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Boundary integral equations of the logarithmic potential theory for domains with peaks

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Analisi matematica. — *Boundary integral equations of the logarithmic potential theory for domains with peaks.* Nota (*) di VLADIMIR MAZ'YA e ALEXANDER A. SOLOVIEV, presentata dal Socio G. Fichera.

ABSTRACT. — Integral equations of boundary value problems of the logarithmic potential theory for a plane domain with several peaks at the boundary are studied. We present theorems on the unique solvability and asymptotic representations for solutions near peaks. We also find kernels of the integral operators in a class of functions with a weak power singularity and describe classes of uniqueness.

KEY WORDS: Boundary integral equation; Logarithmic potential; Asymptotics of solution.

RIASSUNTO. — *Equazioni integrali al contorno della teoria del potenziale logaritmico per domini con cuspidi*. Vengono studiate le equazioni integrali dei problemi al contorno della teoria del potenziale logaritmico per un dominio piano con diverse cuspidi sul contorno. Vengono presentati teoremi sull'unicità della soluzione e sulle rappresentazioni asintotiche delle soluzioni in prossimità delle cuspidi. Vengono anche considerati nuclei di operatori integrali in una classe di funzioni con singolarità debole e descritte le classi per l'unicità della soluzione.

1. INTRODUCTION

1.1. A classical method for solving Dirichlet and Neumann boundary value problems for the Laplace equation is the representation of their solutions in the form of double layer potentials $W\sigma$ and simple layer potentials $V\tau$. For the internal Dirichlet problem and for the external Neumann problem the densities of the corresponding potentials can be found from the boundary integral equations

$$-\pi\sigma + W\sigma = g \tag{1}$$

and

$$-\pi\tau + \frac{\partial}{\partial n} V\tau = h\,, \tag{2}$$

respectively, where $\partial/\partial n$ is the derivative with respect to the outward normal to the contour S. Equations (1) and (2) for domains with non-zero angles, *i.e.* without peaks, were studied by many authors in various function spaces by methods of the Fredholm operator theory. (For a historical survey and a bibliography see [1]).

In this paper we develop a theory of equations (1) and (2) on contours with several peaks. Since in the presence of peaks the Fredholm theory is not applicable (cf. [2]), we use another approach proposed by one of the authors (cf. [1]) which is based on representations of solutions to (1) and (2) by means of solutions to certain auxiliary boundary value problems. We obtain conditions for solvability of (1) and (2), find classes of uniqueness, and describe kernels of the integral operators in a certain class $\mathfrak{M}$ of functions with a weak power singularity. We also give asymptotic formulae for solutions of (1) and (2) near peaks. Such formulae were obtained in our papers [3, 4] but

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the present proof is independent and simpler. We restrict ourselves to contours with peaks of first order tangency. This requirement is unimportant for the method, but facilitates calculations.

We give a short qualitative description of our results concerning equations (1) and (2).

We show that the number of linearly independent solutions in $\mathfrak{M}$ of the homogeneous equation (1) is equal to the number of outward peaks. We prove that (1) is solvable in $\mathfrak{M}$ provided the right-hand side belongs to a certain class $\mathfrak{N}$ of continuous functions with prescribed asymptotics near peaks. We give an example of equation (1) with continuous right-hand side on the contour with exterior peak, which is unsolvable in the class $\mathfrak{M}$. In the presence of exterior peaks we achieve the unique solvability of (1) reducing both classes of solutions and right-hand sides. These new smaller classes will be denoted by $\mathfrak{M}_{\text{ext}}$ and $\mathfrak{N}_{\text{ext}}$ respectively.

We turn to equation (2). It appears that the presence of peaks does not violate the uniqueness in the class $\mathfrak{M}$. If the contour has no exterior peaks, equation (2) is solvable in $\mathfrak{M}$ for an arbitrary $h \in \mathfrak{N}$ with zero mean value. If S has exterior peaks the solvability in $\mathfrak{M}$ holds under the orthogonality of h to zeros of (1) from the class $\mathfrak{M}$. Therefore for the contour which contains exterior peaks it is preferable to express a solution of the exterior Neumann problem as the sum of $V\tau$ and a linear combination of explicitly written functions. The resulting integral equation proves to be solvable in $\mathfrak{M}$.

We introduce our basic notation.

We consider a plane simply connected domain Ω with compact closure bounded by the piecewise C^∞-smooth contour S. Let S have the outward peaks e_n, $1 \leq n \leq N$, and the inward peaks i_m, $1 \leq m \leq M$. The set of all peaks will be denoted by T.

To each peak z_0 we attach a Cartesian coordinate system, in which either Ω or its complementary domain Ω^c are given by the inequalities $\kappa_-(x) < y < \kappa_+(x)$, $0 < x < \delta$, where $\kappa_\pm$ are C^∞-functions on $[0, \delta]$ satisfying conditions: $\kappa_\pm(0) = \kappa'_\pm(0) = 0$, $\kappa''_+(0) > \kappa''_-(0)$. The arcs $\{(x, \kappa_\pm(x)) : x \in [0, \delta]\}$ will be denoted by $S_\pm(z_0)$.

The above mentioned classes $\mathfrak{M}$, $\mathfrak{M}_{\text{ext}}$ and $\mathfrak{N}$, $\mathfrak{N}_{\text{ext}}$ of solutions and right-hand sides respectively are defined as follows.

By $\mathfrak{M}$ we denote the class of C^∞-functions on $S \setminus T$ such that

$$\sigma(z) = O((z - z_0)^{\beta(z_0)}), \qquad \beta(z_0) > -1,$$

for each peak z_0.

The subset $\mathfrak{M}_{\text{ext}}$ of $\mathfrak{M}$ is defined by the additional condition $\beta(e_p) > -1/2$ for exterior peaks e_p, $p = 1, \ldots, N$.

Let $\mathfrak{N}$ denote the class of functions on S admitting the representation

$$\varphi_\pm(x) = x^{\nu(z_0)} \psi_\pm(x) \qquad \text{on } S_\pm(z_0)$$

for each peak z_0. Here $\psi_\pm$ are C^∞-functions on $[0, \delta]$, $|\psi_+(0)| + |\psi_-(0)| \neq 0$, and $\nu(z_0) > 0$.

The subclass of $\mathfrak{N}$ with $\nu(e_p) > 1/2$, $p = 1, \ldots, N$ will be denoted by $\mathfrak{N}_{\text{ext}}$.

Now we are in a position to give a more precise account of our results.

Let $u^{(i)}$ denote the solution of the internal Dirichlet problem $\mathcal{D}^{(i)}$

$$\Delta u^{(i)} = 0 \quad \text{in } \Omega, \qquad u^{(i)} = g \quad \text{on } S, \ g \in \mathfrak{N}.$$

If we are seeking $u^{(i)}$ in the form of a double layer potential $W\sigma(z)$ then the density σ will be found from the integral equation (1) valid on $S \setminus T$. The kernel of the operator $\pi I - W$ in the space $\mathfrak{M}$ consists of linear combinations of the functions $\mathrm{Re}(1/\zeta_k)$, where ζ_k is the conformal mapping of Ω^c onto the upper half-plane subjected to the conditions

$$\zeta_k(e_k) = 0, \ \ \mathrm{Re}\,\zeta_k(\infty) = 0 \quad \text{and} \quad \mathrm{Re}(1/\zeta_k)(z) = \pm x^{-1/2} + O(1)$$

on $S_{\pm}(e_k)$ in the local coordinate system (see Theorem 1, Sect. 2.1). The solvability of (1) with right-hand side $g \in \mathfrak{N}$ in the class $\mathfrak{M}$ is proved in Theorem 2 (Sect. 2.2). There we also study the asymptotic behaviour of the solution near the peak. In the same Section we prove Theorem 3 on the unique solvability of equation (1) in the class $\mathfrak{M}_{\mathrm{ext}}$ and with right-hand side from $\mathfrak{N}_{\mathrm{ext}}$.

In order to apply the last theorem to the solution of problem $\mathcal{D}^{(i)}$ with $g \in \mathfrak{N}$ we proceed in the following way. For the function g we construct a special harmonic function u_0 such that $g - u_0|_S \in \mathfrak{N}_{\mathrm{ext}}$. Then we represent the solution $u^{(i)}$ in the form

$$u^{(i)} = u_0(z) + W\sigma(z),$$

and obtain the equation

$$-\pi\sigma + W\sigma = g - u_0|_S,$$

which is uniquely solvable in $\mathfrak{M}_{\mathrm{ext}}$ (see Remark in Sect. 2.3).

In the third part we deal with equation (2) and the external Neumann problem $\mathcal{N}^{(e)}$

$$\Delta v^{(e)} = 0 \quad \text{in } \Omega, \qquad (\partial/\partial n) v^{(e)} = h \quad \text{on } S,$$
$$v(z) = O(1 + |z|^{-1}), \qquad |z| \to +\infty,$$

where h is a function from $\mathfrak{N}$. We are looking for the solution $v^{(e)}$ in the form

$$v^{(e)}(z) = V\tau(z) + \sum_{n=1}^{N} t_n \varrho_n(z), \tag{3}$$

where $\varrho_n(z) = \mathrm{Re}\,[(z - e_n)(z_0 - e_n)/(z_0 - z)]^{1/2}$, z_0 is a fixed point in Ω, and the branch of the square root has the positive real part in the upper half-plane. The boundary equation corresponding to representation (3) takes the form

$$-\pi\tau + \frac{\partial}{\partial n} V\tau + \sum_{k=1}^{N} t_k \frac{\partial}{\partial n} \varrho_k = h. \tag{4}$$

Its solution is a pair (τ, t), where $t = (t_1, \dots, t_N)$. The uniqueness theorem for equation (4) in the class of pairs (τ, t) with $\tau \in \mathfrak{M}$ and $t \in \boldsymbol{R}^N$ is proved in Sect. 3.1. Theorem 5 on solvability of (4) is proved in Sect. 3.2, where also asymptotic formulae for solutions near peaks are given. At the end of Sect. 3.2 we obtain the above mentioned information on equation (2) from our previous results on equation (4). In Theorem 6 we prove that (2) is uniquely solvable in $\mathfrak{M}$ for every $h \in \mathfrak{N}$ subjected to the orthogonality conditions

$$\int_S h ds = 0 \quad \text{and} \quad \int_S h \operatorname{Re}(1/\zeta_k) ds = 0, \quad k = 1, \dots, N.$$

The integral equation for the exterior Dirichlet problem $\mathcal{D}^{(e)}$ and that for the interior Neumann problem $\mathcal{N}^{(i)}$ are briefly discussed in Sect. 4.1 and 4.2.

