ij}(u_m)_{x_j})_{x_i} + cu_m = V_m(x)e^{u_m} \quad \text{in } \Omega\,, \tag{3.6}$$

with $u_m = 0$ *on* $\partial\Omega$, $c(x) \geq 0$, *such that*

$$\|V_m\|_{L^p} \leq C \quad \text{for some } 1 < p \leq \infty \tag{3.7}$$

and

$$\int_\Omega |V_m|e^{u_m}\,dx \leq \varepsilon_0 < 4\pi/p' \quad \forall m\,. \tag{3.8}$$

Then $\|u_m\|_{L^\infty} \leq C$.

COROLLARY 3.4. *Assume* $(u_m) \subset L^1(\Omega)$ *is a sequence of solutions of* (3.6) *such that, for some* $1 < p \leq \infty$,

$$\|V_m\|_{L^p} \leq C\,, \qquad \|u_m^+\|_{L^1} \leq C\,,$$

and (3.8) *holds. Then* (u_m^+) *is bounded in* $L^\infty_{\mathrm{loc}}(\Omega)$.

As already observed, the proof of the above corollaries follows the arguments in [8]. We only sketch the proof of Corollary 3.3.

PROOF OF COROLLARY 3.3. Let $\beta < 4\pi$ be such that $\beta > \varepsilon_0(p' + 4\pi - \beta)$. By Theorem 3.1 and (3.8) we have:

$$\int_\Omega e^{(p'+4\pi-\beta)|u_m(x)|}\,dx \leq C(p,\beta)\,.$$

Therefore e^{u_m} is bounded in $L^{p'+4\pi-\beta}(\Omega)$ and, because of (3.7), $V_m e^{u_m}$ is bounded in $L^q(\Omega)$, for some $q > 1$. Using standard estimates we obtain that u_m is bounded in $L^\infty(\Omega)$. ∎

REMARK 3.3. Corollaries 3.3 and 3.4 improve similar results in [10]. In fact in [10] it is only proven that under hypothesis (3.7) there exists $\varepsilon_0 > 0$ such that if

$$\int_\Omega |V_m| e^{u_m(x)}\, dx < \varepsilon_0 ,$$

then the boundedness of u_m or u_m^+ in L^∞ holds. On the other hand, as already observed in [8] (in the case $a_{ij} = \delta_{ij}$), hypothesis (3.8) is optimal. In fact, if $a_{ij} = \delta_{ij}$ and

$$\int_\Omega |V_m| e^{u_m}\, dx = 4\pi/p' , \qquad \forall m , \tag{3.9}$$

then one can construct a sequence (u_m) of solutions of (3.6) satisfying (3.7) and (3.9) such that $\|u_m\|_{L^\infty} \to \infty$.

4. THE CASE $f \in L^1$

In this section we will consider solutions of (1.1) in the case $f \in L^1(\Omega)$. We still suppose that $\Omega \subset \boldsymbol{R}^2$ is a smooth, bounded, open set. As in the previous section, a weak solution of (1.1), vanishing on $\partial\Omega$, is a function $u \in L^1(\Omega)$ such that

$$\int_\Omega [-(a_{ji}\,\varphi_{x_j})_{x_i} + c\varphi]\, u\, dx = \int_\Omega \varphi f\, dx \tag{4.1}$$

for any $\varphi \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$ such that $[-(a_{ji}\,\varphi_{x_j})_{x_i} + c\varphi] \in C^0(\overline{\Omega})$. A comparison result as Theorem 2.2 can be stated also for solutions of (4.1).

THEOREM 4.1. *Let $u \in L^1(\Omega)$ be solution of* (4.1) *under the assumptions* (1.2), (1.3), *and $f \in L^1(\Omega)$. The following estimate holds*:

$$u^*(s) \leq (4\pi)^{-1} \int_s^{|\Omega|} \bar{f}(t)\, dt , \qquad s \in]0, |\Omega|] . \tag{4.2}$$

We soon observe that under the hypotheses (1.2) and (1.3) the problem (4.1) admits a unique solution (see [17]).

PROOF. Let $f_m \in L^\infty(\Omega)$ be a sequence of functions such that $f_m \to f$ in $L^1(\Omega)$ and $\|f_m\|_1 \leq \|f\|_1$. If $u_m \in H_0^1(\Omega)$ is solution of (1.1) with f replaced by f_m, it is well known that, passing to a subsequence (still denoted by u_m), $u_m \to u$ in L^1 and a.e. (see *e.g.*, [17]). By Theorem 2.2 the following estimate holds:

$$u_m^*(s) \leq v_m^*(s) = (4\pi)^{-1} \int_s^{|\Omega|} \bar{f}_m(t)\, dt , \qquad \forall s \in]0, |\Omega|] .$$

