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Structural discontinuities to approximate some optimization problems with a nonmonotone impulsive character

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Meccanica. — *Structural discontinuities to approximate some optimization problems with a nonmonotone impulsive character.* Nota di Aldo Bressan e Monica Motta, presentata (*) dal Socio A. Bressan.

Abstract. — In some preceding works we consider a class $O\mathcal{P}$ of Boltz optimization problems for Lagrangian mechanical systems, where it is relevant a line $l = l_{\gamma(\cdot)}$, regarded as determined by its (variable) curvature function $\gamma(\cdot)$ of domain $[s_0, s_1]$. Assume that the problem $\tilde{\mathcal{P}} \in O\mathcal{P}$ is regular but has an *impulsive monotone character* in the sense that near each of some points δ_1 to $\delta_\nu \gamma(\cdot)$ is monotone and $|\gamma'(\cdot)|$ is very large. In [10] we propose a procedure belonging to the theory of impulsive controls, in order to simplify $\tilde{\mathcal{P}}$ into a structurally discontinuous problem $\mathcal{P}$. This is analogous to treating a biliard ball, disregarding its elasticity properties, as a rigid body bouncing according to a suitable restitution coefficient. Here the afore-mentioned treatment of $\tilde{\mathcal{P}}$ is extended to the case where its impulsive character fails to be monotone. Let $c_{r,0}$ to c_{r,m_r} be the successive maxima and minima of $\gamma(\cdot)$ or $-\gamma(\cdot)$ near $\delta_r (r = 1, \ldots, \nu)$. In constructing the problem $\mathcal{P}$, which simplifies and approximates $\tilde{\mathcal{P}}$, as well as in [10] it is essential to approximate $l_{\gamma(\cdot)}$ by means of a line $l_{c(\cdot)}$ with $c(\cdot)$ discontinuous only at $\delta_1, \ldots, \delta_\nu$ and with $|c'(\cdot)|$ never very large; furthermore now we must take the quantities $c_{r,0}$ to c_{r,m_r} into account, *e.g.*, by adding a «nonmonotonicity» type at δ_r, which vanishes in the monotone case $(r = 1, \ldots, \nu)$. Starting from [10] we extend to the afore-mentioned general situation the notions of *weak lower limit* J^* of the functional to minimize, *extended admissible process* (which has an additional part in each $[c_{r,i-1}, c_{r,i}]$) and *extended solution* of the problem $\mathcal{P}$, or better $(\mathcal{P}_\nu; \sigma_{r,1}, \ldots, \sigma_{r,m_r})$ where $\sigma_{r,i} = c_{r,i} - c_{r,i-1}$ $(i = 1, \ldots, m_r; r = 1, \ldots, \nu)$. In the general case we consider the *extended* (*impulsive*) *original problem* and the *extended functional* to minimize. This has an impulsive part at each of the points δ_1 to δ_ν, as well as the differential constraints, complementary equations, and Pontrjagin's optimization conditions. Besides the end conditions at s_0 and s_1, there are *junction conditions* at δ_1 to δ_ν. In the general case being considered we state a version of Pontrjagin's maximum principle and an existence theorem for the extended (impulsive) problem. We also study some properties of J^*, *e.g.* when J^* is a weak minimum. In particular, within both the monotone case and the nonmonotone one, we show that the quantity J^*, defined as a certain lower limit, equals the analogous limit; and this is practically a necessary and sufficient condition for the present approximation theory, started in [10], to be satisfactory.

Key words: Analytical mechanics; Lagrangian systems; Control theory.

