On homogenization problems for the Laplace operator in partially perforated domains with Neumann’s condition on the boundary of cavities.

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Analisi matematica. — On homogeneization problems for the Laplace operator in partially perforated domains with Neumann's condition on the boundary of cavities. Nota (*)

ABSTRACT. — In this paper the problem of homogeneization for the Laplace operator in partially perforated domains with small cavities and the Neumann boundary conditions on the boundary of cavities is studied. The corresponding spectral problem is also considered.

KEY WORDS: Homogeneization; Perforated domains; Small cavities; Neumann's condition; Spectral problem.

INTRODUCTION

The problem of homogeneization for the Laplace operator in perforated domains with a small density of cavities and the Dirichlet boundary conditions on the boundary of cavities was considered in many papers (see, for example, [1-5]). In this paper we study the problem of homogeneization for the Laplace operator in partially perforated domains with small cavities and the Neumann boundary conditions on the boundary of cavities. The corresponding spectral problems are also considered.

1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \), \( Q = \{ x : 0 < x_j < 1, j = 1, \ldots, n \} \), \( G_0 \) is a domain in \( Q \), \( G_0 \subset Q \) and \( G_0 \) is diffeomorphic to a closed ball.

We set \( G_\varepsilon = \bigcup_{z \in \mathbb{Z}} (a_\varepsilon G_0 + \varepsilon z) \), where \( \varepsilon \) is a small positive parameter, \( a_\varepsilon \) is a constant which depends on \( \varepsilon \) and \( a_\varepsilon \varepsilon^{-1} \to 0 \) as \( \varepsilon \to 0 \), \( \mathbb{Z} \) is the set of vectors \( z \) with integer components, \( aB = \{ x: \alpha^{-1} x \in B \} \), \( Y_\varepsilon = \varepsilon Q \setminus \overline{a_\varepsilon G_0} \). We assume that \( \Omega \cap \{ x_1 = 0 \} = = \gamma \neq 0 \).

We denote
\[
\Omega^+ = \Omega \cap \{ x_1 > 0 \}, \quad \Omega^- = \Omega \cap \{ x_1 < 0 \}, \quad \Omega_\varepsilon^+ = \Omega^+ \setminus \overline{G_\varepsilon}, \\
S_0 = \partial G_0, \quad \Omega_\varepsilon = \Omega_\varepsilon^+ \cup \gamma \cup \Omega^-, \quad S_\varepsilon = \partial \Omega_\varepsilon \cap \Omega, \quad \Gamma_\varepsilon = \partial \Omega_\varepsilon \setminus S_\varepsilon, \\
\langle u \rangle_\omega = |\omega|^{-1} \int_\omega u \, dx, \text{ where } |\omega| \text{ is the volume of the domain } \omega.
\]

As usual we denote by \( H_1(\Omega, \Gamma) \) the space of functions which is obtained by completion of the set of infinitely differentiable in \( \overline{\Omega} \) functions \( u(x) \), equal to zero in a
neighborhood of \( \Gamma \), by the norm

\[
\|u\|_{H_1(\Omega)} = \left( \int_{\Omega} \left( u^2 + |\nabla u|^2 \right) \, dx \right)^{1/2}.
\]

In partially perforated domain \( \Omega_\varepsilon \) we study the Neumann boundary value problem:

(1) \( \Delta u_\varepsilon = f \) in \( \Omega_\varepsilon \), \( \frac{\partial u_\varepsilon}{\partial \nu} = 0 \) on \( S_\varepsilon \), \( u_\varepsilon = 0 \) on \( \Gamma_\varepsilon \),

where \( \nu \) is the exterior unit normal vector to \( S_\varepsilon \), \( f \in C^\alpha(\overline{\Omega}) \), \( \alpha > 0 \).

We consider a weak solution \( u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon) \) of the problem (1) and study the behaviour of \( u_\varepsilon \) as \( \varepsilon \to 0 \).

Let us introduce the function \( N^\varepsilon_f(y) \) \((j = 1, \ldots, n)\) as a 1-periodic solution in \( \varepsilon^{-1}Y_\varepsilon \) of the problem:

\[
\begin{aligned}
\Delta_y N^\varepsilon_f = 0 & \quad \text{in } \varepsilon^{-1}Y_\varepsilon, \\
\frac{\partial N^\varepsilon_f}{\partial \nu} = -v_j & \quad \text{on } a_\varepsilon \varepsilon^{-1}S_0, \\
\langle N^\varepsilon_f \rangle_{\varepsilon^{-1}Y_\varepsilon} = 0.
\end{aligned}
\]

In order to estimate \( N^\varepsilon_f \), we need some auxiliary results.

