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Teoria dei gruppi. — *On absolutely-nilpotent of class* k *groups.* Nota (*) di PATRIZIA LONGOBARDI, TRUEMAN MACHENRY, MERCEDE MAJ e JAMES WIEGOLD, presentata dal Socio G. Zappa.

ABSTRACT. — A group G in a variety $\mathfrak{V}$ is said to be absolutely-$\mathfrak{V}$, and we write $G \in A\mathfrak{V}$, if central extensions by G are again in $\mathfrak{V}$. Absolutely-abelian groups have been classified by F. R. Beyl. In this paper we concentrate upon the class $A\mathfrak{N}_k$ of absolutely-nilpotent of class k groups. We prove some closure properties of the class $A\mathfrak{N}_k$ and we show that every nilpotent of class k group can be embedded in an $A\mathfrak{N}_k$-group. We describe all metacyclic $A\mathfrak{N}_k$-groups and we characterize 2-generator and infinite 3-generator $A\mathfrak{N}_2$-groups. Finally we study extensions $1 \to N \to H \to G \to 1$, with $N \leq \zeta_n(H)$, the n-centre of H, with $n > 1$.

KEY WORDS: Variety; Central extension; Nilpotent group.

RIASSUNTO. — *Gruppi assolutamente-nilpotenti di classe* k. Un gruppo G in una varietà $\mathfrak{V}$ vien detto assolutamente-$\mathfrak{V}$ (e si scrive $G \in A\mathfrak{V}$) se ogni estensione centrale mediante G appartiene ancora a $\mathfrak{V}$. I gruppi assolutamente-abeliani sono stati caratterizzati da F. R. Beyl. In questa *Nota* si studiano i gruppi assolutamente-nilpotenti di classe k. Si provano alcune proprietà di chiusura della classe $A\mathfrak{N}_k$, e si mostra che ogni gruppo nilpotente di classe k si può immergere in un $A\mathfrak{N}_k$-gruppo. Si descrivono i gruppi metaciclici assolutamente-nilpotenti di classe k ed i gruppi 2-generati e quelli infiniti 3-generati nella classe $A\mathfrak{N}_2$. Infine si esaminano estensioni $1 \to N \to H \to G \to 1$, con $N \leq \zeta_n(H)$, l'ennesimo centro di H.

1. INTRODUCTION

A group G in a variety $\mathfrak{V}$ is said to be *absolutely-*$\mathfrak{V}$ if central extensions by G are again in $\mathfrak{V}$. We denote the class of absolutely-$\mathfrak{V}$ groups by $A\mathfrak{V}$.

Special cases of such groups have been considered by several authors, *e.g.* Varadarajan [9], Evens [5] and Beyl [1-3].

In this paper we concentrate upon the class $A\mathfrak{N}_k$ of absolutely-nilpotent of class k groups. In [1] Beyl has classified all absolutely abelian groups; they are just those abelian groups having trivial multipliers. Conditions sufficient to ensure that groups in nilpotent varieties are absolute are studied in Passi and Vermani [7].

In sect. 2 of this paper we collect some general results about $A\mathfrak{N}_k$-groups. Obviously the class $A\mathfrak{N}_k$ is not a variety, for example, it is not necessarily subgroup-closed, but it does have some interesting closure properties: for instance, it is closed under some nilpotent products (see 2.1). We do not know if $A\mathfrak{N}_k$ is closed under Cartesian products, the best we can say here is that Cartesian powers of finite $A\mathfrak{N}_k$-groups are $A\mathfrak{N}_k$-groups (see 2.3). We have not been able to recognize if epimorphic images of an $A\mathfrak{N}_k$-group G are always $A\mathfrak{N}_k$, we have only proved that this is true if G is finitely generated (see 2.4).

Every nilpotent n-generator group of class k can be embedded in a $2n$-generator $A\mathfrak{N}_k$-group (see 2.2), in particular the class of n-generator $A\mathfrak{N}_k$-groups is not a small class. In sect. 3, 4 and 5 we concentrate upon $\mathfrak{N}_k$-groups with 2 or 3 generators.

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We describe all metacyclic $A\mathfrak{N}_k$-groups (see [4], for $k = 2$). Moreover we characterize the 2-generator and the infinite 3-generator class 2 groups which are absolute-there are no infinite 2-generator non abelian $A\mathfrak{N}_2$-groups, and, in a sense, very few infinite 3-generator $A\mathfrak{N}_2$-groups.

Finally, in sect. 6 we study extensions $1 \to N \to H \to G \to 1$ with $N \leqslant \zeta_n(H)$, $n > 1$.

Notation is as in [8].