Riassunto. — *Discontinuità strutturali per approssimare certi problemi di ottimizzazione con carattere impulsivo non monotono.* In precedenti lavori abbiamo considerato una classe $O\mathcal{P}$ di problemi di ottimizzazione di Boltz per sistemi meccanici Lagrangiani, nei quali è rilevante una linea $l = l_{\gamma(\cdot)}$, considerata come determinata dalla sua funzione (variabile) di curvatura $\gamma(\cdot)$ di dominio $[s_0, s_1]$. Il problema $\tilde{\mathcal{P}} \in O\mathcal{P}$ sia regolare ma abbia *carattere impulsivo monotono* nel senso che $\gamma(\cdot)$ sia monotona e con $|\gamma'(\cdot)|$ molto grande vicino a ciascuno di alcuni punti $\delta_1, \ldots, \delta_\nu$. In [10] abbiamo costruito un procedimento entro la teoria del controllo impulsivo, atto a semplificare $\tilde{\mathcal{P}}$ in un problema strutturalmente discontinuo $\mathcal{P}$. Ciò è analogo al trattare una palla da bigliardo, anziché per es. con la teoria dell'elasticità, considerandola come un corpo rigido rimbalzante secondo un opportuno coefficiente di restituzione. Qui estendiamo la suaccennata trattazione in [10] al caso che il carattere impulsivo di $\tilde{\mathcal{P}}$ sia non monotono. Siano $c_{r,0}, \ldots, c_{r,m_r}$ i successivi massimi e minimi di $\gamma(\cdot)$ o di $-\gamma(\cdot)$ nella vicinanza di $\delta_r (r = 1, \ldots, \nu)$. Nel costruire il problema $\mathcal{P}$ semplificante e approssimante $\tilde{\mathcal{P}}$, come in [10] è ora essenziale considerare una linea $l_{c(\cdot)}$ approssimante $l_{\gamma(\cdot)}$ con $c(\cdot)$ discontinua solo in $\delta_1, \ldots, \delta_\nu$ e con $|c'(\cdot)|$ mai molto grande; inoltre ora si deve tener conto delle suddette quantità $c_{r,0}, \ldots, c_{r,m_r}$ per es., attraverso il «tipo di non monotonia» in δ_r, che svanisce nel caso monotono

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$(r = 1, \ldots, \nu)$. Partendo da [10] estendiamo alla suddetta situazione generale le nozioni di *estremo inferiore debole* J^* del funzionale da minimizzare, *processo ammissibile esteso* (che ha parti addizionali in $[c_{r,i-1}, c_{r,i}]$) e *soluzione estesa* del problema $\mathcal{P}$, o meglio $(\mathcal{P}_\nu; \sigma_{r,1}, \ldots, \sigma_{r,m_r})$ ove $\sigma_{r,i} = c_{r,i} - c_{r,i-1}$ $(i = 1, \ldots, m_r;$ $r = 1, \ldots, \nu)$. Nel caso generale consideriamo pure il *problema originale (impulsivo) esteso* e il *funzionale esteso* da minimizzare. Questo ha parti impulsive nei punti $\delta_1, \ldots, \delta_\nu$, al pari dei vincoli differenziali, delle equazioni complementari e delle condizioni di ottimizzazione di Pontrjagin. Oltre alle condizioni ai limiti in s_0 ed s_1 vi sono *condizioni di giunzione* in $\delta_1, \ldots, \delta_\nu$. Nel detto caso generale enunciamo una versione del principio di massimo di Pontrjagin e un teorema di esistenza per il problema (impulsivo) esteso. Studiamo anche alcune proprietà di J^*, tra l'altro quando esso è minimo debole. In particolare, nel caso monotono o no, mostriamo che la quantità J^*, definita come un certo limite inferiore, eguaglia l'analogo limite; e ciò è praticamente una condizione necessaria e sufficiente affinché la presente teoria di approssimazione, iniziata in [10], sia soddisfacente.

1. Introduction

In [10] we consider a certain class $O\mathcal{P}$ of Boltz optimization problems that can be represented by means of a differential manifold, where it is relevant a line l considered as determined by its (variable) curvature $\gamma(\cdot)$ that has the arclength as argument and the domain $[s_0, s_1]$ (1).

Let $\widetilde{\mathcal{P}}$ be a regular problem in $O\mathcal{P}$ for which in particular $\gamma'(\cdot)$ is continuous; but let it have a *monotone impulsive character* in that $|\gamma'(\cdot)|$ is very large near each among some points δ_1 to δ_ν and $\gamma(\cdot)$ is monotone there. In [10] we show a procedure of impulsive control theory, useful to approximate and to simplify $\widetilde{\mathcal{P}}$ into a structurally discontinuous problem $\mathcal{P}$: we replace $\gamma(\cdot)$ with a convenient function $c(\cdot)$, that together with $c'(\cdot)$ is piecewise continuous and has at most the discontinuities $\sigma_r \doteq c(\delta_r^+) - c(\delta_r^-)$ at $\delta_r (r = 1, \ldots, \nu)$ (2).