**Lemma 1.** If \( u \in H_1(Y_\varepsilon) \) and \( \langle u \rangle_{Y_\varepsilon} = 0 \), then

\[
\|u\|_{L_2(Y_\varepsilon)} \leq K_1 \varepsilon \|\nabla_x u\|_{L_2(Y_\varepsilon)},
\]

where \( \nabla_x u \equiv (u_{x_1}, \ldots, u_{x_n}) \), the constant \( K_1 \) does not depend on \( \varepsilon \).

**Lemma 2.** Let \( u \in H_1(Y_\varepsilon) \). Then

\[
\|u\|_{L_2(a_\varepsilon S_0)} \leq K_2 \left\{ a_\varepsilon^{(n-1)/2} \varepsilon^{-n/2} \|u\|_{L_2(Y_\varepsilon)} + \sqrt{a_\varepsilon} \|\nabla_x u\|_{L_2(Y_\varepsilon)} \right\},
\]

if \( n \geq 3 \), and

\[
\|u\|_{L_2(a_\varepsilon S_0)} \leq K_3 \left\{ \sqrt{a_\varepsilon} \varepsilon^{-1} \|u\|_{L_2(Y_\varepsilon)} + \sqrt{a_\varepsilon} \ln \frac{\varepsilon}{2a_\varepsilon} \|\nabla_x u\|_{L_2(Y_\varepsilon)} \right\},
\]

if \( n = 2 \), where all constants \( K_j \) here and in what follows do not depend on \( \varepsilon \).

We shall give proofs of these lemmas in the appendix.

Using the integral identity for the problem (2), we obtain the inequality

\[
\|\nabla_y N^\varepsilon_f\|_{L_2(\varepsilon^{-1}Y_\varepsilon)} \leq K_4 \varepsilon^{-(n-1)/2} a_\varepsilon^{(n-1)/2} \|N^\varepsilon_f\|_{L_2(a_\varepsilon S_0)}.
\]

By virtue of inequalities (3)-(5) we have

\[
\|N^\varepsilon_f\|_{L_2(a_\varepsilon S_0)} \leq K_5 (a_\varepsilon^{n/2} \varepsilon^{-n/2} + \sqrt{a_\varepsilon}) \|\nabla_x N^\varepsilon_f\|_{L_2(Y_\varepsilon)} \leq \leq K_6 (a_\varepsilon^{n/2} + \sqrt{a_\varepsilon} \varepsilon^{n/2-1}) \|\nabla_y N^\varepsilon_f\|_{L_2(\varepsilon^{-1}Y_\varepsilon)},
\]

if \( n \geq 3 \), and

\[
\|N^\varepsilon_f\|_{L_2(a_\varepsilon S_0)} \leq K_7 \left( \sqrt{a_\varepsilon} + \sqrt{a_\varepsilon} \ln \frac{\varepsilon}{2a_\varepsilon} \right) \|\nabla_y N^\varepsilon_f\|_{L_2(\varepsilon^{-1}Y_\varepsilon)},
\]

if \( n = 2 \).
From (6)-(8) we obtain the following estimates:

\[(9) \| \nabla_y N_j^\varepsilon \|_{L_2(\varepsilon^{-1} Y_\varepsilon)} \leq K_8 \left( \frac{a_\varepsilon}{\varepsilon} \right)^{n/2}, \quad \text{if } n \geq 3, \]

\[(10) \| \nabla_y N_j^\varepsilon \|_{L_2(\varepsilon^{-1} Y_\varepsilon)} \leq K_9 \frac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}}, \quad \text{if } n = 2. \]