If $n \geqslant 1$ is an integer, x, y elements of a group G, H a subgroup of G, we put $[x, {}_n y] = [x, \underbrace{y, \ldots, y}_{n}]$, $[H, {}_n G] = [H, \underbrace{G, \ldots, G}_{n}]$.

If $F = \langle x, y \rangle$ is a free group, we write

$$d_{i,n} = [x, \underbrace{y, x, \ldots, x}_{i}, \underbrace{y, \ldots, y}_{n-i}], \qquad \text{for any } n \geqslant 0, \ 0 \leqslant i \leqslant n .$$

Obviously $[d_{i,n}, x] \equiv d_{i+1,n+1} \pmod{F''}$ and $[d_{i,n}, y] \equiv d_{i,n+1} \pmod{F''}$.

Moreover the set $\{d_{i,n} \mid n \in \mathbb{N}_0, i \in \{0, 1, \ldots, n\}\}$ is a basis for $F' \bmod F''$.

The following result will be used frequently:

1.1. (see [7, Theorem 5.1]) Let $\mathfrak{V}$ be a variety of exponent 0. Then $G \in A\mathfrak{V}$ if and only if $\mathfrak{V}(F) \leqslant [R, F]$ for all free presentations $1 \to R \to F \to G \to 1$.

2. General results

It is easy to see that $A\mathfrak{N}_k$ is closed under restricted direct products. In fact the result holds even for generalized nilpotent products.

2.1. Let $\{A_i : i \in I\}$ be any set of class k groups.

If $A_i \in A\mathfrak{N}_k$ for any $i \in I$ and $l < k$, then the l-th nilpotent product of the A_i is in $A\mathfrak{N}_k$.

Proof. Let G be such a group, and let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a free presentation of G. By 1.1 we have to prove that $\gamma_{k+1} F \leqslant [R, F]$.

Write $X_i = \pi^{-1}(A_i)$, for any $i \in I$; obviously it suffices to prove that $[y_1, y_2, \ldots, y_{k+1}] \in [R, F]$ for any $y_1, y_2, \ldots, y_{k+1} \in \bigcup_{i \in I} X_i$. If there exist $r, s \leqslant k, r \neq s$, such that $y_h \in X_r, y_j \in X_s$, for some $h, j \leqslant k$, then $[y_1, y_2, \ldots, y_k] \in [X_r, X_s]^F \cap \gamma_k F \leqslant R$ and $[y_1, y_2, \ldots, y_k, y_{k+1}] \in [R, F]$. If $y_1, y_2, \ldots, y_k$ are in one and the same X_i, and $y_{k+1} \in X_i$, then $[y_1, y_2, \ldots, y_k, y_{k+1}] \in [R \cap \langle X_i \rangle, \langle X_i \rangle] \subseteq [R, F]$, since $A_i \in A\mathfrak{N}_k$. Now assume that $y_1, \ldots, y_k \in X_r$ and $y_{k+1} \in X_s$, for suitable $r \neq s$. Then $[y_1, y_2, \ldots, y_k, y_{k+1}] \in [\gamma_k X_r, X_s]$. But, for any $j < k$, $[X_s, \underbrace{X_r, \ldots, X_r}_{j}, \gamma_{k-j-1} X_r] \leqslant$ $\leqslant \gamma_k F \cap [X_s, X_r]^F \leqslant R$ and $[X_s, \underbrace{X_r, \ldots, X_r}_{j}, \gamma_{k-j-1} X_r, X_r] \leqslant [R, F]$.

Hence, by induction on i, using the three subgroups lemma, it is easy to verify that $[\gamma_i X_r, [X_s, \underbrace{X_r, \ldots, X_r}_{k-i}]] \leqslant [R, F]$ for $1 \leqslant i \leqslant k$. Hence with $i = k$ we have $[\gamma_k X_r, X_s] \leqslant [R, F]$, as required. ■

For some nilpotent products we can drop the condition that the factors be absolute.

2.2. Let $\{A_i : i \in I\}$ be any set of class k groups such that $\gamma_k A_i \cong \gamma_k A_j$, $i, j \in I$. Then, with $l < k$, the l-th nilpotent product of all A_i with $\gamma_k A_i$ amalgamating is in $A\mathfrak{N}_k$.

PROOF. Similar to the proof of 2.1. ■

For Cartesian products we have:

2.3. Let A be a finite $A\mathfrak{N}_k$-group. Then every Cartesian power of A is in $A\mathfrak{N}_k$.

PROOF. Let A^I be the Cartesian power of A with index set I, and G a group having a central subgroup N such that $G/N \cong A^I$.

To show that G is in $\mathfrak{N}_k$, it is enough to show that every finitely generated subgroup of G is in $\mathfrak{N}_k$.