In the present paper we extend [10] to the case when $\widetilde{\mathcal{P}}$'s impulsive character fails to be monotone near some δ_i. We do this rather quickly – practically without using any corresponding auxiliary problem $\widehat{\mathcal{P}}$ such as $(3.19\text{-}20)_\eta$ for $\eta = 0$ in [10] – by means of a process which is based on the (results obtained just in the) monotone case and turns out to be a limit process – see P3.6 (*a*).

Thus, in order to solve the problem $\mathcal{P}$ (or $\widetilde{\mathcal{P}}$) in this general case, we first put $\nu = 1$ and $\delta = \delta_1$, we assume that $|\gamma'(\cdot)|$ is very large only in $[\delta, \delta + \varepsilon_0]$, and we approximate the given problem $\widetilde{\mathcal{P}}$ by means of a monotone impulsive problem, say $\mathcal{P}_d$, with $\nu = m$ and $(\delta_1, \ldots, \delta_\nu)$ replaced by $d \doteq (d_1, \ldots, d_m)$ where $\delta = d_1 < d_2 < \ldots < d_m < \delta +$

(1) In [3] to [5] Aldo Bressan started a systematic (non linear) application of control theory to Lagrangian mechanical systems, by using coordinates as controls. This is based on the purely mathematical paper [1] (extended by [2]). A. Bressan's afore-mentioned work has been further developed by himself and other researchers: F. Rampazzo, M. Favretti, M. Motta and B. Piccoli – see [6-17]. The present paper belongs to this research line.

(2) To associate to $\widetilde{\mathcal{P}}$ the discontinuous problem $\mathcal{P}$ is analogous to treating a billiard ball, unlike using, *e.g.*, the elasticity theory, by considering it as a bouncing rigid body with a fixed restitution coefficient (that can be determined only approximately). [10] has been made in view of applications to any mechanical Lagrangian system belonging to the class introduced in [6, sect 5], say Γ_5, or to its extension Γ defined in [9].

$+ \varepsilon_0$. Briefly speaking, we consider the discontinuities $\sigma_i = c_d(d_i^+) - c_d(d_i^-)$ ($i = 1, \ldots, m$) of the corresponding curvature function $c_d(\cdot)$ (with $\sigma_j \sigma_{j-1} < 0$ for $j = 2, \ldots, m$); and we let d_m tend to δ^+, keeping σ_1 to σ_m fixed. By means of this limit, in section 3 we determine, up to a small arbitrariness, a new curvature function, say $c(\cdot)$, with a first order discontinuity only at δ; and we extend to the present case the definition of the *weak infimum* J^* of the functional to minimize. Furthermore in section 4 we introduce (for $\nu = 1$) suitable *extended admissible processes*, which have some additional parts in the intervals $[c(d_i^-), c(d_i^+)]$ ($i = 1, \ldots, m$), as well as the *extended solution* to $\mathcal{P}$. In effect we also consider an *extended original problem* with an *extended functional* to minimize. This has at δ some *impulsive parts* (connected with the above intervals) as well as the differential constraints, the complementary equations, and Pontrjagin's optimization conditions. In section 4 we also state (for $\nu = 1$) the PMP (Pontrjagin's maximum principle) with border and junction conditions. In sections 5-6 the above results are briefly extended to the general nonmonotone case with $\nu \geq 1$. In section 6 an existence theorem for the solution to the extended problem is also considered.

In accord with what was said in [10], we write a well posed extension of the optimization problems, considered in [10] within the monotone case, to the nonmonotone one; to do this we must add *e.g.* the *nonmonotonicity type* $(\sigma_{i,2}, \ldots, \sigma_{i,m_i})$ being considered at each discontinuity point δ_i ($i = 1, \ldots, \nu$). Any change of it generally affects the solutions. Furthermore in the monotone case (for δ_i) it becomes empty. Therefore in [10], where only this case is dealt with, the above type is not mentioned explicitly; while here – see sections 4 to 6 – we speak of problem $(\mathcal{P}_\nu; \sigma_{r,1}, \ldots, \sigma_{r,m_r})$ instead of $\mathcal{P}$.

Let us briefly add, first, that the afore-mentioned weak infimum J^* is defined in [10] by considering a certain functional J_η^* depending on a curvature function $c_\eta(\cdot)$ ($\eta \in \boldsymbol{R}$) that is linear in the small intervals $[\delta_i, \delta_i + \eta_i]$ ($i = 1, \ldots, \nu$); then J^* is identified with the lim inf J_η^* for $\eta \to 0^+$. In Remark 2.1 we briefly show on the basis of [10] that $J^* = \lim J_\eta^*$; and the existence of such a limit is basilar for the possibility of simplifying the given (regular) optimization problem into the impulsive problem $\mathcal{P}$ hinted at above.