From Lemma 1 and estimates (9), (10) we obtain

\[(11) \begin{cases} \| N_j^\varepsilon \|_{L_2(Y_\varepsilon)} + \| \nabla_y N_j^\varepsilon \|_{L_2(Y_\varepsilon)} \leq K_{10} \left( \frac{a_\varepsilon}{\varepsilon} \right)^{n/2}, \\ \| N_j^\varepsilon \|_{L_2(\Omega_\varepsilon^+)} + \| \nabla_y N_j^\varepsilon \|_{L_2(\Omega_\varepsilon^+)} \leq K_{11} \left( \frac{a_\varepsilon}{\varepsilon} \right)^{n/2}, \end{cases} \]

for \( n \geq 3 \) and

\[(12) \begin{cases} \| N_j^\varepsilon \|_{L_2(Y_\varepsilon)} + \| \nabla_y N_j^\varepsilon \|_{L_2(Y_\varepsilon)} \leq K_{12} \left( \frac{a_\varepsilon}{\varepsilon} \right)^{n/2} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}}, \\ \| N_j^\varepsilon \|_{L_2(\Omega_\varepsilon^+)} + \| \nabla_y N_j^\varepsilon \|_{L_2(\Omega_\varepsilon^+)} \leq K_{13} \frac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}}, \end{cases} \]

for \( n = 2. \)

Thus, we have

**Lemma 3.** Let \( N_j^\varepsilon (j = 1, \ldots, n) \) be a solution of the problem (2). Then the estimates (9)-(12) are valid.

**Corollary 1.** For the functions \( N_j^\varepsilon \) we have the following estimates:

\[(13) \begin{cases} \max_{y_1} |N_j^\varepsilon| \leq K_{14} \left( a_\varepsilon \varepsilon^{-1} \right)^{n/2}, & \max_{y_1} \left| \nabla_y N_j^\varepsilon \right| \leq K_{15} \left( a_\varepsilon \varepsilon^{-1} \right)^{n/2}, \quad \text{if } n \geq 3, \\ \max_{y_1} |N_j^\varepsilon| \leq K_{16} \frac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}}, & \max_{y_1} \left| \nabla_y N_j^\varepsilon \right| \leq K_{17} \frac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}}, \quad \text{if } n = 2. \end{cases} \]

**Proof.** Taking into account that \( \frac{\partial N_j^\varepsilon}{\partial y_i} \) is a harmonic 1-periodic in \( y \) function, we can use the mean value theorem for harmonic functions: if \( P \in \gamma \), then

\[ \left. \frac{\partial N_j^\varepsilon}{\partial y_i} \right|_p = \left. \left( \frac{\partial N_j^\varepsilon}{\partial y_i} \right) \right|_{V_{r_0}^P}, \]

where \( V_{r_0}^P \) is a ball of radius \( r_0 \) and \( P \) is the center of \( V_{r_0}^P \).

By virtue of the estimates (9), (10) we obtain

\[ \max_{\gamma} \left| \frac{\partial N_j^\varepsilon}{\partial y_i} \right| \leq |V_{r_0}^P|^{-1/2} \| \nabla_y N_j^\varepsilon \|_{L_2(V_{r_0}^P)} \leq K_{18} \left( a_\varepsilon \varepsilon^{-1} \right)^{n/2}, \]

for \( n \geq 3 \), and

\[ \max_{\gamma} \left| \frac{\partial N_j^\varepsilon}{\partial y_i} \right| \leq K_{19} \frac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}}, \]

if \( n = 2. \)

Other estimates (13) are obtained in a similar way.
Let \( v_0 \in C^{2+a}(\overline{\Omega}) \) be a solution of the problem

(14) \[ \Delta v_0 = f \quad \text{in} \quad \Omega , \quad v_0 = 0 \quad \text{on} \quad \partial \Omega . \]

Consider the function

\[
u^1_e = v_0 + \varepsilon \sum_{j=1}^n \widetilde{N}_j^e \left( \frac{x_i}{\varepsilon} \right) \frac{\partial v_0}{\partial x_j}, \quad x \in \Omega^- \cup \Omega^+_e,
\]

where \( \widetilde{N}_j^e \equiv N_j^e \), if \( y_1 > 0 \) and \( \widetilde{N}_j^e \equiv 0 \), if \( y_1 < 0 \), as an approximate solution for \( u_e \).

In what follows we use the usual convention of repeated indices.

