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Brief comments on the Note «Qualche osservazione sulle forze centrali in Relatività Ristretta» of Luigi Stazi

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Fisica matematica. — *Brief comments on the* Note *«Qualche osservazione sulle forze centrali in Relatività Ristretta» of Luigi Stazi.* Nota di JOACHIM STUBBE e VALENTINO L. TELEGDI, presentata (*) dal Socio V. L. Telegdi.

ABSTRACT. — The central «pseudopotentials» yielding, in relativistic mechanics, closed (bounded) orbits for any given energy are derived by inspection (of the algebraic form of the hamiltonian).

KEY WORDS: Special Relativity; Particle dynamics; Central forces; Closed orbits.

RIASSUNTO. — *Alcuni commenti sulla* Nota *«Qualche osservazione sulle forze centrali in Relatività Ristretta» di Luigi Stazi.* I «pseudopotenziali» centrali per cui, nel quadro della relatività ristretta, le orbite al finito sono necessariamente chiuse per qualunque energia data vengono derivati per semplice ispezione (della forma algebrica dell'hamiltoniana).

In a recent *Note* in these Proceedings [1], L. Stazi has determined, in the framework of special relativity, the central forces under which a mass point moves along a closed trajectory for any given value of the energy. He essentially follows a differential approach, generalizing Binet's formula to relativity. We present here a much shorter derivation, based on a comparison of the algebraic forms of the non-relativistic and relativistic hamiltonians. Our results confirm Stazi's, except that the derived «pseudopotentials» have no explicit dependence on the angular momentum of the orbit. Our approach is inspired by Schiff's [2, chapt. 13] recasting, in quantum mechanics, of the relativistic Coulomb problem into the algebraic form of the non-relativistic one.

We consider the classical hamiltonian function for a central potential with a generalized kinetic energy $T(p^2)$, *i.e.*

$$H = T(p^2) + V(\varrho)\,. \tag{1}$$

The angular momentum L is evidently a conserved quantity; parametrizing the radial coordinate $\varrho(t)$ by the polar angle θ and using $\xi(\theta) = 1/\varrho(t)$, one has

$$H = T[(L^2(\xi'^2 + \xi^2))] + V(1/\xi) = E\,. \tag{2}$$

Nonrelativistically $T(p^2) = p^2/2m_0$, and by Bertrand's theorem there are only two potentials such that all bounded orbits are closed, namely $V = -\lambda\xi$ and $V = \omega^2/2\xi^2$, leading to the equations

$$(3_1)\;(3_2)\qquad L^2(\xi'^2 + \xi^2)/2m_0 = \begin{cases} E + \lambda\xi\,, \\ E - \omega^2/2\xi^2\,. \end{cases}$$

Taking the inverse of the generalized kinetic energy function, one has for any E and L

$$L^2(\xi'^2 + \xi^2) = T^{-1}[E - V(1/\xi)]\,. \tag{4}$$

(*) Nella seduta del 15 giugno 1995.

If one chooses V such that the right hand side of (4) is one of the forms $(3_1, 3_2)$ then that V implies a closed orbit. Since that V depends on E, one should properly call it a pseudopotential; we shall hence denote it by $\tilde{V}$. Taking for T the relativistic expression $T = (p^2c^2 + m_0c^2)^{1/2}$, one readily finds

$$\tilde{V}(1/\xi) = \begin{cases} E - (E^2 + 2\tilde{\lambda}\xi)^{1/2}\,, & (5_1) \\ E - (E^2 - \tilde{\omega}^2/\xi^2)^{1/2}\,, & (5_2) \end{cases}$$

where $\tilde{\lambda} = m_0c^2\lambda$ and $\tilde{\omega}^2 = m_0c^2\omega^2$ (with this choice one recovers eqs. (3) in the non-relativistic limit) The above relations resemble eqs. (19) of [1], but differ in fact substantially in that our $\tilde{V}$ does not depend on L. That L-dependence in [1] is spurious and has its origin in the arbitrary normalization of the centrifugal term in eq. (18) of that *Note*.

A problem related to the one treated here is the transformation of the hamiltonian for the relativistic Kepler problem into the algebraic form (3_1). A term in $\lambda^2\xi^2$ arises relativistically, and can be eliminated by the scale transformation $\tilde{\xi}(\theta) = \xi(\gamma\theta)$, with $\gamma^2 = (1 - \lambda^2/L^2c^2)$. This is tantamount to introducing [3] a «pseudo angular momentum» $\tilde{L}^2 = L^2 - \lambda^2/c^2 = L^2\gamma^2$. As is well known, γ is the angular frequency of the perihelion precession. For the harmonic oscillator, no analogous transformation is possible.

References

[1] L. Stazi, *Qualche osservazione sulle forze centrali in Relatività Ristretta.* Rend. Mat. Acc. Lincei, s. 9, v. 6, 1995, 111-115.

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[3] P. R. Auvil - L. M. Brown, *The relativistic hydrogen atom: A simple solution.* Am. J. Phys., 46(6), 1978, 679-681.

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