1.2. *The Dirichlet and Neumann problems for domains with peaks.*

In the sequel we need several auxiliary facts concerning the solvability of the boundary value problems as well as asymptotic formulae for their solutions near peaks.

LEMMA 1. *Suppose that Ω has an outward peak at the origin. Let $u \in \mathring{W}_2^1(\Omega)$ and let*

$$\Delta u(z) = O(|z|^\mu), \qquad \mu > 0.$$

Then $u(z)$ satisfies the relations

$$u(z) = O(|z|^\mu) \quad \text{and } \nabla u(z) = O(|z|^{\mu - 2}).$$

LEMMA 2. *Suppose that Ω has an inward peak at the origin. Let $u \in \mathring{W}_2^1(\Omega)$ and let*

$$\Delta u(z) = O(|z|^\mu), \qquad \mu > 0.$$

Then $u(z)$ admits the representation

$$u(z) = \sum_{m=1}^{2([\mu]+1)} P_m(\log z) z^{m/2} + O(|z|^{[\mu]+1}),$$

where P_m is a polynomial of degree $[(m-1)/2]$ and ε is a small positive number. This equality can be differentiated at least once.

Proof of these Lemmas can be obtained by conformal mapping from well-known asymptotic representations for solutions of the Dirichlet problem in the strip (see for example [5]).

PROPOSITION 1. *Let g be infinitely differentiable on $S \setminus T$. Suppose that g vanishes outside a small neighbourhood of the peak e_p and admits one of the following asymptotic representations near e_p*

$$q_0^{(\pm)} x^\nu + \sum_{k=1}^{n+1} q_k^{(\pm)} x^{k+\nu} + O(x^{n+\nu+2})$$

for $\nu \geq 0$, $\nu \neq l/2$, $l \in \mathbf{Z}$ and

$$(q_{0,0}^{(\pm)} + q_{0,1}^{(\pm)} \log x) x^\nu + \sum_{k=1}^{n+1} P_{k+2}^{(\pm)} (\log x) x^{k+\nu} + O(x^{n+2+\nu-\varepsilon})$$

for $\nu \geq 0$, $\nu = l/2$, where ε is a small positive number and $P_j^{(\pm)}$ are polynomials *of degree j. Suppose that these representations are differentiable at least twice. Then the problem $\mathcal{D}^{(i)}$ has a solution with the following asymptotic properties:*

a) In a neighbourhood of e_p the following holds

$$u(z) = \operatorname{Re}\varphi_{p,n}^{(\text{ext})}(z) + O(|z-e_p|^{n+[\nu]}), \tag{5}$$

where either

$$\varphi_{p,n}^{(\text{ext})}(z) = \sum_{k=0}^{n} \beta_k (z-e_p)^{k+\nu-1} \quad \text{for } \nu \neq l/2, \tag{6}$$

or

$$\begin{aligned}\varphi_{p,n}^{(\text{ext})}(z) = \sum_{r=0}^{2} \beta_{0,r}(\log(z-e_p))^r (z-e_p)^{\nu-1} + \\ + \sum_{k=1}^{n} Q_{k+1}(\log(z-e_p))(z-e_p)^{k+\nu-1} \quad \text{for } \nu = l/2;\end{aligned} \tag{7}$$

here Q_j are polynomials of degree j. The coefficient β_0 is given by

$$\beta_0 = -i\frac{2}{\nu-1}\frac{q_0^{(+)}-q_0^{(-)}}{\kappa''_+(0)-\kappa''_-(0)}.$$

The coefficients $\beta_{0,r}$ are defined differently for $\nu \neq 1$ and $\nu = 1$. Namely,

$$\beta_{0,0} = i\left(\frac{2}{\nu-1}\frac{q_{0,1}^{(+)}-q_{0,1}^{(-)}}{\kappa''_+(0)-\kappa''_-(0)} - \frac{q_{0,0}^{(+)}-q_{0,0}^{(-)}}{\kappa''_+(0)-\kappa''_-(0)}\right),$$

$$\beta_{0,1} = -i\frac{2}{\nu-1}\frac{q_{0,1}^{(+)}-q_{0,1}^{(-)}}{\kappa''_+(0)-\kappa''_-(0)}, \qquad \beta_{0,2}=0$$

for $\nu \neq 1$, and

$$\beta_{0,0}=0, \qquad \operatorname{Im}\beta_{0,1} = -2i\frac{q_{0,0}^{(+)}-q_{0,0}^{(-)}}{\kappa''_+(0)-\kappa''_-(0)}, \qquad \beta_{0,2} = -i\frac{q_{0,1}^{(+)}-q_{0,1}^{(-)}}{\kappa''_+(0)-\kappa''_-(0)},$$

for $\nu = 1$.

b) In a neighbourhood of e_r, $r \neq p$,

$$u(z) = O(|z-e_r|^{n+[\nu]}). \tag{8}$$

c) In a neighbourhood of i_m we have

$$u(z) = \operatorname{Re}\psi_{m,n}^{(\text{int})}(z) + O(|z-i_m|^{n+[\nu]}), \tag{9}$$

where $\psi_{m,n}^{(\text{int})}$ is given by

$$\psi_{m,n}^{(\text{int})}(z) = \sum_{k=1}^{2(n+[\nu])} R_k(\log(z-i_m))(z-i_m)^{k/2}.$$

Here R_k are polynomials of degree $[(k-1)/2]$. The equalities (5), (8), (9) can be differentiated at least once.

PROOF. Let

$$u_n(z) = \operatorname{Re}[\chi(z-e_p)\varphi_{p,n+3}^{(\text{ext})}(z)], \qquad z \in \Omega.$$

Here $\varphi^{(\mathrm{ext})}_{p,n+3}$ is given by (5) for $\nu \neq l/2$ and by (6) for $\nu = l/2$, and χ is a cut-off C^∞-function supported in a small neighbourhood of the point e_p. Coefficients of $\varphi^{(\mathrm{ext})}_{p,n+3}$ are chosen to satisfy

$$(g - u_n)(z) = O(|z - e_p|^{n+3+\nu}), \qquad z \in S_\pm(e_p).$$

We represent the harmonic extension of $g - u_n$ as the sum $u^{(2)} + u^{(3)}$ where $u^{(2)} = g - u_n$ on $S \setminus \{0\}$ and

$$\nabla^k u^{(2)}(z) = O(|z - e_p|^{n+3+\nu-k}), \qquad k = 0, 1, 2.$$

The function $u^{(3)}$ is found from the boundary value problem

$$\Delta u^{(3)} = -\Delta u^{(2)} \quad \text{in } \Omega, \qquad u^{(3)} = 0 \quad \text{on } S.$$

It remains to refer to Lemmas 1 and 2.

Proposition 2. *Let g be a C^∞-function on $S \setminus T$. Suppose that g vanishes outside a small neighbourhood of the peak i_p and admits one of the following two asymptotic representations near i_p:*

$$q_0^{(\pm)} x^\nu + \sum_{k=1}^{n+1} q_k^{(\pm)} x^{k+\nu} + O(x^{n+\nu+2})$$

for $\nu > 0$, and

$$(q_{0,0}^{(\pm)} + q_{0,1}^{(\pm)} \log x) + \sum_{k=1}^{n+1} q_k^{(\pm)} x^k + O(x^{n+2})$$

for $\nu = 0$. We assume that both representations are differentiable at least twice. Then problem $\mathcal{D}^{(i)}$ has a solution with the following properties:

a) In a neighbourhood of e_r

$$u(z) = O(|z - e_r|^{n+[\nu]}). \tag{10}$$

b) In a neighbourhood of i_m, $m \neq p$,

$$u(z) = \operatorname{Re} \psi^{(\mathrm{int})}_{m,n}(z) + O(|z - i_m|^{n+[\nu]}). \tag{11}$$

c) In a neighbourhood of i_p we have

$$u(z) = \operatorname{Re}(\varphi^{(\mathrm{int})}_{p,n}(z) + \psi^{(\mathrm{int})}_{p,n}(z)) + O(|z - i_p|^{n+[\nu]}), \tag{12}$$

where either

$$\varphi^{(\mathrm{int})}_{p,n}(z) = \sum_{k=0}^{n} \alpha_k (z - i_p)^{k+\nu} \quad \textit{for } \nu \neq l/2, \ l \in \mathbf{Z} \tag{13}$$

or

$$\varphi^{(\mathrm{int})}_{p,n}(z) = \sum_{r=0}^{2} \alpha_{0,r} (\log(z - i_p))^r (z - i_p)^\nu + \sum_{k=1}^{n} Q_{k+1}(\log(z - i_p))(z - i_p)^{k+\nu} \quad \textit{for } \nu = l/2, \tag{14}$$

and

$$\psi_{r,n}^{(\mathrm{int})}(z) = \sum_{k=1}^{2(n+[\nu])} R_k(\log(z-i_r))(z-i_r)^{k/2}.$$

Here R_k are polynomials of degree $[(k-1)/2]$ and Q_j are polynomials of degree j. The coefficient α_0 is equal to

$$\alpha_0 = i\,\frac{q_0^{(+)}e^{-2\pi\nu i} - q_0^{(-)}}{\sin 2\pi\nu}.$$

The coefficients $\alpha_{0,r}$ are calculated differently for $\nu \neq 0$ and $\nu = 0$. Namely,

$$\operatorname{Re}\alpha_{0,0} = q_0^{(+)}, \qquad \alpha_{0,1} = i\,\frac{q_0^{(+)} - (-1)^{2\nu} q_0^{(-)}}{2\pi}, \qquad \alpha_{0,2} = 0$$

for $\nu \neq 0$, and

$$\operatorname{Re}\alpha_{0,0} = q_{0,0}^{(+)}, \qquad \alpha_{0,1} = q_{0,1}^{(+)} + i\,\frac{q_{0,0}^{(+)} - q_{0,0}^{(-)}}{2\pi}, \qquad \alpha_{0,2} = i\,\frac{q_{0,1}^{(+)} - q_{0,1}^{(-)}}{4\pi}$$

for $\nu = 0$. Equalities (10), (11), (12) *can be differentiated at least once.*