Second, in [10, sect.5] the meaning of J^* is enriched by showing that the above $c_\eta(\cdot)$'s linearity property can, briefly speaking, be weakened into $c_\eta(\cdot)$'s regularity and monotonocity on $[\delta_i, \delta_i + \eta_i]$ ($i = 1, \ldots, \nu$). We note that this enrichment appears to hold also in the nonmonotone case, by its «quick reduction» to the monotone one hinted at above, notwithstanding this reduction involves only curvature functions having the above linearity property, for simplicity reasons.

Third, in section 3, first J^* is defined in the new case, again as a simple lower limit of a certain family J_d^*. Then its meaning is enriched by broadening the family, say to $J_{\eta,d}^*$: $J^* = \liminf J_{\eta,d}^*$. Furthermore in section 3 it is shown that all these lower limits equal their corresponding limits; and more refined limit properties of J^* are proved, which become simpler in case a solution to a certain auxiliary problem exists – see P3.6.

2. On the problems having an impulsive character without monotonicity. A first step of their reduction to the monotone case. Corresponding auxiliary problems

We consider the Cartesian frame $Oc_1c_2c_3$ ($c_r \cdot c_s = \delta_{rs}$, Kronecker's delta) and the line l in the plane Oc_1c_2, of equation $P = P(s)$ where s is the arclength on l. Let $\gamma(s)$ be l's curvature at $P(s)$ with the sign relative to c_3 (when it exists). We assume that for some points $\{\delta_0, \dots, \delta_{\nu+1}\}$ such that $s_0 \doteq \delta_0 < \delta_1 < \dots < \delta_\nu < \delta_{\nu+1} \doteq s_1$, $\gamma(\cdot)$'s restriction to (δ_i, δ_{i+1}) has a continuously differentiable extension to $[\delta_i, \delta_{i+1}]$ ($i = 0, \dots, \nu$).

One can determine l by means of the two-dimensional Cauchy problem – see [10, p. 37]

$$dP/ds = T, \qquad dT/ds = \gamma(s)\, c_3 \times T \qquad \text{for } a.e. \ s \in [s_0, s_1], \tag{2.1}$$

$$P(s_0) = P_0, \qquad T(s_0) = T_0 \qquad (|T_0| = 1, c_3 \cdot T_0 = 0 = OP_0 \cdot c_3) \tag{2.2}$$

in the unknown function $\Theta(s, \gamma(\cdot)) = (P(s), P'(s))$ of s.

In connection with the line $l_{\gamma(\cdot)}$ we consider the Boltz optimization problem (3)

$$\mathfrak{J}[\mathcal{P}(\cdot), u(\cdot)] \doteq \int_{s_0}^{s_1} L[s, \mathcal{P}(s), u(s)]\, ds + \Psi[\mathcal{P}(s_1)] \to \inf \tag{2.3}$$

under the differential constraint

$$dP/ds = \varphi[s, \mathcal{P}(s), u(s)] \ (\in \boldsymbol{R}) \qquad \text{for } a.e. \ s \in [s_0, s_1] \tag{2.4}$$

and the initial and control constraints

$$P(s_0) = P_0, \qquad u(\cdot) \in \mathcal{U} \doteq \mathcal{B}([s_0, s_1], U), \tag{2.5}$$

where U is a compact subset of $\boldsymbol{R}^m$, $\mathcal{B}([s_0, s_1], U)$ denotes the set of Borel measurable functions from $[s_0, s_1]$ to U, $\Psi(\cdot) \in C^1(\boldsymbol{R})$, while $\varphi(\cdot)$ and $L(\cdot)$ have the forms

$$\begin{cases} \varphi(s, \mathcal{P}, u) \doteq \varphi^1[s, \mathcal{P}, u, \Theta(s, \gamma(\cdot)), \gamma(s)]\, \gamma'(s) + \varphi^0[s, \mathcal{P}, u, \Theta(s, \gamma(\cdot)), \gamma(s)], \\ L(s, \mathcal{P}, u) \doteq L^1[s, \mathcal{P}, u, \Theta(s, \gamma(\cdot)), \gamma(s)]\, \gamma'(s) + L^0[s, \mathcal{P}, u, \Theta(s, \gamma(\cdot)), \gamma(s)], \\ \forall (s, \mathcal{P}, u) \in \mathcal{D} \doteq [s_0, s_1] \times \boldsymbol{R} \times U, \ \text{ where } s \mapsto \Theta(s, \gamma(\cdot)) \text{ solves problem (2.1-2)}. \end{cases} \tag{2.6}$$