Taking into account the definition of the functions \( v_0, N_j^e \) \( (j = 1, \ldots, n) \), we obtain that \( (u_e^1 - u_e) \) is a weak solution of the problem:

\[ \Delta (u_e^1 - u_e) = 0, \quad \text{if} \quad x \in \Omega^- , \]
\[ \Delta (u_e^1 - u_e) = \varepsilon \frac{\partial}{\partial x_i} \left( N_j^e \frac{\partial^2 v_0}{\partial x_j \partial x_i} + \varepsilon N_j^e \frac{\partial^2 v_0}{\partial y_s \partial x_j} \frac{\partial^2 v_0}{\partial x_s \partial x_i} \right), \quad \text{if} \quad x \in \Omega^+_e , \]

\[
[u_e^1 - u_e] \bigg|_{y} = \varepsilon N_j^e \frac{\partial v_0}{\partial x_i} \bigg|_{x_1 = +0} , \]

\[
\frac{\partial}{\partial \nu} (u_e^1 - u_e) = \varepsilon N_j^e \frac{\partial^2 v_0}{\partial y_j \partial x_i} \frac{\partial}{\partial x_i} v_i \quad \text{on} \quad \Sigma_e , \quad u_e^1 - u_e = \varepsilon \widetilde{N}_j^e \frac{\partial v_0}{\partial x_j} \quad \text{on} \quad \Gamma_e , \]

where \([\varphi]_{\partial e} \equiv \varphi |_{\partial e^+} - \varphi |_{\partial e^-}\) for any point \( P \in \gamma \) and function \( \varphi \).

We set \( u_e^1 - u_e = v_e^1 + v_e^2 \), where \( v_e^1 \) is a weak solution of the following problem:

(15) \[
\begin{align*}
\Delta v_e^1 &= 0 , \quad \text{if} \quad x \in \Omega^- , \\
\Delta v_e^1 &= F^+_e + \varepsilon \frac{\partial}{\partial x_i} f_{i,e} , \quad \text{if} \quad x \in \Omega^+_e , \\
[v_e^1] \bigg|_{y} &= 0 , \\
\frac{\partial v_e^1}{\partial x_1} \bigg|_{y} &= \varepsilon f_{1,e} \big|_{x_1 = +0} + l_e \big|_{x_1 = +0} , \\
\frac{\partial v_e^1}{\partial \nu} &= \varepsilon f_{1,e} v_i , \quad \text{on} \quad \Sigma_e , \quad v_e^1 = 0 \quad \text{on} \quad \Gamma_e ,
\end{align*}
\]

where

\[
F^+_e \equiv \frac{\partial N_j^e}{\partial y_s} \frac{\partial^2 v_0}{\partial x_j \partial x_i} , \quad f_{i,e} \equiv N_j^e \frac{\partial^2 v_0}{\partial x_j \partial x_i} \quad (i = 1, \ldots, n),
\]
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\[ l_\varepsilon = \partial N_j^\varepsilon \partial v_0 \partial x_j, \] and \( v_\varepsilon^2 \) is a weak solution of the problem

\[
\begin{cases}
\Delta v_\varepsilon^2 = 0, & \text{if } x \in \Omega^- \cup \Omega_+^\varepsilon, \\
[v_\varepsilon^2]_\gamma = \varepsilon N_j^\varepsilon \frac{\partial v_0}{\partial x_j} |_{x_1 = +0},
\end{cases}
\]

(16)
\[
\left[ \frac{\partial v_\varepsilon^2}{\partial x_1} \right]_\gamma = 0, \quad \frac{\partial v_\varepsilon}{\partial v} = 0 \text{ on } S_\varepsilon, \quad v_\varepsilon^2 = \varepsilon N_j^\varepsilon \frac{\partial v_0}{\partial x_j} \text{ on } \Gamma_\varepsilon.
\]

Now we obtain estimates for \( v_\varepsilon^1, v_\varepsilon^2 \). Using the integral identity for the problem (15) and taking the test-function \( \varphi = v_\varepsilon^1 \), we deduce

(17)
\[
\int_{\Omega_\varepsilon} |\nabla_x v_\varepsilon^1|^2 \, dx - \int_{\Omega_\varepsilon} l_\varepsilon v_\varepsilon^1 \, dx = \int_{\Omega_\varepsilon^+} F_\varepsilon^+ v_\varepsilon^1 \, dx - \varepsilon \int_{\Omega_\varepsilon^+} \frac{\partial v_\varepsilon^1}{\partial x_1} \, dx.
\]

For the function \( u \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon) \) one can prove the Friedrichs type inequality

(18)
\[
\|u\|_{L_2(\Omega_\varepsilon)} \leq C_0 \|\nabla_x u\|_{L_2(\Omega_\varepsilon)},
\]

where the constant \( C_0 \) does not depend on \( \varepsilon \). This inequality can be proved in the same way as the Friedrichs type inequality is proved in [4, p.53, Theor. 4.5] for perforated domains.