Proof. Let

$$u_n(z) = \operatorname{Re}[\chi(z-i_p)\,\varphi_{p,n+1}^{(\mathrm{int})}(z)], \qquad z \in \Omega.$$

Here $\varphi_{p,n+1}^{(\mathrm{int})}$ is given by (13) for $\nu \neq l/2$ and by (14) for $\nu = l/2$, and χ is a cut-off C^∞-function supported in a small neighbourhood of point i_p. Coefficients of $\varphi_{p,n+1}^{(\mathrm{int})}$ are chosen to satisfy

$$(g-u_n)(z) = O(|z-e_p|^{n+1+\nu}), \qquad z \in S_\pm(i_p).$$

We are seeking the harmonic extension of $g-u_n$ as a sum $u^{(2)} + u^{(3)}$ where $u^{(2)} = g - u_n$ on $S\setminus\{0\}$ and

$$\nabla^k u^{(2)}(z) = O(|z-i_p|^{n+1+\nu-k}), \qquad k = 0, 1, 2.$$

The function $u^{(3)}$ is found from the boundary value problem

$$\Delta u^{(3)} = -\Delta u^{(2)} \quad \text{in } \Omega, \qquad u^{(3)} = 0 \quad \text{on } S.$$

It remains to apply Lemmas 1 and 2 to complete the proof. ∎

We introduce the following notation:

$$n_0 = [2(n+\nu)+1].$$

Proposition 3. *Let h be a C^∞-function on $S\setminus T$, vanishing outside a small neighbourhood of the peak e_p, and having the following asymptotic representation in local coordinates on the arcs $S_\pm(e_p)$:*

$$h_0^{(\pm)}x^\nu + \sum_{k=1}^{n+1} h_k^{(\pm)} x^{k+\nu} + O(x^{n+\nu+2}), \qquad \nu > -2.$$

Suppose this decomposition can be differentiated at least one time. Assume also that

$$V.P.\int_S h\,ds = 0\,.$$

Then the problem $\mathcal{N}^{(e)}$ *has a solution* v, *with the following asymptotic properties:*

a) *In a neighbourhood of* e_p

$$v(z) - v(e_p) = \operatorname{Re}\left(\varphi_{p,n}^{(\mathrm{ext})}(z) + \psi_{p,n}^{(\mathrm{ext})}(z)\right) + O(|z-e_p|^{n+[\nu]})\,, \tag{15}$$

where either

$$\varphi_{p,n}^{(\mathrm{ext})}(z) = \sum_{k=1}^{n} \beta_k \left(\frac{(z-e_p)(z_0-e_p)}{z_0-z}\right)^{k+\nu} \tag{16}$$

for $\nu \neq l/2$, $l \in \mathbf{Z}$, *or*

$$\varphi_{p,n}^{(\mathrm{ext})}(z) = \sum_{r=0}^{2} \beta_{1,r}\left(\log\frac{(z-e_p)(z_0-e_p)}{z_0-z}\right)^r \left(\frac{(z-e_p)(z_0-e_p)}{z_0-z}\right)^{\nu+1} + \tag{17}$$

$$+ \sum_{k=2}^{n} Q_{k+1}\left(\log\frac{(z-e_p)(z_0-e_p)}{z_0-z}\right)\left(\frac{(z-e_p)(z_0-e_p)}{z_0-z}\right)^{k+\nu}$$

for $\nu = l/2$. *The coefficient* β_1 *equals*

$$\beta_1 = \frac{h_0^{(-)} + h_0^{(+)}\cos 2\pi\nu}{(\nu+1)\sin 2\pi\nu} - i\,\frac{h_0^{(+)}}{\nu+1}\,.$$

Here Q_j *are polynomials of degree* j. *The coefficients* $\beta_{1,r}$ *are defined in different ways in the cases* $\nu \neq -1$, *and* $\nu = -1$. *They are defined by*

$$\operatorname{Im}\beta_{1,0} = -\frac{h_0^{(+)}}{\nu+1}\,,\quad \beta_{1,1} = \frac{h_0^{(+)} + (-1)^{2\nu}h_0^{(-)}}{2\pi(\nu+1)}\,,\qquad \beta_{1,2}=0$$

for $\nu \neq -1$, *and*

$$\operatorname{Im}\beta_{1,0} = 0\,,\quad \beta_{1,1} = -ih_0^{(+)}\,,\quad \beta_{1,2} = \frac{h_0^{(+)} + h_0^{(-)}}{4\pi}$$

for $\nu = -1$.

b) *In a neighbourhood of* e_r, $r \neq p$, *we have*

$$v(z) - v(e_r) = \operatorname{Re}\psi_{r,n}^{(\mathrm{ext})}(z) + O(|z-e_r|^{n+[\nu]})\,. \tag{18}$$

c) *In a neighbourhood of* i_m

$$v(z) - v(i_m) = O(|z-i_m|^{n+[\nu]})\,. \tag{19}$$

Here $\psi_{r,n}^{(\mathrm{ext})}$ *is given by*

$$\psi_{r,n}^{(\mathrm{ext})}(z) = \sum_{k=1}^{n_0} R_k\left(\log\frac{(z-e_r)(z_0-e_r)}{z_0-z}\right)\left(\frac{(z-e_r)(z_0-e_r)}{z_0-z}\right)^{k/2},$$

$r = 1, \dots, N$, R_k *are polynomials of degree* $[(k-1)/2]$, *and* z_0 *is a fixed point in* Ω. *Equalities* (15), (18), (19) *can be differentiated at least once.*

PROOF. Let

$$v_n(z) = \mathrm{Re}\,[\chi(z-e_p)\,\varphi^{(\mathrm{ext})}_{p,n+1}(z)]\,, \qquad z \in \Omega^c$$

where $\varphi^{(\mathrm{ext})}_{p,n+1}$ is given by (16) with $\nu \neq l/2$ and by (17) with $\nu = l/2$, and χ is a smooth cut-off function supported in a small neighbourhood of point e_p. Coefficients of $\varphi^{(\mathrm{ext})}_{p,n+1}$ are chosen to satisfy

$$h_n(z) = (h - \partial v_n/\partial n)(z) = O(|z-e_p|^{n+\nu})\,.$$

Let $\widetilde{v}^{(2)}$ be such that

$$\widetilde{v}^{(2)}(z) = \mp \int_{e_p}^{z} h_n(s)\,ds \qquad \text{on } S_\pm(e_p)$$

and

$$\nabla^k \widetilde{v}^{(2)}(z) = O(|z-e_p|^{n+\nu+1-k})\,, \qquad k = 0,\,1,\,2\,,$$

in the vicinity of the point e_p in Ω^c. Denote by $\widetilde{v}^{(3)}$ the solution of the problem

$$\Delta\widetilde{v}^{(3)} = -\Delta\widetilde{v}^{(2)}\,, \qquad \widetilde{v}^{(3)} \in \overset{\circ}{W}{}^1_2(\Omega^c \cap \{|z|<\varepsilon\})\,.$$

Then the function $\widetilde{v}^{(1)} = \widetilde{v}^{(2)} + \widetilde{v}^{(3)}$ is harmonic in a small neighbourhood of point e_p. A conjugate function $v^{(1)}$ satisfies Neumann condition $\partial v^{(1)}/\partial n = h_n$ in the same neighbourhood. According to Lemma 2, $v^{(1)}$ admits the estimate

$$v^{(1)}(z) = \sum_{m=1}^{2(n+[\nu])} P_m(\log(z-e_p))(z-e_p)^{m/2} + O(|z-e_p|^{n+\nu+1-\varepsilon})\,,$$

where P_m are polynomials of degree $[(m-1)/2]$ and ε is a small positive number. The last equality is differentiable once. It remains to refer to Lemmas 1 and 2. ■

PROPOSITION 4. *Let h be infinitely differentiable on $S \setminus T$, vanishing outside a small neighbourhood of the peak i_p, and having one of the following asymptotic representations in local coordinates on the arcs $S_\pm(i_p)$*

$$\sum_{k=0}^{n} h_k^{(\pm)} x^{k+\nu} + \sum_{k=1}^{n_0} T_k^{(\pm)}(\log x)\,x^{-1+k/2} + O(x^\gamma)$$

for $\nu > -1$, $\nu \neq l/2$, $l \in \mathbf{Z}$, and

$$(h_{0,0}^{(\pm)} + h_{0,1}^{(\pm)}\log x)\,x^\nu + \sum_{k=1}^{n} P_{k+1}^{(\pm)}(\log x)\,x^{k+\nu} + \sum_{k=1}^{n_0} T_k^{(\pm)}(\log x)\,x^{-1+k/2} + O(x^\gamma)$$

for $\nu > -1$, $\nu = l/2$. Here $T_k^{(\pm)}$ are polynomials of degree $[(k-1)/2]$, $P_j^{(\pm)}$ are polynomials of degree j, and $n + \nu < \gamma < n_0$. We assume that these representations can be differentiated at least once and that

$$\int_S h\,ds = 0\,.$$

Then the problem $\mathcal{N}^{(e)}$ has a solution v which can be represented as follows:

a) In a neighbourhood of e_m we have

$$v(z) - v(e_m) = \operatorname{Re}\psi^{(\mathrm{ext})}_{m,n}(z) + O(|z - e_m|^{n+[\nu]}). \tag{20}$$

The function $\psi^{(\mathrm{ext})}_{m,n}$ in (20) is defined by

$$\psi^{(\mathrm{ext})}_{m,n} = \sum_{k=1}^{n_0} R_k\left(\log \frac{(z-e_m)(z_0-e_m)}{z_0-z}\right)\left(\frac{(z-e_m)(z_0-e_m)}{z_0-z}\right)^{k/2},$$

R_k are polynomials of degree $[(k-1)/2]$, and z_0 is a point in Ω.