Thus, let H be a finitely generated subgroup of G. Then HN/N is a finitely generated subgroup of A^I, and the finiteness of A now implies that HN/N is a subgroup of the direct product X of finitely many groups isomorphic to A, namely of diagonals of powers A^J with $J \subseteq I$. (The reader will recognize the genesis of this type of argument in B. H. Neumann's proof [6] that Cartesian products of finite groups are locally finite). Note next that X is in $A\mathfrak{N}_k$. Thus, if K is the subgroup of G such that $K/N = X$, it follows that K is in $\mathfrak{N}_k$, and thus H is in $\mathfrak{N}_k$ since $K \geqslant H$.

Therefore G is in $\mathfrak{N}_k$ and A^I is in $A\mathfrak{N}_k$, as required. ■

It would be nice to know if epimorphic images of $A\mathfrak{N}_k$-groups are always $A\mathfrak{N}_k$. We have not been able to confirm or deny this. For finitely generated groups, we have:

2.4. If G is a finitely generated $A\mathfrak{N}_k$-group and N is a normal subgroup of G, then G/N is in $A\mathfrak{N}_k$.

PROOF. Let G be a finitely generated nilpotent group. Then $M(G)$ is finitely generated and so *Hopf*. Write $\mathfrak{V} = \mathfrak{N}_k$.

Obviously it suffices to show that G is in $A\mathfrak{V}$ if and only if $\mathfrak{V}(F) \leqslant [R, F]$ for some free presentation $1 \to R \to F \to G \to 1$ of G. For, let $1 \to R_1 \to F_1 \to G \to 1$ any free presentation of G. Then $(R_1 \cap F_1')/([R_1, F_1]V(F_1)) \cong M_V(G) \cong M(G) \cong (R_1 \cap F_1')/[R_1, F_1]$, so that $\mathfrak{V}(F_1) \leqslant [R_1, F_1]$. Then the result follows from 1.1. ■

We end this section with the following theorem, which in some sense reduces the study of finitely generated $A\mathfrak{N}_k$-groups to the cases of torsion-free groups and finite groups (see also [1, Theorem 2.1]).

2.5. Let $G \in A\mathfrak{N}_k$. If G is finitely generated, then G can be embedded as a subgroup of finite index in an $A\mathfrak{N}_k$-group which is the direct product of a finite $A\mathfrak{N}_k$-group and a finitely generated torsion-free $A\mathfrak{N}_k$-group.

Proof. Let τG be the torsion subgroup of G. Then τG is finite and $G/\tau G$ is a finitely generated torsion-free $A\mathfrak{N}_k$-group by 2.4. Since G is residually finite, there exists a normal subgroup N in G such that $N \cap \tau G = 1$ and G/N is a finite group in $A\mathfrak{N}_k$, again by 2.4. Then $\chi : x \in G \mapsto (xN, x\tau G) \in G/N \times G/\tau G$ is an embedding. By 2.1 $H = G/N \times G/\tau G \in A\mathfrak{N}_k$ if $k > 1$. Moreover G^χ has finite index in H since N^χ has finite index in $G/\tau G$. If $k = 1$, the result is trivial. ■

3. Metacyclic $A\mathfrak{N}_k$-groups

Theorem 3.1. Let G be a metacyclic p-group of class k. Then

($\mathcal{A}$) $G \in A\mathfrak{N}_k$ if and only if one of the following holds:

(*i*) $G = \langle x, y \mid |x| = p^a, x^{p^b} = y^{p^c}, [x, y] = x^{p^d s}\rangle$, with $(p, s) = 1$, $a \leqslant b + d$, $b \leqslant c$, and $b \leqslant (k-1)d$, $d > 0$;

(*ii*) $p = 2$, $k > 2$ and $G = \langle x, y \mid |x| = 2^k, y^{2^{k-m-1}} = x^{2^k} = 1, x^y = x^{-1+2^m}\rangle$, with $0 < m < k-1$;

(*iii*) $p = 2$, $k > 2$ and $G = \langle x, y \mid |x| = 2^k, x^{2^{k-1}} = y^{2^c}, x^y = x^{-1+2^m s}\rangle$, with $s \not\equiv 0(2)$, $0 < m < k-1$, $k-1-m \leqslant c \leqslant k-1$;

(*iv*) $p = 2, k > 2$ and $G = \langle x, y \mid |x| = 2^k, x^{2^{k-1}} = y^{2^h}, x^y = x^{-1}\rangle$, with $0 < h < k-1$.

($\mathcal{B}$) For any $n \geqslant 2$ there exists a group H, with a normal subgroup $M \leqslant \zeta_n H$ such that $H/M \cong G$ and $cl H \geqslant n + k - 1$.