From the inequality (18), the imbedding theorem and (17) it follows that

(19)
\[
\int_{\Omega_\varepsilon} |\nabla_x v_\varepsilon^1|^2 \, dx \leq K_{20} \|\nabla_x v_\varepsilon^1\|_{L_2(\Omega_\varepsilon)} \cdot \\
\cdot \left\{ \max_\gamma \left| l_\varepsilon \right| + \|F_\varepsilon^+\|_{L_2(\Omega_\varepsilon^+)} + \varepsilon \sum_{i=1}^n \|f_{i,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \right\}.
\]

Using the Friedrichs inequality (18), estimates (11)-(13) and (19), we get

(20)
\[
\begin{cases}
\|v_\varepsilon^1\|_{H_1(\Omega_\varepsilon)} \leq K_{21} (a_\varepsilon \varepsilon^{-1})^{n/2} & \text{for } n \geq 3, \\
\|v_\varepsilon^1\|_{H_1(\Omega_\varepsilon)} \leq K_{22} \frac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}} & \text{for } n = 2.
\end{cases}
\]

We set

(21)
\[
\phi_n(\varepsilon) = \begin{cases}
(a_\varepsilon \varepsilon^{-1})^{n/2}, & \text{if } n \geq 3, \\
\frac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \frac{\varepsilon}{2a_\varepsilon}}, & \text{if } n = 2.
\end{cases}
\]

Thus we have

**Lemma 4.** Let \( v_\varepsilon^1 \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon) \) be a weak solution of the problem (15). Then the following estimate is valid

(22)
\[
\|v_\varepsilon^1\|_{H_1(\Omega_\varepsilon)} \leq K_{23} \phi_n(\varepsilon).
\]

Now we derive the estimate for the solution \( v_\varepsilon^2 \). We define the function \( \varphi(x_1) \) as a smooth function for \( x_1 \geq 0 \) and \( \varphi \equiv 1 \) for \( x_1 \in [0, \delta \varepsilon] \), \( \varphi \equiv 0 \) for \( x_1 \geq 2\delta \varepsilon \), \( |\varphi'| \leq K_{24} \varepsilon^{-1} \) and the constant \( \delta \) is chosen in such a way that \( \varphi \equiv 0 \) for \( x \in S_\varepsilon \). We set \( \varphi \equiv 0 \) for \( x_1 < 0 \). We say that \( v_\varepsilon^2 \) is a weak solution of the problem (16) if the function
\[ \tilde{v}_e^2 = v_e^2 - \varepsilon \varphi \tilde{N}_j x_j \frac{\partial v_0}{\partial x_j} \in H_1(\Omega_e) \]

and for any function \( \psi \in H_1(\Omega_e, \Gamma_e) \) the following integral identity is valid

\[
\int_{\Omega_e} (\nabla_x \tilde{v}_e^2, \nabla_x \psi) \, dx = -\varepsilon \int_{\Omega_e} \left( \nabla_x \left( \varphi \tilde{N}_j x_j \frac{\partial v_0}{\partial x_j} \right), \nabla_x \psi \right) \, dx.
\]

We represent the function \( \tilde{v}_e^2 \) in the form

\[ \tilde{v}_e^2 = w_e^1 + w_e^2, \]

where \( w_e^1 \) is a weak solution of the problem

\[
\begin{cases}
\Delta w_e^1 = -\varepsilon \Delta \left( \varphi N_j \frac{\partial v_0}{\partial x_j} \right) \quad & \text{in } \Omega_e^+, \quad \Delta w_e^1 = 0 \quad & \text{in } \Omega_e^-,
\\
[w_e^1]_\gamma = 0, \quad \left[ \frac{\partial w_e^1}{\partial x_1} \right]_\gamma = -\varepsilon \frac{\partial}{\partial x_1} \left( N_j \frac{\partial v_0}{\partial x_j} \varphi \right) \bigg|_{x_1 = +0},
\\
\frac{\partial w_e^1}{\partial v} = 0 \quad & \text{on } S_e, \quad w_e^1 = 0 \quad & \text{on } \Gamma_e,
\end{cases}
\]

and \( w_e^2 \) is a weak solution of the problem

\[
\begin{cases}
\Delta w_e^2 = 0 \quad & \text{in } \Omega_e, \quad \frac{\partial w_e^2}{\partial v} = 0 \quad & \text{on } S_e,
\\
w_e^2 = \varepsilon (1 - \varphi) \tilde{N}_j x_j \frac{\partial v_0}{\partial x_j} \quad & \text{on } \Gamma_e.
\end{cases}
\]