b) In a neighbourhood of i_r, $r \neq p$,

$$v(z) - v(i_r) = O(|z - i_r|^{n+[\nu]}). \tag{21}$$

c) In a neighbourhood of i_p

$$v(z) - v(i_p) = \operatorname{Re}\varphi^{(\mathrm{int})}_{p,n}(z) + O(|z - i_p|^{n+[\nu]}), \tag{22}$$

where we use the notations

$$\varphi^{(\mathrm{int})}_{p,n}(z) = \sum_{k=0}^{n} \alpha_k\left(\frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)^{k+\nu} + \sum_{k=1}^{n_0} U_k\left(\log \frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)\left(\frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)^{-1+k/2} \tag{23}$$

for $\nu \neq l/2$, and

$$\begin{aligned}\varphi^{(\mathrm{int})}_{p,n}(z) = {}& \sum_{r=0}^{2} \alpha_{0,r}\left(\log \frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)^{r}\left(\frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)^{\nu} + \\ &+ \sum_{k=1}^{n} Q_{k+1}\left(\log \frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)\left(\frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)^{k+\nu} + \\ &+ \sum_{k=1}^{n_0} U_k\left(\log \frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)\left(\frac{(z-i_p)(z_0-i_p)}{z_0-z}\right)^{-1+k/2}\end{aligned} \tag{24}$$

for $\nu = l/2$. Here U_k are polynomials of degree $[(k-1)/2]$ and Q_j are polynomials of degree j. The coefficient α_0 is equal to

$$\alpha_0 = \frac{2}{\nu(\nu+1)} \frac{h_0^{(+)} + h_0^{(-)}}{\kappa''_+(0) - \kappa''_-(0)}.$$

The coefficients $\alpha_{0,r}$ are defined differently for $\nu \neq 0$ and $\nu = 0$. They are defined by

$$\alpha_{0,0} = \frac{2}{\nu(\nu+1)} \frac{h_{0,0}^{(+)} + h_{0,0}^{(-)}}{\kappa''_+(0) - \kappa''_-(0)} - 2\,\frac{2\nu+1}{\nu^2(\nu+1)^2} \frac{h_{0,1}^{(+)} + h_{0,1}^{(-)}}{\kappa''_+(0) - \kappa''_-(0)},$$

$$\alpha_{0,1} = \frac{2}{\nu(\nu+1)} \frac{h_{0,1}^{(+)} + h_{0,1}^{(-)}}{\kappa''_+(0) - \kappa''_-(0)}, \qquad \alpha_{0,2} = 0$$

for $\nu \neq 0$, *and*

$$\alpha_{0,0} = 0\,, \qquad \alpha_{0,1} = 2\,\frac{h_{0,0}^{(+)} + h_{0,0}^{(-)}}{\kappa_+''(0) - \kappa_-''(0)} - 2\,\frac{h_{0,1}^{(+)} + h_{0,1}^{(-)}}{\kappa_+''(0) - \kappa_-''(0)}\,,$$

$$\alpha_{0,2} = \frac{h_{0,1}^{(+)} + h_{0,1}^{(-)}}{\kappa_+''(0) - \kappa_-''(0)}\,,$$

for $\nu = 0$. *The equalities* (20), (21), (22) *can be differentiated once.*

PROOF. Let

$$v_n(z) = \mathrm{Re}\,[\chi(z - i_p)\,\varphi_{p,n+3}^{(\mathrm{int})}(z)]\,, \qquad z \in \Omega^c$$

where $\varphi_{p,n+3}^{(\mathrm{int})}$ is given by (23) with $\nu \neq l/2$ and by (24) with $\nu = l/2$, and χ is a smooth cut-off function supported in a small neighbourhood of i_p. Coefficients of $\varphi_{p,n+3}^{(\mathrm{int})}$ are chosen to satisfy

$$h_n(z) = (h - \partial v_n/\partial n)(z) = O(|z - i_p|^{n+\nu+2})\,.$$

Let a function $\tilde{v}^{(2)}$ be such that

$$\tilde{v}^{(2)}(z) = \pm \int_{i_p}^{z} h_n(s)\,ds \qquad \text{on } S_\pm(i_p)$$

and

$$\nabla^k \tilde{v}^{(2)}(z) = O(|z - i_p|^{n+\nu+3-k})\,, \qquad k = 0, 1, 2\,,$$

in the neighbourhood of point i_p in domain Ω^c. Denote by $\tilde{v}^{(3)}$ the solution of the problem

$$\Delta\tilde{v}^{(3)} = -\Delta\tilde{v}^{(2)}\,, \qquad \tilde{v}^{(3)} \in \mathring{W}_2^1(\Omega^c \cap \{|z| < \varepsilon\})\,.$$

Then the function $\tilde{v}^{(1)} = \tilde{v}^{(2)} + \tilde{v}^{(3)}$ is harmonic in a small neighbourhood of point i_p. A conjugate function $v^{(1)}$ satisfies Neumann condition $\partial v^{(1)}/\partial n = h_n$ in the same neighbourhood. According to Lemma 2, $v^{(1)}$ admits the estimate

$$v^{(1)}(z) = O(|z - i_p|^{n+\nu+1})\,.$$

The last equality is differentiable once. It remains to apply Lemmas 1 and 2 in order to complete the proof. ■

2. THE INTEGRAL EQUATION OF THE PROBLEM $\mathcal{D}^{(i)}$

2.1. *On the number of solutions to the homogeneous integral equation of the problem* $\mathcal{D}^{(i)}$.

LEMMA 3. *a) Let* Ω *be a domain with outward peak. Then for any* $\sigma \in \mathfrak{M}$ *the representation*

$$\int_{S_\pm} \sigma(q)\,\frac{\partial}{\partial n_q}\log\frac{1}{r}\,ds_q = -\pi\sigma_\pm(x) + O(1)\,,$$

is valid on the arcs $S_\pm$. Here $r = |q - z|$, $z = x + iy$ and $\partial/\partial n$ is the derivative in the direction of the outward normal.

b) Let Ω be a domain with inward peak and let the arcs $S_\pm$ be given by $y = \kappa_\pm(x)$, $x \in [0, \delta]$. We set $\alpha = \max(|\kappa''_+(0)|, |\kappa''_-(0)|)$. Then for any $\sigma \in \mathfrak{M}$

$$\int_{S_\pm} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{r} ds_q = \mp\pi\sigma_\pm(x) + O(1),$$

if $\kappa_+(x) < y < \alpha x^2$,

$$\int_{S_\pm} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{r} ds_q = \pm\pi\sigma_\pm(x) + O(1),$$

if $-\alpha x^2 < y < \kappa_-(x)$ and

$$\int_{S_\pm} \sigma(q) \frac{\partial}{\partial s} \log \frac{1}{r} ds_q = \mp \int_0^\delta \sigma_\pm(\tau) \frac{1}{x - \tau} d\tau + O(1),$$

if $-\alpha x^2 < y < \alpha x^2$.

The proof of Lemma 3 is given in [3].

LEMMA 4. *Let g coincide with the restriction to S of a C^∞-function defined on $\boldsymbol{R}^2$. Then the integral equation of the problem $\mathcal{D}^{(i)}$*

$$-\pi\sigma + W\sigma = g$$

has a solution in the class $\mathfrak{M}$ bounded in a neighbourhood of each point e_k, $k = 1, \ldots, N$.

PROOF. Suppose that e_k coincides with the origin. Then g has the representation

$$g(z) = c^{(0)} + c^{(1)}x + c_\pm^{(2)}x^2 + \ldots$$

on the arcs $S_\pm(0) = S_\pm$.

By Proposition 1, the potential of the problem $\mathcal{D}^{(i)}$ can be written in the form

$$\varphi(z) = a + \beta z + \gamma z^2 + \ldots,$$

where

$$\operatorname{Im}\beta = -2\,\frac{c_+^{(2)} - c_-^{(2)}}{\kappa''_+(0) - \kappa''_-(0)}, \quad \operatorname{Re}\beta = c^{(1)}, \, a = c^{(0)}, \quad \operatorname{Re}\gamma = 2\,\frac{\alpha_+ c_-^{(2)} - \alpha_- c_+^{(2)}}{\kappa''_+(0) - \kappa''_-(0)}.$$

The normal derivative $\partial(\operatorname{Re}\varphi)/\partial n$ of the solution admits the decomposition

$$\frac{\partial \operatorname{Re}\varphi}{\partial n} = \frac{\partial \operatorname{Im}\varphi}{\partial s} = \mp\operatorname{Im}\beta \pm d_\pm x + \ldots.$$

Therefore, the boundary function h of the problem $\mathcal{N}^{(e)}$ has the representation

$$h(z) = \pm\operatorname{Im}\beta \pm d_\pm x + \ldots$$

on the arcs $S_\pm$.

By Proposition 3, the holomorphic function $\varphi(z)$, whose real part is a solution of the problem $\mathcal{N}^{(e)}$ has the form

$$\varphi(z) = c_0 + c_1 z^{1/2} + (\beta_{1,1} \log z + \beta_{1,0}) z + \dots .$$

From the expression for $\beta_{1,1}$ it is clear that $\beta_{1,1} = 0$, *i.e.*

$$\varphi(z) = c_0 + c_1 z^{1/2} + \beta_{1,0} z + \dots .$$

Hence the function $\sigma = (\mathrm{Re}\, \varphi - g)/2\pi$ has the estimate

$$\sigma(z) = O(1)$$

on the arcs $S_\pm(e_n)$. According to Propositions 2 and 4 we have

$$\sigma(z) = \delta_m x^{-1/2} + O(1)$$

on the arcs $S_\pm(i_m)$, $m = 1, \dots, M$.

Therefore the function σ is a solution of the integral equation of the problem $\mathcal{D}^{(i)}$ (cf. [1]) with required properties. ■

THEOREM 1. *Let $\sigma \in \mathfrak{M}$ be a solution of the homogeneous integral equation of the problem $\mathcal{D}^{(i)}$*

$$-\pi\sigma + W\sigma = 0 .$$

Then

$$\sigma = \sum_{k=1}^{N} c_k \mathrm{Re}\,(1/\zeta_k) .$$

PROOF. (*i*) Let σ be a solution of the homogeneous equation in the class $\mathfrak{M}$. Consider C^∞-functions χ_k, $k = 1, \dots, N$, such that $\chi_k = 1$ in a small neighbourhood of e_k $\mathrm{supp}\,\chi_k \cap \mathrm{supp}\,\chi_n = \emptyset$, $k \neq n$, and $i_m \notin \mathrm{supp}\,\chi_k$, $m = 1, \dots, M$.