Proof. Since G is metacyclic, $G = \langle x, y\rangle$, with $|x| = p^a$, $x^{p^b} = y^{p^c}$, $[x, y] = x^{p^d s}$, $a \leqslant b + d$, $(p, s) = 1$, $d \geqslant 1$.

Let $1 \to R \to F \to G \to 1$ be a free presentation of G, with $F = \langle \bar{x}, \bar{y}\rangle$.

Assume first that $p \neq 2$. Then $(xy)^{p^{a-1}} = x^{p^{a-1}} y^{p^{a-1}}$, and we may suppose $|x| \leqslant |y|$ and $b \leqslant c$ (replacing eventually y by xy). Then $[R, F] = \langle d_{i,n}, d_{0,h}^{p^b}, d_{0,r}, d_{0,m}^{-1} d_{0,0}^{p^{md} s^m}\rangle F''$, with $1 \leqslant n$, $1 \leqslant i \leqslant n$, $1 \leqslant m \leqslant k-1$, $0 \leqslant h \leqslant k-1$, $r \geqslant s$. From $\gamma_{k+1} F \leqslant [R, F]$ it follows that $d_{0,k-1} \in [R, F]$ and

$$d_{0,k-1} = (d_{0,k-1}^{-1} d_{0,0}^{p^{(k-1)d} s^{k-1}})^\beta d_{0,0}^{p^b \alpha} (d_{0,k-1})^{p^b \gamma}$$

since the $d_{h,l}$ are independent mod F''.

Hence $\beta - p^b \gamma - 1 = 0$, $p^{(k-1)d} s^{k-1} \beta + p^b \alpha = 0$ and that happens if and only if $b \leqslant (k-1)d$.

Therefore if G is in $A\mathfrak{N}_k$ then $b \leqslant (k-1)d$.

Conversely it is easy to see that G is in $A\mathfrak{N}_k$ if (*i*) holds.

Furthermore, for any $l \geqslant 2$ we have

$$[N, {}_l F] \leqslant F'' \langle d_{i,n}, d_{0,b+l}^{p^b}, d_{0,r+l}, d_{0,m+l}^{-1} d_{0,l}^{p^{md} s^m} \rangle$$

$1 \leqslant n$, $k \leqslant r$, $1 \leqslant m \leqslant k-1$, $1 \leqslant i$.

From $\gamma_{k+l-2} F = [N, {}_{l-1} F]$ it follows $d_{0,k-2+l-1} \in [N, {}_{l-1} F]$ and

$$d_{0,k+l-3} = (d_{0,k-2+(l-1)}^{-1} d_{0,l-1}^{p^{(k-2)d} s^{k-2}})^{\beta} (d_{0,l}^{p^b})^{\alpha} (d_{0,k+l-2}^{p^b})^{\gamma}$$

and from that $b \leqslant (k-2)d$. Thus $a \leqslant b + d \leqslant (k-1)d$ and G is of class $\leqslant k-1$, a contradiction.

Then, with $T = [N, {}_{l-1} F]$, the group $H = F/T$ has class $\geqslant s + l - 1$ and $N/T \leqslant \zeta_l H$, hence $(\mathcal{B})$ holds.

Now assume that $p = 2$.

If $|x| \leqslant |y|$, arguing as before we can prove that G has the structure in (*i*) and that $(\mathcal{B})$ holds. Now let $|x| > |y|$; then we can assume $x^y = x^{1+2t}$, with t odd (if $[x, y] = x^{4s}$, then $(yx)^{2^{a-1}} = x^{2^{a-1}} \neq 1$ and, replacing y by xy, we get $|x| \leqslant |y|$).

Then $[x, {}_i y] = x^{2^i t}$, for every i, and $|x| = 2^k$ since G has class k, also $b \geqslant k-1$.

First suppose that $t \neq -1$; then $t = -1 + 2^m s$ with $s \not\equiv 0(2)$. By induction on h it is easy to prove that, for every $h \in \mathbb{N}$, we have $[x, y^{2^h}] = [x, y]^{2^{h+m}\beta}$, $\beta \not\equiv 0(2)$. Thus $c \geqslant k - m - 1$, and $[N, F] = \langle d_{i,n}, d_{0,h}^{2^{c+m}}, d_{0,b}^{2^b}, d_{0,r}, d_{0,l}^{-1} d_{0,0}^{2^l t^l} \rangle F''$ with $n \in \mathbb{N}$, $1 \leqslant i \leqslant n$, $0 \leqslant h \leqslant k-1$, $r \geqslant k$, $1 \leqslant l \leqslant k-1$. If $b > c + m$ and $G \in A\mathcal{N}_k$ then $c + m \leqslant k - 1$, and so $c + m = k - 1$, $c = k - m - 1$, $b = k$ and (*ii*) holds. If $b \leqslant c + m$ and $G \in A\mathcal{N}_k$, then $b \leqslant k - 1$, thus $b = k - 1$ and (*iii*) holds.