We also assume that for $j = 0, 1$ the functions $\varphi^j(s, \mathcal{P}, u, \Theta, \gamma)$, $\varphi^j_{,\mathcal{P}}(s, \mathcal{P}, u, \Theta, \gamma)$, $L^j(s, \mathcal{P}, u, \Theta, \gamma)$ and $L^j_{,\mathcal{P}}(s, \mathcal{P}, u, \Theta, \gamma)$ of $(s, \mathcal{P}, u, \Theta, \gamma)$ are continuous on $\mathcal{D}_1 \doteq [s_0, s_1] \times \boldsymbol{R} \times U \times \boldsymbol{R}^4 \times \boldsymbol{R}$, and for some constant C

$$|\varphi^j(s, \mathcal{P}, u, \Theta, c)| \leq C(1 + |\mathcal{P}|) \quad \forall (s, \mathcal{P}, u, \Theta, c) \in \mathcal{D}_1 \quad (j = 0, 1). \tag{2.7}$$

Briefly, here we consider again the regular physical system $\widetilde{S}$ studied in [10] and having the properties P1.1-3 in [10, sect. 1], but we disregard its monotonicity property

(3) We shall often use *e.g.* $(2.1)^{\chi(\cdot)}$ or $(2.1)^{c(\cdot)}$ for (2.1) where the function $\gamma(\cdot)$ is replaced by $\chi(\cdot)$ or $c(\cdot)$, respectively. More generally, we denote by $(\text{r.s})^{\chi(\cdot)}$ or $(\text{r.s})^{c(\cdot)}$ the formula obtained from (r.s) in the above way.

P1.4 there; and we aim at extending the results of [10] to this general case. More explicitly we consider a regular instance of the optimization problem (2.1-5) – *i.e.* an instance of the latter problem with $\nu = 0$ and with $\gamma(\cdot)$ replaced by a given curvature function $\chi(\cdot) \in C^1([s_0, s_1])$, and we first assume that $\chi(\cdot)$ has an impulsive character, for the sake of simplicity, near one instant $\delta \in (s_0, s_1)$. This means that, *e.g.*, for some small $\varepsilon_0 > 0$, $|\chi'(\cdot)|$ is very large in $(\delta, \delta + \varepsilon_0)$ while it has an ordinary size on $[s_0, \delta) \cup \cup (\delta + \varepsilon_0, s_1]$. Furthermore we assume that $\chi(\cdot)$ may be very complex on $[\delta, \delta + \varepsilon_0]$ and that, for some $m > 1$, it has there the following *monotonicity type of order m*:

(*i*) *for some* $\tilde{d}_0$ *to* $\tilde{d}_m$ *with* $\delta = \tilde{d}_0 < \tilde{d}_1 < \ldots < \tilde{d}_m = \delta + \varepsilon_0$, *in* $[\delta, \delta + \varepsilon_0]\, \chi(\cdot)$ *is monotone* (*only*) *on each of the intervals* $[\tilde{d}_{i-1}, \tilde{d}_i]$ $(i = 1, \ldots, m)$, *so that*

$$\sigma_{i+1}\sigma_i < 0 \text{ for } i < m\,, \text{ where } \sigma_i = c_i - c_{i-1}\,(i = 1, \ldots, m) \quad \text{with } c_j = \chi(\tilde{d}_j)(j = 0, \ldots, m)\,. \tag{2.8}$$

Of course, the interval $[\delta, \delta + \varepsilon_0]$ has now the role of the interval $[\tilde{a}_1, \tilde{b}_1]$ mentioned in *e.g.* [10, P1.4] for $\nu = 1$. In the case $m > 1$ being considered, one may call $(\sigma_2, \ldots, \sigma_m)$ $\chi(\cdot)$'s *nonmonotonicity type* in $[\delta, \delta + \varepsilon_0]$ – see (v) above (3.2). In the case $m = 1$ this type becomes empty and $\chi(\cdot)$ just has the monotonicity property [10, P1.4], σ_1 being always determined by $c(\cdot)$ and σ_2 to σ_m.