Since

$$(\pi I - W)(\sigma\chi_j) = -(\pi I - W)((1 - \chi_j)\sigma)$$

then, by Lemma 4, the equation

$$(\pi I - W)\sigma_j = -(\pi I - W)(\sigma\chi_j)$$

has the solution σ_j belonging to $\mathfrak{M}$ and bounded in a neighbourhood of each point e_k, $k = 1, \dots, N$.

The function $\theta_j = \sigma_j + \sigma\chi_j$ is a solution of the homogeneous integral equation, bounded in a neighbourhood of each point e_k, $k = 1, \dots, N$, $k \neq j$. By $W_\pm \theta_j$ we denote the double layer potential in domains Ω and Ω^c respectively.

According to statement *a*) of Lemma 3 one has

$$W_+ \theta_j(z) = -\pi((\theta_j)_+ + (\theta_j)_-)(x) + O(1), \qquad z = x + iy \to e_j ,$$

in the appropriate coordinate system. Since the boundary values of $W_+ \theta_j$ are zero on $S \setminus T$, then

$$((\theta_j)_+ + (\theta_j)_-)(x) = O(1) \quad \text{as } z \to e_j .$$

This implies that $W_+ \theta_j$ is bounded in a neighbourhood of each e_k, $k = 1, \dots, N$.

(*ii*) We fix an integer $m \geqslant 1$ not exceeding M and choose a coordinate system so that in a neighbourhood of i_m the domain Ω^c is defined by $\{\kappa_-(x) < y < \kappa_+(x),\ 0 < x < \delta\}$ with functions $\kappa_\pm$ described above. According to statement *b*) of Lemma 3, the potential $W_+\,\theta_j(z)$ is (up to a sign) $\pi[(\theta_j)_+ - (\theta_j)_-](x) + O(1)$ on the set

$$\Omega \cap \{z = x + iy\colon \text{ either } \kappa_+(x) < y < \alpha x^2 \text{ or } -\alpha x^2 < y < \kappa_-(x)\},$$

where $\alpha = \max(|\kappa_+''(0)|,\ |\kappa_-''(0)|)$. This implies that $W_+\,\theta_j$ is bounded on the set $\Omega \cap \{z\colon -\alpha x^2 < y < \alpha x^2\}$.

Further, let χ be a cut-off function on the plane with small support which is equal to unity near the origin. We introduce the function σ_e on the arcs $S_\pm(i_m)$ as

$$\sigma_e(x) = [(\theta_j)_+(x) + (\theta_j)_-(x)]/2.$$

In view of Lemma 3, $W_+(\chi\sigma_e)$ and its harmonic conjugate $\widetilde{W}_+(\chi\sigma_e)$ are bounded on the set $\Omega \cap \{z\colon -\alpha x^2 < y < \alpha x^2\}$. Besides, using explicit expressions for these functions, one can show that

$$W_+(\chi\sigma_e)(z) = O(|z|^{-2}), \qquad \widetilde{W}_+(\chi\sigma_e)(z) = O(|z|^{-2})$$

in Ω, as $|z| \to 0$.

By the Phragmén-Lindelöf principle (cf. [6, p. 262]), the holomorphic function $W_+(\chi\sigma_e) + i\widetilde{W}_+(\chi\sigma_e)$ is bounded near the origin. Since

$$(\theta_j)_+(x) - (\theta_j)_-(x) = O(1) \quad \text{ as } x \to 0,$$

then $(W_+\,\theta_j)(z)$ is also bounded near the origin. Thus, the potential is bounded in a neighbourhood of each point i_m, $m = 1, \dots, M$, and hence it is bounded in Ω. Since $W_+\,\theta_j$ is a harmonic function vanishing on $S \setminus T$, then $W_+\,\theta_j = 0$ in Ω. Therefore its conjugate $\widetilde{W}_+\,\theta_j$ is a constant.

For each n, $n \leqslant M$, according to Lemma 3,

$$\int_0^\delta [((\theta_j)_+ - (\theta_j)_-)(x+\tau) - ((\theta_j)_+ - (\theta_j)_-)(x-\tau)]\,\frac{d\tau}{\tau} = O(1).$$

Since in Ω^c one has

$$(\widetilde{W}_-\,\theta_j)(z) = -\int_0^\delta [((\theta_j)_+ - (\theta_j)_-)(x+\tau) - ((\theta_j)_+ - (\theta_j)_-)(x-\tau)]\,\frac{d\tau}{\tau} + O(1)$$

as $x \to +0$, then $(\widetilde{W}_-\,\theta_j)$ is a bounded function in a neighbourhood of each point i_m, $m = 1, \dots, M$.

(*iii*) In a neighbourhood of each e_k, $k = 1, \dots, N$, $k \neq j$, the function $(W_-\,\theta_j)(z)$ is bounded. On the other hand, we have $\widetilde{W}_+\,\theta_j = \text{const}$. Therefore, according to Lemma 3, the function $(\widetilde{W}_-\,\theta_j)(z)$ is bounded on the set $\Omega^c \cap \{z\colon -\alpha x^2 < y < \alpha x^2\}$ where $\alpha = \max(|\kappa_+''(0)|,\ |\kappa_-''(0)|)$. Since $(\widetilde{W}_-\,\theta_j)(z) = O(|z|^{-2})$, the Phragmén-Lindelöf principle applied to the function $W_-\,\theta_j + i\widetilde{W}_-\,\theta_j$ implies the boundedness of the potential in each neighbourhood in question. Thus $\widetilde{W}_-\,\theta_j$ is bounded inside a small neighbourhood of e_k.

(*iv*) Let $\widetilde{W}_+ \theta_j$ be equal to a constant C in Ω. Consider the function

$$W(z) = W_- \theta_j(z) + i(\widetilde{W}_- \theta_j(z) - C).$$

It is holomorphic in Ω^c and its imaginary part vanishes on $S \setminus T$.

Let $z = \lambda_j(\zeta)$ ($\zeta = \eta + i\xi$) be the conformal mapping of the upper half-plane onto Ω^c, the inverse of $\zeta = \zeta_j(z)$. The imaginary part $v(\zeta)$ of $W(\lambda_j(\zeta)) = u(\zeta) + iv(\zeta)$ vanishes at the boundary of the upper half-plane except for the points $\zeta_j(T)$.

By $\widetilde{v}$ we denote an odd extension of the function v. Clearly, $\widetilde{v}$ is harmonic on the punctured complex plane $C \setminus \{0\}$ and bounded at infinity. Therefore $\widetilde{v}$ can be expanded as

$$\widetilde{v}(re^{i\gamma}) = \sum_{k \geq 1} d_k r^{-k} \sin k\gamma .$$

The conjugate function $\widetilde{u}$ has the expansion

$$\widetilde{u}(re^{i\gamma}) = \sum_{k \geq 0} d_k r^{-k} \cos k\gamma .$$

Since

$$W(\lambda_j(\zeta)) = O(|\zeta|^{-4}) \quad \text{as } \zeta \to 0,$$

then the function $\widetilde{u}(re^{i\gamma}) + i\widetilde{v}(re^{i\gamma})$, which is holomorphic in $\overline{C} \setminus \{0\}$, has a pole at the origin. At the boundary of the upper half-plane $\widetilde{u}$ equals $-2\pi(\theta_j \circ \lambda_j)(\eta + i0)$, $\eta \in \boldsymbol{R}$. Hence $\widetilde{u} = O(|\eta|^{2\beta})$ as $\eta + i0 \to 0$, where $\beta > -1$. Then

$$\widetilde{u}(\zeta) = d_0 + d_1 r^{-1} \cos \gamma = d_0 + d_1 \operatorname{Re}(1/\zeta)$$

and, consequently,

$$\theta_j(z) = -(2\pi)^{-1}(d_0 + d_1(\operatorname{Re} 1/\zeta_j)(z)), \qquad z \in S.$$

Since a non-zero constant does not satisfy the homogeneous integral equation of the problem $\mathcal{D}^{(i)}$, then

$$\theta_j(z) = -(2\pi)^{-1} d_1 (\operatorname{Re} 1/\zeta_j)(z) = c_j (\operatorname{Re} 1/\zeta_j)(z).$$

(*v*) We set

$$\widetilde{\sigma} = \sigma - \sum_1^N \sigma\chi_j$$

and obtain

$$\sigma = \widetilde{\sigma} + \sum_1^N \sigma\chi_j = \widetilde{\sigma} - \sum_1^N \sigma_j + \sum_1^N c_j (\operatorname{Re} 1/\zeta_j).$$

The density $\widetilde{\sigma} - \sum_{j=1}^N \sigma_j$ is a solution of the homogeneous equation

$$(\pi I - W)\left(\widetilde{\sigma} - \sum_1^N \sigma_j\right) = (\pi I - W)\widetilde{\sigma} - \sum_I^N (\pi I - W)\sigma_j =$$

$$= (\pi I - W)\widetilde{\sigma} + \sum_1^N (\pi I - W)(\sigma\chi_j) = (\pi I - W)\left(\widetilde{\sigma} + \sum_1^N \sigma\chi_j\right) = (\pi I - W)\sigma = 0 .$$

The function $\widetilde{\sigma} - \sum_{j=1}^N \sigma_j$ is a bounded function in a neighbourhood of each e_k,

$k = 1, \dots, N$. Repeating arguments in (*ii*) and (*iii*) for the function $\widetilde{\sigma} - \sum_{j=1}^{N} \sigma_j$ we prove that,

$$W_+\left(\widetilde{\sigma} - \sum_{j=1}^{N} \sigma_j\right) = 0 \quad \text{in } \Omega \text{ and } W_-\left(\widetilde{\sigma} - \sum_{j=1}^{N} \sigma_j\right) = 0 \quad \text{in } \Omega^c .$$

The limit relations for the double layer potential imply

$$\widetilde{\sigma} - \sum_{j=1}^{N} \sigma_j = \frac{1}{2\pi}\left[W_-\left(\widetilde{\sigma} - \sum_{j=1}^{N} \sigma_j\right) - W_+\left(\widetilde{\sigma} - \sum_{j=1}^{N} \sigma_j\right)\right] = 0 .$$

Thus,

$$\sigma = \sum_1^N c_j \operatorname{Re} 1/\zeta_j. \qquad \blacksquare$$

As a corollary of the Theorem 1 we can state the following result.