The function $v_m^*(s)$ is the decreasing rearrangement of the solution v_m of the problem:

$$\begin{cases} -\Delta v_m = f_m^\# & \text{in } \Omega^\# , \\ v_m = 0 & \text{on } \partial\Omega^\# , \end{cases}$$

where $f_m^\#(x) = f_m^*(\pi|x|^2)$ is the spherically decreasing rearrangement of f_m and $\Omega^\#$ is the circle centered at the origin such that $|\Omega^\#| = |\Omega|$. The fact that $f_m \to f$ in L^1 implies (see for example [11]) that $f_m^\# \to f^\#$ in $L^1(\Omega^\#)$. Using the arguments above it follows that (passing to a subsequence) $v_m \to v$ in $L^1(\Omega^\#)$ and a.e., where v is the solution of the problem

$$\begin{cases} -\Delta v = f^\# & \text{in } \Omega^\# , \\ v = 0 & \text{on } \partial\Omega^\# . \end{cases}$$

Being $\Omega^\#$ a circle, it is possible to write explicitely v and v^*:

$$v(x) = (4\pi)^{-1} \int_{\pi|x|^2}^{|\Omega|} \bar{f}(t)\,dt , \qquad v^*(s) = (4\pi)^{-1} \int_s^{|\Omega|} \bar{f}(t)\,dt .$$

Using, for example, Proposition 4.3 in [12], one deduces:

$$u^*(s) \leqslant v^*(s) , \qquad \forall s \in]0, |\Omega|],$$

that is the assert. ∎

Remark 4.1. Obviously Theorem 4.1 recovers and sharpens Theorem 3.1 in the case the right-hand side in (3.1) is in $L^1(\Omega)$. Indeed, if $f \in L^1(\Omega)$ and $\|f\|_1 \leqslant 1$, the estimate (4.2) gives

$$u^*(s) \leqslant (4\pi)^{-1} \log(|\Omega|/s) , \qquad s \in]0, |\Omega|],$$

that is (3.3).

Remark 4.2. It is possible to define a weak solution of (1.1) when $f \in L^1(\Omega)$ for example as a function $u \in W_0^{1,1}(\Omega)$ such that (see, for example, [9, 6]):

$$\int_\Omega (a_{ij} u_{x_j} \varphi_{x_i} + cu\varphi)\,dx = \int_\Omega f\varphi\,dx , \qquad \forall \varphi \in W_0^{1,\infty}(\Omega) .$$

It is well known (see [16]) that in such a case the solution does not need to be unique. This means that Theorem 4.1 cannot be proven for this kind of weak solutions, unless the solution itself is obtained as «limit of approximations». A definition of solution having this property is contained, for example, in [6, 7] and we remind that in our case such a solution is unique.

In the same spirit of Section 4 in [4], the above result can be used to obtain sharp Moser-like estimates for solutions of (4.1) when f belongs to spaces which are intermediate between $L^1(\Omega)$ and $L \log L$. Let us define the spaces $L(1,p)$, $1 \leqslant p \leqslant \infty$, as the set of the functions $\varphi \in L^1(\Omega)$ such that (see [5]):

$$\|\varphi\|_{1,p} = \begin{cases} \left(\int_0^{|\Omega|} \left(\int_0^s \varphi^*(t)\, dt \right)^p \frac{ds}{s} \right)^{1/p} & \text{if } 1 \leqslant p < \infty , \\ \|\varphi\|_1 & \text{if } p = \infty \end{cases}$$

is finite. The following inclusions hold: $L \log L = L(1, 1) \subset L(1,p) \subset L(1,q) \subset \subset L(1, \infty) = L^1(\Omega)$, for $1 < p < q < \infty$.

The cases $p = 1$ and $p = \infty$ have been treated in Sections 2 and 3 respectively. In the intermediate cases we have the following:

THEOREM 4.2. *Let $u \in L^1(\Omega)$ be solution of* (4.1) *under the assumptions* (1.2), (1.3), *with $f \in L(1,p)$, $1 < p < \infty$ and $\|f\|_{1,p} \leqslant 1$. Then a constant $c_0 = c_0(p)$ exists such that*:

$$\int_\Omega e^{\beta |u(x)|^{p'}} dx \leqslant c_0 |\Omega| , \qquad \forall \beta \leqslant \beta_p = (4\pi)^{p'} ,$$

where, as usual, $p' = p/(p-1)$.

Before proving Theorem 4.2 we recall a technical lemma due to Adams [1].