As in the previous case we prove that $(\mathcal{B})$ holds and that $G \in A\mathcal{N}_k$ if either (*ii*) or (*iii*) hold.

Finally suppose $x^y = x^{-1}$. Then, if $b = k - 1$, we get

$$G = \langle x, y \,|\, x^{2^k} = 1, x^{2^{k-1}} = y^{2^b}, x^y = x^{-1} \rangle, \qquad 1 \leqslant b < k - 1,$$

and $[N, F] = \langle d_{i,n}, d_{0,h}^{2^{k-1}}, d_{0,r}, d_{0,m} d_{0,0}^{2^m} \rangle F''$, with $n \in \mathbb{N}$, $1 \leqslant i \leqslant n$, $0 \leqslant h \leqslant k-1$, $r \geqslant k$, $1 \leqslant m \leqslant k-1$. Thus $d_{0,k-1} \in [N, F]$ and arguing as in the previous case $d_{0,k+l-3} \notin [N, {}_{l-1} F]$. If $b = k$, then $G = \langle x \rangle \rtimes \langle y \rangle$, with $|x| = 2^k$, $x^y = x^{-1}$, $|y| = 2^b$, $0 < h < k$. Then, with $G_n = \langle a \rangle \rtimes \langle b \rangle$, $|a| = 2^{k+h}$, $a^b = a^{-1}$, $|b| = 2^b$, we have $a^{2^k} \in \zeta_n G_n$, and G_n has class $k + n$. From $G_n / \langle a^{2^k} \rangle \cong G$ it follows that G is not in $A\mathcal{N}_k$ and that $(\mathcal{B})$ holds. ■

4. 2-GENERATOR $A\mathcal{N}_2$-GROUPS

We give a complete description of 2-generator $A\mathcal{N}_2$-groups.

We start with an easy Lemma:

LEMMA 4.1. Let G be a 2-generator finite p-group and $1 \to R \to F \to G \to 1$ a free presentation of G, with F 2-generator. Then either G is metacyclic or $R \leqslant F^p [F', F]$ and, if $p = 2$, $R \leqslant \langle x^2, y^2 \rangle (F')^2 [F', F]$, with $F = \langle x, y \rangle$.

Proof. There exist free generators x, y of F such that

$$R = \langle x^{\alpha}[x,y]^{\gamma}, y^{\beta}[x,y]^{\delta}, [x,y]^{\mu}\rangle [F',F],$$

where $\alpha, \beta, \gamma, \delta, \mu \in \mathbb{Z}$. Obviously $p \mid \alpha$ and $p \mid \beta$; if $p \nmid \mu$, then G is abelian. If either $p \nmid \gamma$ or $p \nmid \delta$, then either $\langle x\rangle[G',G]$ or $\langle y\rangle[G',G]$ is normal in $G/[G',G]$ and $G/[G',G]$ is metacyclic; so G is metacyclic.

Finally if $p \mid \gamma$, $p \mid \mu$, $p \mid \delta$, then $R \leqslant F^p[F',F]$, and, if $p=2$, $R \leqslant \langle x^2, y^2\rangle (F')^2[F',F]$. ■

Theorem 4.2. Let G be a 2-generator finite p-group.
Then $G \in A\mathfrak{N}_2$ if and only if G is metacyclic and

$$G = \langle a, b \mid |a| = p^{\alpha}, a^{p^{\beta}} = b^{p^{\gamma}}, [a,b] = a^{kp^{\delta}}\rangle,$$

where $(k,p)=1$, $\beta \leqslant \gamma$, $\alpha \leqslant \beta + \delta$ and $\beta \leqslant \delta$.

Proof. Such a group is in $A\mathfrak{N}_2$ (see Theorem 3.1).
Now assume that $G = \langle a, b\rangle$ is a p-group in $A\mathfrak{N}_2$.
If G is metacyclic, then G has the required structure (see Theorem 3.1).
Assume for a contradiction that G is non-metacyclic, and let $1 \to R \to F \to G \to 1$ be a free presentation of G. Thus we have $R \leqslant F^p[F',F]$ by Lemma 4.1; hence if $p \neq 2$, $[R,F] = [F^p,F][F',F,F] = (F')^p[F',F,F]$ and $[F',F] \leqslant [R,F]$, contradicting 1.1.