In order to simplify the treatment of the above regular problem, it is convenient – roughly speaking – , first, to schemetize or approximate this by means of a problem such as [10, $(2.1\text{-}2)^{c(\cdot)} \cup (2.4\text{-}6)^{c(\cdot)}$] with both a monotone impulsive character and $\nu = = m > 1$; and second, to take the limit of this as $\varepsilon_0 \to 0^+$.

In more details, in analogy with the replacement of the regular problem $\tilde{\mathcal{P}}$ with the impulsive one $\mathcal{P}$ made in [10], as a FIRST STEP we replace problem $(2.1\text{-}5)^{\chi(\cdot)}$ with an impulsive problem $(2.1\text{-}5)^{c(\cdot)}$ where now $c(\cdot)$ is regarded to depend on d_1 to d_m,

$$s_0 < \delta = d_1 < d_2 < \ldots < d_m < \delta + \varepsilon_0 < d_{m+1} \doteq s_1 \tag{2.9}$$

holds, and – see $(2.8)_3$

$$c(d_i^{\pm}) = c_i^{\pm}\,, \quad \text{where } c_i^- = c_{i-1}\,, \quad c_i^+ = c_i \qquad (i = 1, \ldots, m)\,; \tag{2.10}$$

furthermore $c(\cdot)$ satisfies (at least approximately) conditions such as

$$\begin{cases} c(s) = \chi(s) \;\; \forall s \in [s_0, \delta)\,, \\ c(s) = \tilde{\chi}[\delta + (s - d_m)(s_1 - \delta)/(s_1 - d_m)] \quad \forall s \in (d_m, s_1]\,, \end{cases} \tag{2.11}$$

(*ii*) $\tilde{\chi}(\cdot)$ *being an extension to* $(\delta, s_1]$ *for* $\chi(\cdot)$*'s restriction to* $(\delta + \varepsilon_0, s_1]$, *with* $|\tilde{\chi}'(\cdot)|$ *of ordinary size.*

Of course the function $c(\cdot)$ is required to have a continuously differentiable extension on any interval $[d_{l-1}, d_l]$, which leaves it a large arbitrariness in any interval (d_{l-1}, d_l) $(l = 2, \ldots, m)$ (4). In order to reach our results quicker, *we can use this arbitrariness to assume that* $c(\cdot)$ *is constant on* (d_{l-1}, d_l) $(l = 2, \ldots, m)$.

(4) In analogy with [10, ftn.1], this arbitrariness has a counterpart in the widespread treatment of a billiard ball as a rigid body bouncing according to a certain restitution coefficient: the (precise) choice of this.

Later it will be clear that the same results can be reached under much weaker assumptions.

As well as in [10, sect.2], we regard the above problem $(2.1\text{-}5)^{c(\cdot)}$ as a limit of a sequence formed by some suitable problems that involve implementable processes (we mean connected with continuous curvature functions). Therefore we consider the m-tuples $\eta = (\eta_1, \ldots, \eta_m) \in D \doteq D_d$ with $d = (d_1, \ldots, d_m)$, where

(2.12) $\quad D_d \doteq \{\eta\colon \delta = d_1 < d_1 + \eta_1 < d_2 < d_2 + \eta_2 < \ldots < d_m < d_m + \eta_m < \delta + \varepsilon_0\}$,

and for any $\eta \in D$ we use the function

$$(2.13) \qquad c_{\eta, d}(s) \doteq c_\eta(s) \doteq \begin{cases} c(s), & s \in [s_0, d_1], \\ c_{i-1} + \sigma_i (s - d_i)/\eta_i, & s \in (d_i, d_i + \eta_i], \\ c[\alpha_\eta(s)], & s \in \mathfrak{J}_\eta \doteq \bigcup_{i=1}^{m} (d_i + \eta_i, d_{i+1}], \end{cases}$$

where i runs on $\{1, \ldots, m\}$ while the function $\alpha_\eta(\cdot)\colon \overline{\mathfrak{J}_\eta} \to \boldsymbol{R}$ reads

$$(2.14) \quad \alpha_\eta(s) \doteq d_{i+1} - \frac{d_{i+1} - d_i}{d_{i+1} - d_i - \eta_i}(d_{i+1} - s) \quad s \in [d_i + \eta_i, d_{i+1}] \quad (i = 1, \ldots, m).$$