COROLLARY. *The homogeneous integral equation of the problem* $\mathcal{D}^{(i)}$ *has only trivial solution in the class* $\mathfrak{M}_{\text{ext}}$.

2.2. *The solvability of the integral equation of the problem* $\mathcal{D}^{(i)}$ *and asymptotic formulae for solutions.*

THEOREM 2. *Let g be a function from* $\mathfrak{N}$. *Then the boundary equation*

$$-\pi\sigma(p) + W\sigma(p) = g(p), \qquad p \in S \setminus T, \tag{25}$$

has a solution in the class $\mathfrak{M}$ *with the following representation in local coordinates:*

a) on the arcs $S_\pm(e_n)$

$$\sigma(z) = \pm(\beta_0 + \beta_1 \log x) x^{-1+\nu(e_n)} + O(1), \tag{26}$$

b) on the arcs $S_\pm(i_m)$

$$\sigma(z) = (\alpha_0 + \alpha_1 \log x)^{-1+\nu(i_m)} + \delta x^{-1/2} + O(1). \tag{27}$$

Moreover, the space of solutions of the homogeneous equation is N-dimensional.

PROOF. Consider the simple layer potential with the density σ

$$V\sigma(z) = V.P. \int_S \sigma(q) \log \frac{1}{|z-q|}\, ds_q .$$

Let $u^{(i)}$ and $u^{(e)}$ be solutions of the problems $\mathcal{D}^{(i)}$ and $\mathcal{D}^{(e)}$. By Proposition 1, the holomorphic function $\varphi^{(i)}$ whose real part is the harmonic extension of $u^{(i)}$ admits the following decomposition in local variables,

$$\varphi^{(i)}(z) = (\beta_{0,0} + \beta_{0,1} \log z) z^{\nu-1} + \beta_1 z^\nu + O(z)$$

in a neighbourhood of e_n. Here $\beta_{0,1} \neq 0$ only for $\nu = 1$. Hence

$$(\partial u^{(i)}/\partial n)(z) = \pm \beta_0 x^{\nu-2} + \beta_1^{(\pm)} x^{\nu-1} + O(1)$$

on the arcs $S_\pm(e_n)$.

In a neighbourhood of i_m one has

$$\varphi^{(i)}(z) = (\alpha_{0,0} + \alpha_{0,1} \log z) z^{\nu} + \delta_0 z^{1/2} + O(z)$$

in the local coordinate system. Here $\alpha_{0,1} \neq 0$ only for $\nu = l/2$, $l \in \boldsymbol{Z}$ (cf. Proposition 2). This implies the decomposition

$$(\partial u^{(i)} / \partial n)(z) = (\alpha_{0,0}^{(\pm)} + \alpha_{0,1}^{(\pm)} \log x) x^{\nu - 1} + \delta_0 x^{-1/2} + O(1)$$

on the arcs $S_{\pm}(i_m)$. Here $\alpha_{0,1}^{(\pm)} \neq 0$ only for $\nu = l/2$. Then

$$g(p) = \frac{1}{2\pi} V\left(\frac{\partial u^{(i)}}{\partial n} - \frac{\partial u^{(e)}}{\partial n} \right)(p) + u^{(e)}(\infty), \qquad p \in S \setminus T.$$

Let v denote a solution of the problem $\mathcal{N}^{(e)}$:

$$\Delta v = 0 \quad \text{in } \Omega^c, \qquad \partial v / \partial n = \partial u^{(i)} / \partial n \quad \text{on } S \setminus T$$

and v vanishes at infinity.

In a neighbourhood of e_n the holomorphic function φ, whose real part is the solution v, has the form

$$\varphi(z) = \beta_1 \left(\frac{(z - e_n)(z_0 - e_n)}{z_0 - z} \right)^{\nu - 1} + \delta_0 \left(\frac{(z - e_n)(z_0 - e_n)}{z_0 - z} \right)^{-1/2} + O(1)$$

for $\nu \neq l/2$ and

$$\varphi(z) = \left(\beta_{1,0} + \beta_{1,1} \log \frac{(z - e_n)(z_0 - e_n)}{z_0 - z} \right) \left(\frac{(z - e_n)(z_0 - e_n)}{z_0 - z} \right)^{\nu - 1} + \\ + \gamma_0 \left(\frac{(z - e_n)(z_0 - e_n)}{z_0 - z} \right)^{-1/2} + O(1)$$

for $\nu = l/2$. Therefore, by Proposition 3,

$$v(z) = \pm \beta_0 x^{\nu - 1} \pm \gamma_0 x^{-1/2} + O(1)$$

for $\nu \neq l/2$ and

$$v(z) = \pm (\beta_0 + \beta_1 \log x) x^{\nu - 1} \pm \gamma_0 x^{-1/2} + O(1)$$

for $\nu = l/2$ in a local coordinate system on the arcs $S_{\pm}(e_n)$.

According to Proposition 4,

$$\varphi(z) = \left(\alpha_{0,0} + \alpha_{0,1} \log \frac{(z - i_m)(z_0 - i_m)}{z_0 - z} \right) \left(\frac{(z - i_m)(z_0 - i_m)}{z_0 - z} \right)^{\nu - 1} + \\ + \delta_0 \left(\frac{(z - i_m)(z_0 - i_m)}{z_0 - z} \right)^{-1/2} + O(1)$$

in a neighbourhood of i_m. Hence,

$$v(z) = (\alpha_0 + \alpha_1 \log x) x^{\nu - 1} + \delta x^{-1/2} + O(1)$$

in a local coordinate system on the arcs $S_{\pm}(i_m)$.

From the integral representation for the harmonic function $w = v - u^{(e)} + u^{(e)}(\infty)$ in Ω^c and from the formula for limit values of the double layer potential one has

$$\pi w - Ww = -V\left(\frac{\partial u^i}{\partial n} - \frac{\partial n^{(e)}}{\partial n}\right) = 2\pi(u^{(e)}(\infty) - g) \qquad \text{on } S\setminus T.$$

Consequently, the density

$$\sigma = (2\pi)^{-1}(w - u^{(e)}(\infty)) = (2\pi)^{-1}(v - g)$$

is a solution of equation (25).

In view of Theorem 1, the functions of the form

$$\sigma = 2\pi^{-1}\left(v - g + \sum_1^N c_k \operatorname{Re} 1/\zeta_k\right)$$

are solutions of (25) in the class $\mathfrak{M}$. ■

Explicit formulae for the coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1$, and δ in (26), (27) were given in [3].

We can summarize Theorem 1 and Theorem 2 in the following statement.

THEOREM 3. *Let $g \in \mathfrak{N}_{\text{ext}}$. Then equation (25) is uniquely solvable in $\mathfrak{M}_{\text{ext}}$.*

2.3. *Remark.*

The above considerations allow one to conclude that boundary equation (25) is applicable to solving the problem $\mathcal{D}^{(i)}$ in domains with peaks and with boundary data in the class $\mathfrak{N}$.

We look for the solution $u^{(i)}$ of the problem $\mathcal{D}^{(i)}$ as the sum of an explicitly written harmonic function u_0 and of the double layer potential $W\sigma$ with unknown density σ.

By Propositions 1 the solution of the problem $\mathcal{D}^{(i)}$ admits the following representation:

$$u^{(i)}(z) = \operatorname{Re}\varphi_p^{(\text{ext})}(z) + O(|z - e_p|^{1+[\nu(e_p)]}),$$

in a neighbourhood of e_p, where

$$\varphi_p^{(\text{ext})}(z) = (\beta_{0,0} + \beta_{0,1}\log(z - e_p))(z - e_p)^{\nu(e_p)-1} + \beta_1(z - e_p)^{\nu(e_p)}.$$

We set

$$u_0(z) = \operatorname{Re}\left\{\sum_{k=1}^N p_k^{(\text{ext})}(z)\,\varphi_k^{(\text{ext})}(z)\right\},$$

where $p_k^{(\text{ext})}$ are interpolation polynomials such that $p_k^{(\text{ext})}(e_k) = 1$, $D_z^n p_k^{(\text{ext})}(e_k) = 0$, $n = 1, \dots, 1 + [\nu(e_k)]$, $D_z^n p_k^{(\text{ext})}(e_r) = 0$, $n = 0, \dots, 1 + [\nu(e_r)]$, $r \neq k$ (cf. [7]). Then

$$(u^{(i)} - u_0)(z) = O(|z - z_0|^{1+[\nu(z_0)]})$$

in a neighbourhood of each outward peak z_0. By Theorem 3 the boundary integral equation (25) with the right-hand side $g - u_0$ is uniquely solvable in $\mathfrak{M}_{\text{ext}}$.

3. Integral equation of the problem $\mathcal{N}^{(e)}$

3.1. *The homogeneous integral equation of the problem* $\mathcal{N}^{(e)}$.

THEOREM 4. *The homogeneous boundary equation of the problem* $\mathcal{N}^{(e)}$

$$-\pi\tau(p) + \frac{\partial}{\partial n_p} V\tau(p) + \sum_{k=1}^{N} t_k \frac{\partial}{\partial n_p} \varrho_k(p) = 0$$

considered on the set of pairs $\{\tau, t\}$, *where* $\tau \in \mathfrak{M}$ *and* $t \in \mathbf{R}^N$, *has only the trivial solution.*

PROOF. The functions $V\tau(z)$ and $\varrho_n(z)$, $n = 1, \dots, N$, $z \in \Omega^c$ are harmonic, tend to zero at infinity and are bounded. Hence,

$$V\tau(z) - \sum_{n=1}^{N} t_n \varrho_n(z)$$

is a solution of the Neumann problem with the zero boundary condition. By the uniqueness theorem for the problem $\mathcal{N}^{(e)}$,

$$V\tau(z) - \sum_{n=1}^{N} t_n \varrho_n(z) = \text{const}.$$

Since the solution vanishes at infinity, then

$$V\tau(z) = \sum_{n=1}^{N} t_n \varrho_n(z)$$

in Ω^c.