Taking in the integral identity for the problem (23) the solution \( w_e^1 \) as a test-function and using the Friedrichs inequality (18) for functions from \( H_1(\Omega_e, \Gamma_e) \) and estimates (11), (12) we deduce the following estimate

\[
\|w_e^1\|_{H_1(\Omega_e)} \leq K_{25} \Phi_n(\varepsilon).
\]

In order to estimate \( w_e^2 \) we define the function \( \theta \in C^\infty(\overline{\Omega}) \) such that \( \theta \equiv 1 \) when \( \varphi(x, \partial \Omega) \leq \varepsilon, \theta \equiv 0 \) when \( \varphi(x, \partial \Omega) \geq 2\varepsilon, 0 \leq \theta \leq 1. \)

Let \( w_e^3 = w_e^2 - \varepsilon (1 - \varphi) \theta \tilde{N}_j x_j \frac{\partial v_0}{\partial x_j} \) in \( \Omega_e. \) It is easy to see that \( w_e^3 \in H_1(\Omega_e, \Gamma_e). \)

Let us take \( w_e^3 \) as a test-function in the integral identity for the problem (24). Then we have

\[
\int_{\Omega_e} |\nabla_x w_e^2|^2 \, dx = \varepsilon \int_{\Omega_e} \left( \nabla_x \left[ (1 - \varphi) \theta \tilde{N}_j x_j \frac{\partial v_0}{\partial x_j} \right], \nabla_x w_e^2 \right) \, dx.
\]

By virtue of the definition of the function \( \varphi \) and \( \theta \) and estimates (11), (12) we deduce from (26) that

\[
\|\nabla_x w_e^2\|_{L^2(\Omega_e)} \leq K_{26} \Phi_n(\varepsilon).
\]
Using the Friedrichs inequality (18) for \( w^2 \), we obtain
\[
\|w^2\|_{L^2(\Omega_\varepsilon)} \leq K_{27} \phi_n(\varepsilon).
\]
Therefore, from estimates (25), (27), (28) we have

**Lemma 5.** Let \( v^2_\varepsilon \) be a weak solution of the problem (16). Then the following estimate is valid
\[
\|v^2_\varepsilon\|_{H_1(\Omega^-)} + \|v^2_\varepsilon\|_{H_1(\Omega_\varepsilon^+)} \leq K_{28} \phi_n(\varepsilon).
\]

The next theorem follows from (22) and (29).

**Theorem 1.** Let \( u_\varepsilon \) be a solution of the problem (1), \( v_0 \) be a solution of the problem (14). Then
\[
\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \leq K_{29} \phi_n(\varepsilon),
\]
where \( \phi_n(\varepsilon) \) is defined by (21).

The case when \( a_\varepsilon \varepsilon^{-1} \geq C, C = \text{const} \), is considered in [6-8].

2. The spectral problem, corresponding to the boundary-value problem (1), can be considered in the same way as in [6, 7], using the theorem from [4] about the spectrum of a sequence of singularly perturbed operators.

On the base of Theorem 1 we have

**Theorem 2.** Let \( \{\lambda^m_\varepsilon\} \) be a nondecreasing sequence of eigenvalues of the eigenvalue problem
\[
\Delta u^m_\varepsilon + \lambda^m_\varepsilon u^m_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon,
\]
\[
\frac{\partial u^m_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad S_\varepsilon, \quad u^m_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon,
\]
and let \( \{\lambda^m\} \) be a nondecreasing sequence of eigenvalues of the eigenvalue problem
\[
\Delta u^m + \lambda^m u^m = 0 \quad \text{in} \quad \Omega,
\]
\[
u^m = 0 \quad \text{on} \quad \partial \Omega,
\]
and every eigenvalue is counted as many times as its multiplicity. Then
\[
\left| \frac{1}{\lambda^m_\varepsilon} - \frac{1}{\lambda^m} \right| \leq C_1 \phi_n(\varepsilon),
\]
where \( C_1 \) is a constant independent of \( \varepsilon \).