LEMMA 4.3. *Let $1 < p < \infty$ and let $a(s,t)$ be a non-negative measurable function on $(-\infty, +\infty) \times [0, +\infty)$ such that*:

$$a(s,t) \leqslant 1 , \quad \textit{for a.e. } 0 < s < t , \tag{4.3}$$

$$\sup_{t > 0} \left(\int_{-\infty}^0 a(s,t)^{p'} ds + \int_t^\infty a(s,t)^{p'} ds \right)^{1/p'} = b < \infty . \tag{4.4}$$

Then there is a constant $c_0 = c_0(p, b)$ such that if for $\phi \geqslant 0$,

$$\int_{-\infty}^{+\infty} \phi(s)^p ds \leqslant 1 , \tag{4.5}$$

then

$$\int_0^\infty e^{-F(t)} dt \leqslant c_0 , \tag{4.6}$$

where

$$F(t) = t - \left(\int_{-\infty}^{+\infty} a(s,t)\, \phi(s)\, ds \right)^{p'} . \tag{4.7}$$

PROOF OF THEOREM 4.2. By Theorem 4.1, putting $s = |\Omega|e^{-t}$ in (4.2) we have

$$u^*(|\Omega|e^{-t}) \leq (4\pi)^{-1}\int_0^t \bar{f}(|\Omega|e^{-r})|\Omega|e^{-r}\,dr = (4\pi)^{-1}\int_{-\infty}^{+\infty} a(s,t)\,\phi(s)\,ds\,, \tag{4.8}$$

where

$$a(s,t) = \begin{cases} 0 & \text{if } -\infty < s < 0\,, \\ 1 & \text{if } 0 \leq s \leq t\,, \\ 0 & \text{if } t < s < +\infty \end{cases}$$

and

$$\phi(s) = \begin{cases} 0 & \text{if } -\infty < s < 0\,, \\ \bar{f}(|\Omega|e^{-s})|\Omega|e^{-s} & \text{if } 0 \leq s \leq t\,, \\ 0 & \text{if } t < s < +\infty\,. \end{cases}$$

Now we observe that $a(s,t)$ and $\phi(s)$ satisfy conditions (4.3), (4.4), (4.5). Then (4.8) and Lemma 4.3 imply that:

$$\int_0^{+\infty} e^{(4\pi u^*(|\Omega|e^{-t}))^{p'} - t}\,dt \leq c_0\,,$$

where $c_0 = c_0(p)$ is a constant. In other words

$$\int_0^{|\Omega|} e^{(4\pi u^*(s))^{p'}}\,ds \leq c_0\,|\Omega|$$

and theorem follows. ■

REMARK 4.3. Theorem 4.1 is sharp because if $\beta > \beta_p$ then in general the integral

$$\int_\Omega e^{\beta|u(x)|^{p'}}\,dx$$

is finite but it cannot be bounded by a constant which is independent of f. In other words it is possible to find a sequence $(f_k)_{k\in N}$ such that $f_k \in L(1,p)$, $\|f_k\|_{1,p} \leq 1$, and the corresponding sequence of solutions $(u_k)_{k\in N}$ of (1.1) is such that, if $\beta > \beta_p$,

$$\lim_{k\to+\infty}\int_\Omega e^{\beta|u_k(x)|^{p'}}\,dx = +\infty\,. \tag{4.9}$$

As an example one can choose

$$f_k(x) = f_k^{\#}(x) = \begin{cases} 0 & \text{if } e^{-k} \leq \pi|x|^2 \leq 1\,, \\ e^k(1/p + k)^{-1/p} & \text{if } 0 \leq \pi|x|^2 < e^{-k} \end{cases}$$

in the ball B_R centered at the origin and such that $|B_R| = 1$. The solution u_k of the problem

$$\begin{cases} -\Delta u_k = f_k & \text{in } B_R\,, \\ u_k = 0 & \text{on } \partial B_R\,, \end{cases}$$

is given by

$$u_k(x) = \begin{cases} \log(1/\pi|x|^2)/(4\pi(1/p+k)^{1/p}) & \text{if } e^{-k} \leqslant \pi|x|^2 \leqslant 1\,, \\ (k+1-e^k\pi|x|^2)/(4\pi(1/p+k)^{1/p}) & \text{if } 0 \leqslant \pi|x|^2 < e^{-k}\,. \end{cases}$$

It is easy to check that $\|f_k\|_{1,p} = 1$. On the other hand, observing that $u_k(x) = u_k^*(\pi|x|^2)$, we have:

$$\int_{B_R} e^{\beta|u_k(x)|^{p'}}\,dx = \int_0^1 e^{\beta(u_k^*(s))^{p'}}\,ds \geqslant$$

$$\geqslant \int_0^{e^{-k}} \exp\left(\frac{\beta(k+1-se^k)^{p'}}{(4\pi)^{p'}(1/p+k)^{p'/p}}\right)ds \geqslant \int_0^{e^{-k}} \exp\left(\frac{\beta k^{p'}}{(4\pi)^{p'}(1/p+k)^{p'/p}}\right)ds =$$

$$= \exp\left\{\left(\frac{k^p}{k+1/p}\right)^{p'/p}\left[\frac{\beta}{(4\pi)^{p'}} - \left(1+\frac{1}{pk}\right)^{p'/p}\right]\right\}.$$

Then, if $\beta > \beta_p = (4\pi)^{p'}$, (4.9) holds.