Now let $p=2$; then for some free generators x, y of F we have $R \leqslant \langle x^2, y^2\rangle(F')^2[F',F]$ and $[R,F] \leqslant \langle [x^2,y^2],[y^2,x]\rangle(\gamma_3 F)^2\gamma_4 F = \langle [x,y]^2[x,y,x], [x,y]^2[x,y,y]\rangle(\gamma_3 F)^2\gamma_4 F$. From $[x,y,x] \in [R,F]$ it follows $[x,y,x] = [x,y]^{2h} \cdot [x,y,x]^h [x,y]^{2k}[x,y,x]^k a$, with $a \in (\gamma_3 F)^2\gamma_4 F$ and $2h + 2k = 0$, $k \equiv 0(2)$, $h \equiv 1(2)$, contradiction. ■

A finite group is in $A\mathfrak{N}_2$ if and only if its Sylow subgroups are. Furthermore the following result shows that there are no infinite 2-generator non abelian $A\mathfrak{N}_2$-groups. Hence Theorem 4.2 gives a complete description of 2-generator $A\mathfrak{N}_2$-groups.

Theorem 4.3. There are no 2-generator infinite non-abelian $A\mathfrak{N}_2$-groups.

Proof. Assume for a contradiction that G is an infinite 2-generator non-abelian $A\mathfrak{N}_2$-group. Then G/G' is again infinite and 2-generator; we may write $G/G' = \langle xG'\rangle \times \langle yG'\rangle$, with yG' of infinite order. Then $G = \langle x, y\rangle$ and $\langle x\rangle \cap \langle y\rangle = 1$. By 2.4 we can assume $|[x,y]| = p$. Thus x^p, $y^p \in \zeta G$ and we can assume that x is a p-element.

If $\langle x\rangle \cap G' = 1$, then the group $\langle x, y\rangle / \langle x^p, y^p\rangle$ is isomorphic to D_4 if $p=2$ and is not metacyclic if $p \neq 2$, in any case it is not in $A\mathfrak{N}_2$.

If $\langle x\rangle \cap G' = \langle x^{p^{\alpha}}\rangle \neq 1$, then $|x| = p^{\alpha+1}$ and the group $\langle x, y\rangle / \langle y^{p^{\alpha+1}}\rangle$ is not in $A\mathfrak{N}_2$ by Theorem 4.2. ■

By contrast

THEOREM 4.4. Every infinite 2-generator nil-2 group is residually $A\mathcal{N}_2$.

PROOF. Let G be a 2-generator infinite nil-2 group. Then G/G' is 2-generator and infinite; we can assume that $G/G' = \langle aG' \rangle \times \langle bG' \rangle$, with bG' of infinite order. Thus $G = \langle a, b \rangle$ and $\langle a \rangle \cap \langle b \rangle = 1$. Let T be the torsion subgroup of G, then T is finite and we can assume T to be a p-group (p a prime).

Assume first that $|[a,b]| = p^k$. Then $a^{p^k}, b^{p^k} \in \zeta G$. If a is torsion-free, then, for any $\gamma \geqslant k$, with $N_\gamma = \langle a^{p^\gamma}[b,a], b^{p^\gamma}[b,a] \rangle$, we have $G/N_\gamma \in A\mathcal{N}_2$; obviously $\cap N_\gamma = 1$ and G is residually $A\mathcal{N}_2$.

If $|a| = p^h$, then $h \geqslant k$; for any $\gamma \geqslant h+1$, with $N_\gamma = \langle b^{p^\gamma}[b,a] \rangle$, we have $G/N_\gamma = \langle b, ab \,|\, b^{p^{k+\gamma}} = 1, b^{p^h} = (ab)^{p^h}, [b, ab] = b^{p^\gamma} \rangle$, $h \leqslant \gamma$ and $G/N_\gamma \in A\mathcal{N}_2$ by Theorem 4.2; moreover $\cap N_\gamma = 1$ as required.

Now let $[a,b]$ be torsion-free; then, with $M_k = \langle [a,b]^{p^k} \rangle$, we have $\cap M_k = 1$ and G/M_k residually $A\mathcal{N}_2$ by the previous case, thus G is residually $A\mathcal{N}_2$ as required. ■

5. INFINITE 3-GENERATOR $A\mathcal{N}_2$-GROUPS

By 2.1 the direct product of an infinite cyclic group with a finite 2-generator $A\mathcal{N}_2$-group is in $A\mathcal{N}_2$. The aim of this section is to prove that there are no other infinite 3-generator $A\mathcal{N}_2$-groups.

THEOREM 5.1. A non-abelian infinite 3-generator group is in $A\mathcal{N}_2$ if and only if it is isomorphic to $C_\infty \times H$, with $H \in A\mathcal{N}_2$, H finite.

PROOF. As remarked above, such groups are in $A\mathcal{N}_2$.