The homogenization problem for the Laplace operator in partially perforated domain with the mixed type of boundary conditions is considered in [9].

**Appendix**

**Proof of Lemma 1.** First of all we extend a function \( u \) on \( \varepsilon G_0 \); this means that we construct a new function \( \tilde{u} \in H_1(\varepsilon Q) \) such that \( \tilde{u} \equiv u \) when \( x \in \varepsilon Q \setminus \alpha \varepsilon G_0 \) and the fol-
Following inequality is valid:

$$\|\nabla_x \tilde{u}\|_{L_2(\epsilon Q)} \leq K_{30} \|\nabla_x u\|_{L_2(Y_\epsilon)}.$$  

In order to get such an extension we introduce a new variable $y' = a_\epsilon^{-1} x$ and consider the domain $a_\epsilon^{-1} Y_\epsilon = (a_\epsilon^{-1} \epsilon Q) \setminus \overline{G_0}$. Since $a_\epsilon \epsilon^{-1} \to 0$ as $\epsilon \to 0$ we can take a cube $Q_1$ with the length of the edge which does not depend on $\epsilon$. In addition we suppose that the faces of $Q_1$ and the faces of $Q$ are parallel and $\overline{G_0} \subset Q_1$.

Then for any function $u \in H_1(Q_1 \setminus \overline{G_0})$ we can construct such extension $\tilde{v}$ that

$$\|\nabla_{y'} \tilde{v}\|_{L_2(Q_1)} \leq K_{31} \|\nabla_{y'} u\|_{L_2(Q_1 \setminus \overline{G_0})}.$$  

The proof of this inequality can be find in [4].

From the inequality (31) we deduce that

$$\|\nabla_x \tilde{v}\|_{L_2(a_\epsilon Q_1)} \leq K_{32} \|\nabla_x u\|_{L_2(a_\epsilon Q_1 \setminus \overline{G_0})}.$$  

Now we can define the function $\tilde{u}$ setting

$$\tilde{u} = \begin{cases} \tilde{v}, & \text{when } x \in a_\epsilon \setminus \overline{G_0}, \\ u, & \text{when } x \in Y_\epsilon \setminus a_\epsilon \setminus G_0. \end{cases}$$

For simplicity we assume that $u$ and $\tilde{u}$ are smooth functions. Then, for any points $P, P' \in Y_\epsilon$ we have

$$u(P') = u(P) + \int_{x_1^P}^{x_1^{P'}} \tilde{u}_{x_1}(x_1, x_2^P, \ldots, x_n^P) \, dx_1 + \int_{x_2^P}^{x_2^{P'}} \tilde{u}_{x_2}(x_1^P, x_2, x_3^P, \ldots, x_n^P) \, dx_2 +$$

$$+ \ldots + \int_{x_n^P}^{x_n^{P'}} \tilde{u}_{x_n}(x_1^P, x_2^P, \ldots, x_{n-1}^P, x_n) \, dx_n,$$

where $P = (x_1^P, \ldots, x_n^P)$, $P' = (x_1^{P'}, \ldots, x_n^{P'})$.

From this representation we obtain

$$(u(P') - u(P))^2 \leq K_{33} \epsilon \left\{ \int_{x_1^P}^{x_1^{P'}} \tilde{u}_{x_1}^2 \, dx_1 + \ldots + \int_{x_n^P}^{x_n^{P'}} \tilde{u}_{x_n}^2 \, dx_n \right\}.$$  

Now we integrate the last inequality at first with respect to $P' \in Y_\epsilon$ and then with respect to $P \in Y_\epsilon$, and using (30) we deduce

$$2 \left| Y_\epsilon \right| \int_{Y_\epsilon} u^2 \, dx - 2 \left( \int_{Y_\epsilon} u \, dx \right)^2 \leq K_{34} \epsilon^2 \left| Y_\epsilon \right| \int_{Y_\epsilon} |\nabla_x u|^2 \, dx.$$  

Thus, we have

$$\int_{Y_\epsilon} u^2 \, dx \leq \left| Y_\epsilon \right|^{-1} \left( \int_{Y_\epsilon} u \, dx \right)^2 + K_{34} \epsilon^2 \int_{Y_\epsilon} |\nabla_x u|^2 \, dx.$$  