5. Some extensions to higher dimensions

Results analogous to those contained in the previous sections can be obtained also in the case $n \geqslant 3$. First of all we observe that, following the arguments in Section 4, one can extend to any dimension Theorem 4.1. In particular we have:

Theorem 5.1. *Let Ω be an open bounded subset of $\mathbf{R}^n$, $n \geqslant 3$, and let $u \in L^1(\Omega)$ be a solution of* (4.1), *vanishing on $\partial\Omega$, under the assumptions* (1.2), (1.3), $f \in L^1(\Omega)$. *Then the following estimate holds*:

$$u^*(s) \leqslant n^{-2} C_n^{-2/n} \int_s^{|\Omega|} \bar{f}(t)\, t^{2/n}\,\frac{dt}{t}\,, \qquad \forall s \in \,]0, |\Omega|]\,. \tag{5.1}$$

The above comparison result extends Theorem 1 in [18]. Using it (actually it is enough Talenti's version) one can prove Moser-like estimates in the same spirit of Section 4.

If $L(p,q)$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, denotes the set of the measurable functions φ

such that (see [5]):

$$\|\varphi\|_{p,q} = \begin{cases} \left(\int_0^{+\infty} (\overline{\varphi}(s) s^{1/p})^q \frac{ds}{s} \right)^{1/q} & \text{if } 1 \leqslant q < \infty , \\ \sup_{s>0} \overline{\varphi}(s) s^{1/p} & \text{if } q = \infty \end{cases}$$

is finite, we can prove the following:

THEOREM 5.2. *Let Ω be an open bounded subset of $\boldsymbol{R}^n$, $n \geqslant 3$, and let $u \in H_0^1(\Omega)$ be a solution of* (1.1) *under the assumptions* (1.2), (1.3), $f \in L(n/2, p)$, *with* $1 \leqslant p \leqslant \infty$. *We have*:

a) *if* $p = 1$, *then* $u \in L^\infty(\Omega)$ *and* $\|u\|_\infty \leqslant n^{-2} C_n^{-2/n} \|f\|_{n/2,1}$;

b) *if* $1 < p < \infty$ *and* $\|f\|_{n/2,p} \leqslant 1$, *then a constant* $c = c(p)$ *exists such that*:

$$\int_\Omega e^{\beta |u|^{p'}} dx \leqslant c |\Omega| , \qquad \forall \beta \leqslant \beta_{n,p} = (n^2 C_n^{2/n})^{p'} ;$$

c) *if* $p = \infty$ *and* $\|f\|_{n/2,\infty} \leqslant 1$, *then a constant* $c = c(p, \beta)$ *exists such that*:

$$\int_\Omega e^{\beta |u|} dx \leqslant c |\Omega| , \qquad \forall \beta < \beta_{n,\infty} = n^2 C_n^{2/n} .$$

PROOF. Parts *a*) and *c*) are direct consequences of (5.1) (see also [3]). Part *b*) can be proved using arguments similar to those in the proof of Theorem 4.2. One has only to put:

$$a(s,t) = \begin{cases} 0 & \text{if } -\infty < s < 0 , \\ 1 & \text{if } 0 \leqslant s \leqslant t , \\ 0 & \text{if } t < s < +\infty \end{cases}$$

and

$$\phi(s) = \begin{cases} 0 & \text{if } -\infty < s < 0 , \\ \overline{f}(|\Omega| e^{-s})(|\Omega| e^{-s})^{2/n} & \text{if } 0 \leqslant s \leqslant t , \\ 0 & \text{if } t < s < +\infty . \end{cases}$$ ■

REMARK 5.1. Part *b*) of Theorem 5.2 is sharp in the same sense Theorem 4.1 is. As in Remark 4.3 (see also [4]) one can find a sequence $(f_k)_{k \in \boldsymbol{N}}$ of functions such that $f_k \in L(n/2, p)$, $\|f_k\|_{n/2,p} \leqslant 1$, and the corresponding sequence of solutions $(u_k)_{k \in \boldsymbol{N}}$ of (1.1) is such that, if $\beta > \beta_{n,p}$, then

$$\lim_{k \to +\infty} \int_\Omega e^{\beta |u_k(x)|^{p'}} dx = +\infty .$$

The sharpness of parts *a*) and *c*) of Theorem 5.2 has already been exploited in [3].

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Dipartimento di Matematica e Applicazioni «R. Caccioppoli»
Università degli Studi di Napoli «Federico II»
Complesso Monte S. Angelo - Edificio T
Via Cintia - 80126 Napoli