Conversely, assume that $G \in A\mathcal{N}_2$ is a 3-generator infinite group, neither 2-generator nor abelian. Then G/G' is again a 3-generator infinite group and we can write $G/G' = \langle aG' \rangle \times \langle bG' \rangle \times \langle cG' \rangle$, with cG' of infinite order. Then $\langle a, b \rangle \cap \langle c \rangle = 1$. We prove that $G/\langle [a,b] \rangle$ is abelian, and from that it follows $[a,c] = [a, b^\alpha]$, $[b,c] = [a^\beta, b]$ for some $\alpha, \beta \in \mathbb{Z}$ and $[a, ca^\beta b^{-1}] = [b, ca^\beta b^{-\alpha}] = 1$, so $G = \langle a, b \rangle \times \langle cab^{-\alpha} \rangle$, as required.

$G/\langle [a,b] \rangle$ is in $A\mathcal{N}_2$, so we may assume $[a,b] = 1$, $\langle a \rangle \cap \langle b \rangle = 1$ and prove that G is abelian.

It suffices to show that $(*)$ if $[a,c] = 1$, then $[b,c] = 1$.

In fact then $G/\langle [a,c] \rangle$ is abelian, so $[b,c] = [a^\alpha, c]$ for some $\alpha \in \mathbb{Z}$, hence $[ba^{-\alpha}, c] = 1$ and by $(*)[a,c] = 1$.

Then assume $[a,c] = [a,b] = 1$. Since $G/\langle a \rangle$ is a 2-generator infinite $A\mathcal{N}_2$-group, it follows that $G/\langle a \rangle$ is abelian, by Theorem 4.3, hence $[b,c] = a^\gamma$ for some $\gamma \in \mathbb{Z}$. Then $G = (\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle$ and $G = \langle a, b, c \,|\, [a,b] = [a,c] = 1, [b,c] = a^\gamma, a^\alpha = 1, b^\beta = 1 \rangle$ for some $\alpha, \beta \in \mathbb{N}_0$ (α or $\beta = 0$ if a or b torsion-free).

Let $1 \to N \to F \to G \to 1$ be a free presentation of G with $F = \langle x, y, z \rangle$, $N = \langle x^\alpha, y^\beta, [z,y]x^\gamma, [x,y], [x,z] \rangle^F$. Then $N = \langle x^\alpha, y^\beta, [z,y]x^\gamma, [x,y], [x,z], [y,z]^\beta \rangle \cdot$

$\cdot[F', F]$ and $[N, F] = \langle [x, y]^\alpha, [x, z]^\alpha, [y, x]^\beta, [y, z]^\beta, [z, y, z][x, z]^\gamma, [z, y, y][x, y]^\gamma, [x, z, x], [x, z, y], [x, z, z], [x, y, x], [x, y, y], [x, y, z], [z, y, x] \rangle [F', F, F]$.

From $[z, y, z] \in [N, F]$ it follows easily that $[z, y, z] = ([z, y, z][x, z]^\gamma)^\mu ([x, z]^\alpha)^\nu$, and from that $\mu = 1$, $\gamma = \alpha\nu$, so α divides γ, $[b, c] = 1$ and G is abelian, as required. ■

6. Higher nilpotent extensions

Let G be an absolutely nilpotent of class k and H a group possessing a normal subgroup N contained in the n-th term $\zeta_n H$ of the upper central series such that $H/N \cong G$, then $H/[H, N]$ is a central extension by G so has class k, and thus H has class at most $k + n - 1$, one less than is granted by general dispensation. In section 4 we showed that if G is a metacyclic p-group of class k and n any integer $\geqslant 2$, then there exists an extension $1 \to N \to H \to G \to 1$, with $N \leqslant \zeta_n H$ and H of class $s \geqslant n + k - 1$.

We believe that for most constellations of k and n, every nilpotent group G of class k has an «n-th central extension» $1 \to N \to H \to G \to 1$, $N \leqslant \zeta_n H$, $n > 1$, such that H is of class at least $k + n - 1$.

We have been able to prove this only for class 2 finite p-groups and for 2-generator p-groups of class $\leqslant 3$. We start with the following remark:

Theorem 6.1. *Let G be a finite p-group of class $k \geqslant 2$. Then for any $n \geqslant k$ there exists a group H of class $\geqslant n + 1$ with a normal subgroup M such that $M \leqslant \zeta_n H$ and $H/M \cong G$.*