Let $\varrho_n^{(i)}(z)$ denote the bounded harmonic extension of $\varrho_n(p)$, $p \in S$, to Ω, and let $z = \omega(\zeta)$ be a conformal mapping of the strip $\Pi = \{(\eta, \xi)\colon 0 < \xi < 1\}$ onto Ω with $\operatorname{Re} \omega^{-1}(e_n) = \infty$. Then the function

$$\varrho_0(z) = V\tau(z) - \sum_{n=1}^{N} t_n \varrho_n^{(i)}(z)$$

is bounded in Ω and vanishes on S. Therefore the Fourier decomposition of $\omega(\zeta) = \varrho_0(\omega(\zeta))$ has the form

$$w(z) = \sum_{k=1}^{\infty} c_k(\eta) \sin \pi k \xi.$$

The coefficients $c_k(\eta)$ have the form $\alpha_k e^{\pi k \eta} + \beta_k e^{-\pi k \eta}$. Since

$$\int_0^1 |w(\eta, \xi)|^2 \, d\xi = \frac{1}{2} \sum_{k=1}^{\infty} |c_k(\eta)|^2$$

and since the left-hand side of this equality is bounded, then $\alpha_k = 0$, $k = 1, 2, \dots$. It follows that $\varrho_0(z)$ and gradient $\nabla \varrho_0(z)$ exponentially decays as $z \to e_n$. We have

$$V\tau(z) = \sum_{n=1}^{N} t_n \varrho_n^{(i)}(z) + \varrho_0(z).$$

Hence

$$\tau(p)=\sum_{n=1}^{N} t_n \frac{\partial \varrho_n^{(i)}}{\partial n}(p)-\sum_{n=1}^{N} t_n \frac{\partial \varrho_n}{\partial n}(p)+\frac{\partial \varrho_0}{\partial n}(p), \qquad p \in S,$$

where functions $(\partial \varrho_n^{(i)}/\partial n)(p)$ and $(\partial \varrho_n/\partial n)(p)$ have different orders of singularities, and $(\partial \varrho_0/\partial n)(p)$ exponentially decays as $p \to e_n$.

Since $\tau \in \mathfrak{M}$, the coefficients t_n, $n=1,\dots,N$, vanish. Consequently, $V\tau(z)=0$ in Ω^c. The potential $V\tau(z)$ is continuous in $\overline{\Omega}\setminus T$ and bounded in Ω, therefore $V\tau=0$ in Ω. The formula for the jump of the simple layer potential implies $\tau=0$. Thus the density τ vanishes and the vector $t \in \mathbf{R}^N$ is zero. ■

3.2. *On the solvability of integral equation of the problem* $\mathcal{N}^{(e)}$.

THEOREM 5. *Let h be a function from $\mathfrak{N}$ and*

$$\int_S h\, ds = 0 .$$

Then the boundary equation

$$-\pi\tau(p)+\frac{\partial}{\partial n} V\tau(p)+\sum_{k=1}^{N} t_k \frac{\partial}{\partial n}\varrho_k(p)=h(p) \tag{28}$$

considered on the set of pairs $\{\tau, t\}$ with $\tau \in \mathfrak{M}$ and $t \in \mathbf{R}^N$ is uniquely solvable. The density τ has the following representations in local coordinates:

a) on the arcs $S_\pm(e_n)$

$$\tau(z)=\pm(\beta_{1,0}+\beta_{1,1}\log x)x^{\nu-1}\pm(\mu_{1,0}+\mu_{1,1}\log x)x^{-1/2}+O(\log x), \tag{29}$$

b) on the arcs $S_\pm(i_m)$.

$$\tau(z)=(\alpha_{0,0}+\alpha_{0,1}\log x)x^{\nu-1}+\mu_0 x^{-1/2}+O(1). \tag{30}$$

PROOF. By Proposition 3, the holomorphic function whose real part is a solution of problem $\mathcal{N}^{(e)}$ with the boundary condition h has the decomposition

$$\begin{aligned}\varphi(z)=\gamma_0+\gamma_{1,0}&\left(\frac{(z-e_n)(z_0-e_n)}{z_0-z}\right)^{1/2}+\gamma_{2,0}\frac{(z-e_n)(z_0-e_n)}{z_0-z}+\\&+\left(\gamma_{3,0}+\gamma_{3,1}\log\frac{(z-e_n)(z_0-e_n)}{z_0-z}\right)\left(\frac{(z-e_n)(z_0-e_n)}{z_0-z}\right)^{3/2}+\\&+\left(\beta_{1,0}+\beta_{1,1}\log\frac{(z-e_n)(z_0-e_n)}{z_0-z}\right)\left(\frac{(z-e_n)(z_0-e_n)}{z_0-z}\right)^{1+\nu}+\\&+O(|z-e_n|^2\log|z-e_n|)\end{aligned}$$

in a neighbourhood of e_n. Here $\beta_{1,1}\neq 0$ only for $\nu=l/2$, $l\in\mathbf{Z}$.

We choose the parameter t_n to be equal $\operatorname{Re}\gamma_{1,0}$. Then the problem $\mathcal{N}^{(e)}$ with the boundary function

$$h-\sum_{n=1}^{N} t_n \frac{\partial}{\partial n}\varrho_n$$

has a solution v with the following representation in local coordinates on the arcs $S_\pm(e_n)$

$$v(z) = \gamma_0 + \gamma_1 x + (\gamma_{2,0}^{(\pm)} + \gamma_{2,1}^{(\pm)} \log x) x^{3/2} + (\beta_{0,0}^{(\pm)} + \beta_{0,1}^{(\pm)} \log x) x^{1+\nu} + O(x^2 \log x).$$

On the arcs $S_\pm(i_m)$ we have

$$v(z) = \alpha_0 + \alpha_1 x^\nu + \alpha_2^{(\pm)} x^{\nu+1} + O(x^2)$$

in corresponding coordinate systems (cf. Proposition 4).

Moreover, from Proposition 1 it follows that the complex potential of the problem $\mathcal{D}^{(i)}$ with the boundary function $v(p)$, $p \in S \setminus T$, has the representation

$$\varphi^{(i)}(z) = \beta_0 + (\beta_{1,0} + \beta_{1,1} \log(z - e_n))(z - e_n)^\nu +$$
$$+ (\gamma_{1,0} + \gamma_{1,1} \log(z - e_n))(z - e_n)^{1/2} + O(|z - e_n| \log |z - e_n|)$$

in a neighbourhood of e_n. The coefficient $\beta_{1,1}$ does not vanish only for $\nu = l/2$.

Therefore

$$(\partial u^{(i)} / \partial n)(z) = \pm(\beta_{1,0} + \beta_{1,1} \log x) x^{\nu - 1} \pm (\gamma_{1,0} + \gamma_{1,1} \log x) x^{-1/2} + O(\log x),$$

on the arcs $S_\pm(e_n)$, where $\gamma_{1,1} \neq 0$, only for $\nu = l/2$.

For a neighbourhood of i_m one obtains

$$\varphi^{(i)}(z) = \alpha + (\alpha_{0,0} + \alpha_{0,1} \log(z - i_m))(z - i_m)^\nu + \delta_0 (z - i_m)^{1/2} + O(|z - i_m|)$$

(cf. Proposition 2) with $\alpha_{0,1} \neq 0$ only for $\nu = l/2$. This implies the decomposition

$$(\partial u^{(i)} / \partial n)(z) = (\alpha_{0,0} + \alpha_{0,1} \log x) x^{\nu - 1} + \delta_0 x^{-1/2} + O(1)$$

on the arcs $S_\pm(i_m)$. Here $\alpha_{0,1} \neq 0$ only for $\nu = l/2$.

On S we have

$$v = \frac{1}{2\pi} V\left(\partial u^{(i)} / \partial n - h + \sum_1^N t_k \partial \varrho_k / \partial n\right).$$

Consider the difference

$$v_0(z) = v(z) - \frac{1}{2\pi} V\left(\partial u^{(i)} / \partial n - h + \sum_1^N t_k \partial \varrho_k / \partial n\right)(z), \qquad z \in \Omega^c.$$

Let $z = \omega(\zeta)$, $\zeta = \eta + i\xi$, be a conformal mapping of the strip $\Pi = \{(\eta, \xi) : 0 < \xi < 1\}$ onto Ω^c with $\operatorname{Re} \omega^{-1}(i_m) = +\infty$. The function $w(\zeta) = v_0(\omega(\zeta))$ is a solution of the Laplace equation in the strip Π, it grows at infinity not faster than a power function and vanishes on $\partial \Pi$ with exception of a finite set of points. We take the Fourier decomposition of $w(\zeta)$

$$w(\zeta) = \sum_{k=1}^\infty c_k(\eta) \sin \pi k \xi.$$

The coefficients $c_k(\eta)$ have the form $\alpha_k e^{\pi k \eta} + \beta_k e^{-\pi k \eta}$.

Since

$$\int_0^1 |w(\eta, \xi)|^2 d\xi = \frac{1}{2} \sum_{k=1}^{\infty} |c_k(\eta)|^2$$

and since the left-hand side of this equality increases not faster than a power function as η tends to infinity, then $\alpha_k = 0$, $k = 1, 2, \dots$. Hence $v_0(z)$ is bounded in a neighbourhood of i_m. Thus, $v(z)$ is a bounded harmonic function in Ω^c vanishing on $S \setminus T$ and therefore v vanishes in Ω^c.

From this we conclude that the density

$$\tau = (2\pi)^{-1} \left(\partial u^{(i)} / \partial n - h + \sum_{k=1}^{N} t_k \partial \varrho_k / \partial n \right)$$

belongs to the class $\mathfrak{M}$, satisfies the boundary equation of the problem $\mathcal{N}^{(e)}$ and has the required asymptotic representation.

Theorem 4 implies that the solution of equation (28) just constructed is unique. ∎

Explicit formulae for the coefficients $\beta_{1,0}$, $\beta_{1,1}$, $\mu_{1,0}$, $\mu_{1,1}$, $\alpha_{0,0}$, $\alpha_{0,1}$ and μ_0 in (29), (30) were given in [4].

We apply Theorems 4 and 5 to obtain the following result.

THEOREM 6. *Let the function h belong to $\mathfrak{N}$. Suppose that*

$$\int_S h\, ds = 0 \quad \textit{and} \quad \int_S h \operatorname{Re}(1/\zeta_k)\, ds = 0\,, \qquad k = 1, \dots, N\,.$$

Then equation (2) *is uniquely solvable in* $\mathfrak{M}$.