Taking into account that $\langle u \rangle_{Y_\epsilon} = 0$ we obtain the statement of Lemma 1.
Proof of Lemma 2. For simplicity we assume that \( G_0 \) is a ball with the radius \( q < 1/2 \) and its center coincides with the center of \( Q \). Let \( P \in a_e S_0 \), \( P' \in q^{-1} r S_0 \), \( a_e q \leq r < \frac{e}{2} q \) and \( P, P' \) lie at the same vector-radius. We have

\[
\begin{align*}
u^2(P) & \leq 2u^2(P') + 2 \left( \int_{a_e q}^{eq/2} \frac{\partial u}{\partial r} \right)^2 r^{(n-1)/2} r^{(1-n)/2} dr. \\
\end{align*}
\]

From (32) we deduce

\[
\begin{align*}
u^2(P) & \leq 2u^2(P') + 2 \left( \int_{a_e q}^{eq/2} \left( \frac{\partial u}{\partial r} \right)^2 r^{n-1} dr \right) \\
& \leq 2u^2(P') + 2 \left( \int_{a_e q}^{eq/2} \left( \frac{\partial u}{\partial r} \right)^2 r^{n-1} dr \right).
\end{align*}
\]

We have

\[
\begin{align*}
u^2(P) & \leq 2u^2(P') + \frac{2}{n-2} \left( a_e q \right)^{2-n} \left( \int_{a_e q}^{eq/2} \left( \frac{\partial u}{\partial r} \right)^2 r^{n-1} dr \right), \quad \text{if } n \geq 3, \\
u^2(P) & \leq 2u^2(P') + 2 \ln \frac{e}{2a_e} \left( \int_{a_e q}^{eq/2} \left( \frac{\partial u}{\partial r} \right)^2 r dr \right), \quad \text{if } n = 2.
\end{align*}
\]

Multiplying the last inequalities by \( (a_e q)^{n-1} \psi(q_1, \ldots, q_{n-1}) \), where \( J \equiv r^{n-1} \psi(q_1, \ldots, q_{n-1}) \) is the Jacobian for the spherical coordinates and integrating with respect to \( q_1, \ldots, q_{n-1} \), we obtain

\[
\begin{align*}
u^2 ds & \leq K_{35} \left( a_e q \right)^{n-1} \left( \int_{\delta_1} u^2(P') \psi d q_1 \ldots d q_{n-1} + a_e \| \nabla u \|^2_{L^2(T_{a_e} \setminus T_e)} \right), \\
\end{align*}
\]

if \( n \geq 3 \), and

\[
\begin{align*}
u^2 ds & \leq K_{36} \left( a_e q \right) \left( \int_{\delta_1} u^2(P') \psi d q_1 \ldots d q_{n-1} + a_e \ln \frac{e}{2a_e} \| \nabla u \|^2_{L^2(T_{a_e} \setminus T_e)} \right), \\
\end{align*}
\]

if \( n = 2 \), where \( \delta_1 \) is a sphere of radius 1, \( T_e \) is a ball of radius \( \sigma q \) whose center coincides with the center of \( G_0 \).

Then multiplying the inequalities (33), (34) by \( r^{n-1} \) and integrating with respect to \( P' \) over \( r \in \left( a_e q, \frac{eq}{2} \right) \), we deduce estimates:

\[
\begin{align*}K_{37} \left( \frac{e^n}{2^n} - a_e^n \right) \| u \|^2_{L^2(a_e q, 0)} & \leq \leq K_{38} \left( a_e^n \| u \|^2_{L^2(T_{a_e} \setminus T_e)} + a_e \left( \frac{e^n}{2^n} - a_e^n \right) \| \nabla u \|^2_{L^2(T_{a_e} \setminus T_e)} \right),
\end{align*}
\]
if \( n \geq 3 \), and

\[
K_{39} \left( \frac{e^2}{4} - a^2 \right) \|u\|^2_{L^2(O, \Sigma_0)} \leq \leq K_{40} \left\{ a_e \|u\|^2_{L^2(T_{\theta_2} \setminus T_{\theta_1})} + a_e \left( \frac{e^2}{4} - a^2 \right) \ln \frac{e}{2a_e} \|\nabla u\|^2_{L^2(T_{\theta_2} \setminus T_{\theta_1})} \right\},
\]

if \( n = 2 \). From these inequalities we can conclude that Lemma 2 is valid.

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REFERENCES