Proof. Let $1 \to R \to F \to G \to 1$ be a free presentation of G, with $d(F) = d(G)$, where $d(G)$ is the minimum number of generators of the group G. Then $R \leqslant F' F^p$, so $[R, F] \leqslant [F' F^p, F] = [F', F][F^p, F] \leqslant (F')^p \gamma_3 F$ and, by induction on i, for every $i \geqslant 1$, $[R,_i F] \leqslant (F')^p \gamma_{i+2} F$. Therefore, with $N = [R,_n F]$, we get $R/N \leqslant \gamma_n (F/N)$, $(F/N)/(R/N) \cong G$, and the group F/N has class $\geqslant n + 1$ since $\gamma_{n+1} F \nleqslant (F')^p \gamma_{i+2} F$. ■

Theorem 6.2. *Let G be a 2-generator p-group of class $\leqslant 3$. Then for every integer $n > 1$ there exists a group H of class $n + 2$ with a normal subgroup M such that $M \leqslant \gamma_n H$ and $H/M \cong G$.*

Proof. If G is metacyclic, the result follows from Theorem 3.1. Otherwise, by Lemma 4.1, there exists a free presentation of G $1 \to R \to F \to G \to 1$, $F = \langle x, y \rangle$ such that $R \leqslant F' F^p$ and $R \leqslant \langle x^2, y^2 \rangle (F')^2 [F', F]$ if $p = 2$.

Obviously $\gamma_{n+3} F \leqslant [R,_{n-1} F]$, since $\gamma_4 F \leqslant R$. We show that, for every $n \geqslant 1$,

$$\gamma_{n+2} F \nleqslant [R,_n F]. \tag{6.1}$$

From this the result follows. In fact, if $\gamma_{n+3} F \nleqslant [R,_n F]$, then the group $H = F/[R,_{n-1} F]$ has class $n + 2$ and, with $M = R/[R,_{n-1} F]$, we have $M \leqslant \gamma_{n-1} H \leqslant \gamma_n H$ and $H/M \cong F/R \cong G$. If $\gamma_{n+3} F \leqslant [R,_n F]$, then the group $H = F/[R,_n F]$ has class $n + 2$ and, with $M = R/[R,_n F]$ we have $M \leqslant \gamma_n H$ and $H/M \cong G$.

To establish (6.1) assume first $p \neq 2$. Then $[R, F] \leqslant [F^p [F', F], F] = [F^p, F]\gamma_4 F = (F')^p \gamma_4 F$ and, for every $n \geqslant 1$, $[R, {}_n F] \leqslant (\gamma_{n+1} F)^p \gamma_{n+3} F$. Then $\gamma_{n+2} F \cap [R, {}_n F] \leqslant \gamma_{n+2} F \cap (\gamma_{n+1} F)^p \gamma_{n+3} F = \gamma_{n+3} F(\gamma_{n+2} F \cap (\gamma_{n+1} F)^p) = \gamma_{n+3} F(\gamma_{n+2} F)^p$. Therefore $\gamma_{n+2} F \nleqslant [R, {}_n F]$, since $\gamma_{n+2} F \nleqslant \gamma_{n+3} F(\gamma_{n+2} F)^p$.

Assume now $p = 2$. Then $[R, F] \leqslant \langle [x^2, y], [y^2, x] \rangle^F (\gamma_3 F)^2 \gamma_4 F = \langle [x, y]^2 \cdot [x, y, x], [x, y]^2 [x, y, y] \rangle (\gamma_3 F)^2 \gamma_4 F$, and, by induction on i, it is easy to show that, for every $n \geqslant 1$, $[R, {}_n F] \leqslant (\gamma_{n+2} F)^2 \gamma_{n+3} F F'' < d_{i,n} d_{i,n-1}^2, d_{j,n} d_{j-1,n-1}^2 \mid 0 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n \rangle$.

Assume by contradiction $\gamma_{n+2} F \leqslant [R, {}_n F]$ for some $n \geqslant 1$. Then $d_{0,n} \in [N, {}_n F]$ and

$$d_{0,n} = \prod_{i=0}^{n-1} d_{i,n}^{h_i} d_{i,n-1}^{2h_i} \prod_{j=1}^{n} d_{j,n}^{k_j} d_{j-1,n-1}^{2k_j} a \tag{6.2}$$

for some integers h_i, k_j and $a \in F''(\gamma_{n+2} F)^2 \gamma_{n+3} F$.

The elements $d_{i,n}, d_{j,n-1}, i \in \{0, \ldots, n-1\}, j \in \{0, \ldots, n\}$, are free generators of $\gamma_{n+1} F \bmod \gamma_{n+3} F F''$, thus from (6.2) it follows

$$1 \equiv h_0(2), \quad 0 \equiv k_n(2), \quad h_i + k_i \equiv 0(2), \quad \text{for } 1 \leqslant i \leqslant n-1,$$
$$2h_i + 2k_{i+1} \equiv 0(2), \quad \text{for } 0 \leqslant i \leqslant n-1,$$

and that is impossible. ■

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