PROOF. By $\{\tau, t\}$, where $\tau \in \mathfrak{M}$ and $t \in \mathbf{R}^N$, denote the unique solution of (4). We apply the Green formula to the solution $v^{(e)}$ of the problem $\mathcal{N}^{(e)}$ and to the function $\operatorname{Re}(1/\zeta_k)$ in $\Omega^c \setminus \{|z - e_k| < \varepsilon\}$. Passing to the limit as $\varepsilon \to 0$ we obtain

$$t_k = (1/\pi) \int_S h \operatorname{Re}(1/\zeta_k)\, ds = 0\,, \qquad k = 1, \dots, N\,.$$

Therefore τ is the unique solution of (2). ∎

4. APPENDIX

4.1. *The external Dirichlet problem.*

Let g be a function from $\mathfrak{N}$. We look for the solution of the problem $\mathcal{D}^{(e)}$ in the form of the potential

$$u(z) = \int_S \sigma(q) \left[\frac{\partial}{\partial n_q} \log \frac{1}{r} + 1 \right] ds_q\,, \qquad z \in \Omega^c\,,\ r = |z - q|\,,$$

with the density σ satisfying the equation

$$\pi\sigma(p) + \int_S \sigma(q)\left[\frac{\partial}{\partial n_q} \log \frac{1}{r} + 1\right] ds_q = g(p). \tag{31}$$

We refer to $u^{(i)}$, $u^{(e)}$ as harmonic extensions of g to Ω and Ω^c. By Propositions 1 and 2 we have decompositions

$$(\partial u^{(e)}/\partial n)(z) = \pm\beta_0 x^{\nu-2} + \beta_1^{(\pm)} x^{\nu-1} + O(1)$$

on the arcs $S_\pm(i_m)$ and

$$(\partial u^{(e)}/\partial n)(z) = (\alpha_{0,0}^{(\pm)} + \alpha_{0,1}^{(\pm)} \log x) x^{\nu-1} + \delta_0 x^{-1/2} + O(1)$$

on the arcs $S_\pm(e_n)$. Let v be the solution of the Neumann problem

$$\Delta v = 0 \quad \text{in } \Omega, \qquad \partial v/\partial n = \partial u^{(e)}/\partial n \quad \text{on } S,$$

normalized by the condition

$$\int_S v\, ds = \int_S g\, ds - 2u^{(e)}(\infty),$$

where $u^{(e)}(\infty)$ is the limit value of $u^{(e)}$ at infinity. According to Propositions 3 and 4 we have

$$v(z) = \pm\beta_0 x^{\nu-1} \pm \gamma_0 x^{-1/2} + O(1)$$

for $\nu \neq l/2$ and

$$v(z) = \pm(\beta_0 + \beta_1 \log x) x^{\nu-1} \pm \gamma_0 x^{-1/2} + O(1)$$

for $\nu = l/2$ on the arcs $S_\pm(i_m)$ and

$$v(z) = (\alpha_0 + \alpha_1 \log x) x^{\nu-1} + \delta x^{-1/2} + O(1)$$

on the arcs $S_\pm(e_n)$. Then $\sigma = (2\pi)^{-1}(v - g)$ is a solution of equation (31) in the class $\mathfrak{M}$, and furthermore, σ has the following representations

a) $$\sigma(z) = \pm(\alpha_0 + \alpha_1 \log x) x^{-1+\nu(i_m)} + O(1)$$

on the arcs $S_\pm(i_m)$,

b) $$\sigma(z) = (\alpha_0 + \alpha_1 \log x)^{-1+\nu(e_n)} + \delta x^{-1/2} + O(1)$$

on the arcs $S_\pm(i_m)$. As in Theorem 1 solutions of the homogeneous equation (31) in $\mathfrak{M}$ are functions of the form

$$\sum_{k=1}^{M} c_k \operatorname{Re}(1/\zeta_k^{(i)}),$$

where $c_k \in \boldsymbol{R}$ and $\zeta_k^{(i)}$ is the conformal mapping of Ω onto the upper half-plane, normalized by the conditions

$$\zeta_k^{(i)}(i_k) = O, \quad \int_S \operatorname{Re}(1/\zeta_k^{(i)})\, ds = 0, \qquad k = 1, \dots, M.$$

4.2. *The internal Neumann problem.*

For the function h from $\mathfrak{N}$ one seeks the solution of the problem $\mathcal{N}^{(i)}$ as the sum of the simple layer potential and of a linear combination of the functions δ_n,

$n = 1, \dots, M$, with unknown real coefficients

$$v^{(e)}(z) = V\tau(z) + \sum_{n=1}^{M} t_n \delta_n(z),$$

where

$$\delta_n(z) = \operatorname{Re}(z - i_n)^{1/2}, \qquad z \in \Omega .$$

The density τ and the vector $t = (t_1, \dots, t_M)$ satisfy the equation

$$\pi\tau(p) + \int_S \tau(q) \frac{\partial}{\partial n_p} \log \frac{1}{r}\, ds_q + \sum_{k=1}^{M} t_k \delta_k(p) = h(p). \tag{32}$$

Arguing as in Theorems 4 and 5, we prove that equation (32) uniquely solvable on the set of pairs $\{\tau, t\}$, $\tau \in \mathfrak{M}$, $t \in \boldsymbol{R}^M$. The only difference is that the harmonic function $v^{(i)}$ satisfying the boundary condition

$$\partial v^{(i)} / \partial n = h - \sum_{k=1}^{M} t_k \delta_k$$

should be normalized so that the harmonic extension $u^{(e)}$ of $v^{(i)}$ from S to Ω^c vanishes at infinity. The density τ has the following representations

a) $\tau(z) = \pm(\beta_{1,0} + \beta_{1,1} \log x) x^{\nu(i_m)-1} \pm (\mu_{1,0} + \mu_{1,1} \log x) x^{-1/2} + O(\log x)$

on the arcs $S_\pm(i_m)$

b) $\tau(z) = (\alpha_{0,0} + \alpha_{0,1} \log x) x^{\nu(e_n)-1} + \mu_0 x^{-1/2} + O(1)$

on the arcs $S_\pm(e_n)$.

4.3. *Counterexample.*

Here we present an equation (1) having no summable solution, although its right-hand side is continuous.

Consider the function on the contour S with an outward peak at point O defined by

$$\omega(x) = \pm \frac{\cos \log(1/x)}{(\log(1/x))^{\alpha}}, \qquad 0 < \alpha < 1,$$

on the arcs $S_\pm(O)$ in a certain coordinate system. The normal derivative of the solution of the problem $\mathcal{D}^{(i)}$ has the decomposition

$$\frac{\partial u}{\partial n}(x, y) = \pm \frac{\cos \log(1/x)}{x^2 (\log(1/x))^{\alpha}} + O((\log(1/x))^{-\alpha}) \qquad \text{on } S_\pm(O).$$

The solution v of the problem $\mathcal{N}^{(e)}$ with the boundary data $\partial u / \partial n$ can be represented in the form

$$v(x, y) = \mp \frac{1}{2}(\pi) \frac{\sin \log(1/x) - \cos \log(1/x)}{x(\log(1/x))^{\alpha}} + (\text{a summable function}).$$

Then the density

$$\sigma = 2\pi(v - \omega)$$

is a non-summable solution of the integral equation of the problem $\mathcal{D}^{(i)}$ with the right-hand side ω (the integral in the double layer potential is understood in the sense of the principal value

$$\lim_{\varepsilon \to 0} \int_{\{\Gamma: |q| > \varepsilon\}} \sigma(q)\, \partial/\partial n_q \log(1/|z-q|)\, ds_q ,$$

where $z \in \Gamma \setminus \{O\}$).

Among solutions of the homogeneous equation (1) there are no densities represented in the form

$$\pm c\, \frac{\sin \log(1/x) - \cos \log(1/x)}{x(\log(1/x))^{\alpha}} + (\text{a summable function}).$$

Indeed the potential $W\sigma$ has a power growth as $z \to 0$, and it vanishes on $S \setminus \{O\}$. Therefore $W\sigma$ is equal to zero in Ω.

Let $u^{(e)}$ denote the solution of the problem $\mathcal{D}^{(e)}$ with boundary function σ. We have

$$V \frac{\partial}{\partial n} u^{(e)} - W u^{(e)} = 2\pi u^{(e)}(\infty), \qquad z \in \Omega .$$

Since $Wu^{(e)}$ vanishes in Ω, it follows that

$$V \frac{\partial}{\partial n} u^{(e)}(z) = u^{(e)}(\infty), \qquad z \in \Omega .$$

It follows from the limit relation for the simple layer potential

$$\frac{\partial}{\partial n}\left(V \frac{\partial}{\partial n} u^{(e)} + \pi u^{(e)}\right)(z) = 0, \qquad z \in S \setminus \{O\}.$$

So, we have

$$V \frac{\partial}{\partial n} u^{(e)}(z) = -2\pi u^{(e)}(z) + u_0(z), \qquad z \in \Omega^c ,$$

where u_0 is a solution of the problem

$$\Delta u_0 = 0 \quad \text{in } \Omega^c , \qquad \frac{\partial}{\partial n} u_0 = 0 \quad \text{on } S \setminus \{O\}. \tag{33}$$

We substitute the integral representation of $u^{(e)}$ into (33). Then

$$W u^{(e)}(z) = u_0(z) - 2\pi u^{(e)}(\infty), \qquad z \in \Omega^c .$$

Since the potential $Wu^{(e)}$ vanishes at infinity we have

$$u_0(\infty) = 2\pi u^{(e)}(\infty).$$

The formulas for limit values of the double layer potential imply

$$\sigma(z) = u^{(e)}(z) = (2\pi)^{-1}(u_0(z) - u_0(\infty)).$$

The functions $u^{(e)}$ and $Wu^{(e)}$ have a power growth as $z \to 0$. So, the function u_0 grows not faster than a power function. Since

$$\sigma(z) = O\left(\frac{1}{x(\log x)^{\alpha}}\right), \qquad z \in S \setminus \{O\},$$

then the function $u_0(z)$ concides with $\operatorname{Re}(1/\zeta(z))$ where $\zeta(z)$ is the conformal mapping of Ω^c onto the upper half-plane subjected to the conditions

$$\zeta(O) = 0, \qquad \text{and} \qquad \operatorname{Re}\zeta(\infty) = 0.$$

Therefore the equation (1) with right-hand side g coinciding with ω on $S_{\pm}$ is unsolvable in $L(S)$.

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