Incidentally $(2.13)_3$ can be replaced with

$$(2.13)' \qquad \begin{cases} c_\eta(s) = c_i & \forall s \in [d_i + \eta_i, d_{i+1}] \quad (i = 1, \ldots, m-1), \\ c_\eta(s) = c(\alpha_\eta(s)) & \forall s \in [d_m + \eta_m, s_1]. \end{cases}$$

We define $\varphi_\eta(\cdot)$ $[L_\eta(\cdot)]$ to be what φ $[L]$ becomes by replacing $c(\cdot)$ in (2.6) with $c_\eta(\cdot)$ $(\eta \in D)$; more generally we call $(\text{r.s})_\eta$ the formula obtained from (r.s.) by replacing c, φ, L, and $\mathfrak{J}$ with c_η, φ_η, L_η, and $\mathfrak{J}_\eta$, respectively. Furthermore we assume that for any $\eta \in D$ the class, say AdP_η, of the admissible processes $(\mathscr{P}(\cdot), u(\cdot))$ for the regular (approximating) problem $(2.1\text{-}5)_\eta$ is non-empty and we set

$$(2.15) \qquad J^*_{\eta, d} \doteq \mathfrak{J}^*_\eta \doteq \inf\{\mathfrak{J}_\eta(\xi)\colon \xi \in AdP_\eta\}, \quad J^*_d \doteq \mathfrak{J}^* \doteq \liminf_{\eta(\in D) \to 0} \mathfrak{J}^*_\eta .$$

As well as in [10, (2.15)], we regard $(2.3)^{c(\cdot)}$ as the task of determining the above *weak infimum* $\mathfrak{J}^*$. We refer to this in speaking of the *weak optimization* (*or minimization*) *problem* $(2.1\text{-}5)^{c(\cdot)}$.

Remark 2.1. *In the general case* $\nu \geqslant 1$, *by the definitions* (2.15) – *see* [10, (2.15)] – *regarded as valid in this case, we have that*

$$(2.15)' \qquad \mathfrak{J}^* = \lim_{\eta(\in D) \to 0} \mathfrak{J}^*_\eta, \quad i.e. \ J^*_d = \lim_{\eta(\in D) \to 0} \mathfrak{J}^*_{\eta, d} .$$

This is necessary and practically sufficient for the theory started in [10] to be satisfactory. The extension of (2.15′) to the nonmonotone case will be performed explicitly only in the case $\nu = 1$, for the sake of simplicity – see P3.6 (*a*).

The proof of (2.15)′ is short on the basis of [10]. In fact, up to a misprint, [10, $(4.1)_1$] asserts that $\mathfrak{J}^* = \widehat{\mathfrak{J}}^*$. Furthermore in [10] it is in effect deduced that, given any $\varepsilon > 0$, for some $\hat{u} \in \widehat{\mathcal{U}}$ and some $\delta_1(\varepsilon) > 0$, [10, $(4.2)_{1\text{-}3}$] hold for all $\eta \in B(0, \delta_1(\varepsilon)) \cap D$, being understood that $\hat{u}(t) \equiv u[\beta_\eta(t)]$ – see [10, (3.1)] –

and that $\xi \doteq (\mathcal{P}(\cdot), u(\cdot))$ is an admissible process of problem [10, $(1.2\text{-}4)^{c_\eta(\cdot)}$]. Therefore [10, $(4.1)_1$, $(4.2)_{1\text{-}3}$] imply the first of the relations

$$\mathfrak{J}^* + 2\varepsilon > \mathfrak{J}_\eta(\xi) \geqslant \mathfrak{J}^*_\eta \quad \forall \eta \in B(0, \delta_1(\varepsilon)) \cap D$$

– see below [10, (2.13)]. The second follows by the definition $(2.15)_2$ (regarded as valid for $\nu \geqslant 1$). Thus, by the arbitrariness of $\varepsilon(>0)$ we conclude that

$$\mathfrak{J}^* \geqslant \limsup_{\eta(\in D) \to 0} \mathfrak{J}^*_\eta .$$

This and the definition [10, $(2.15)_2$] of $\mathfrak{J}^*$ yield $(2.15)'$. ■

We remember that, as shown in [10, sect. 5], the weak infimum $\mathfrak{J}^*$ remains invariant if we replace the simple family $\mathcal{F}_1 \doteq \{c_\eta(\cdot)\}_{\eta \in D}$ having the linearity property [10, (2.11)] with any family $\mathcal{F} \doteq \{c_\eta(\cdot)\}_{\eta \in D} \in \boldsymbol{F}$ – see [10, Def.5.1] – consisting of continuous approximations of the function $c(\cdot)$ that are increasing and satisfy certain weak conditions but are not necessarily linear in any interval $(d_i, d_i + \eta_i]$ $(i = 1, \dots, m)$.

REMARK 2.2. By the properties of any family $\mathcal{F} \in \boldsymbol{F}$ asserted in [10, sect. 5], *the analogue for $\mathcal{F}$ of equality* $(2.15)'$, *referring to $\mathcal{F}_1$ holds* and it can be proved practically like its original version.

After [10, sects. 3, 4], for $\eta \in D$ we set

$$(2.16) \qquad s = \beta_\eta(t) \doteq$$

$$\doteq \begin{cases} s_0 + (d_1 - s_0)t & (\in [s_0, d_1]), & t \in [0, 1], \\ d_i + (t - 2i + 1)\eta_i & (\in [d_i, d_i + \eta_i]), & t \in [2i - 1, 2i], \\ d_i + \eta_i + (d_{i+1} - d_i - \eta_i)(t - 2i) & (\in [d_i + \eta_i, d_{i+1}]), & t \in [2i, 2i + 1], \end{cases}$$

where i runs on $\{1, \dots, m\}$; hence by (2.13) and (2.16) one has

$$(2.17) \qquad \hat{c}(t) \doteq c_\eta[\beta_\eta(t)] =$$

$$= \begin{cases} c[s_0 + (d_1 - s_0)t], & t \in [0, 1], \\ c_{i-1} + \sigma_i(t - 2i + 1), & t \in [2i - 1, 2i], \quad (i = 1, \dots, m), \\ c[d_i + (d_{i+1} - d_i)(t - 2i), & t \in [2i, 2i + 1], \end{cases}$$

so that $\hat{c}(\cdot)$ is independent of $\eta \in \overline{D}$. Furthermore, for $\nu = m$ we recall the following optimization problem depending on the parameter $\eta \in \overline{D}$ – see [10, $(3.19\text{-}20)_\eta$, p. 43] – and we do this by writing d explicitly in some places, like in the sequel and unlike what we did in [10] where d was fixed at the outset.

$$(2.18)_{\eta, d} \quad \hat{\mathfrak{J}}_{\eta, d}[\hat{\xi}] \doteq \int_0^{2\nu + 1} \hat{L}_{\eta, d}[t, \hat{\mathcal{P}}(t), \hat{u}(t)]\, dt + \Psi[\hat{\mathcal{P}}(2\nu + 1)] \to \inf \; (\hat{\xi} = (\hat{\mathcal{P}}(\cdot), \hat{u}(\cdot))),$$

under the differential, initial, and control constraints

$$(2.19)_{\eta, d} \quad d\hat{\mathcal{P}}/dt = \hat{\varphi}_{\eta, d}(t, \mathcal{P}, \hat{u}(t)), \quad \hat{\mathcal{P}}(0) = \mathcal{P}_0, \quad \hat{u}(\cdot) \in \hat{\mathcal{U}} \doteq \mathcal{B}([0, 2\nu + 1], U),$$

where

$$(2.20)_{\eta,d} \qquad \begin{cases} \hat{\varphi}_{\eta,d}(t,\mathcal{P},u) \doteq \varphi_{\eta,d}[\beta_\eta(t),\mathcal{P},u]\beta'_\eta(t), \\ \hat{L}_{\eta,d}(t,\mathcal{P},u) \doteq L_{\eta,d}[\beta_\eta(t),\mathcal{P},u]\beta'_\eta(t). \end{cases}$$

We remember from [10] that the function $s=\beta_{\eta,d}(t)=\beta_\eta(t)$ is a bijection of $[0, 2\nu+1]$ onto $[s_0, s_1]$, now being $\nu=m$; and it mutually transforms the (optimization) problem $(2.18\text{-}20)_{\eta,d}$ into problem $(2.3\text{-}5)^{c_{\eta,d}(\cdot)}$. More precisely

P2.1. *Under* (2.9) *and* (2.12) *the conditions*

$$(2.21) \quad \hat{u}(t)=u[\beta_\eta(t)], \qquad \hat{\mathcal{P}}(t)=\mathcal{P}[\beta_\eta(t)] \quad \forall t\in[0,2m+1], \qquad \eta\in D$$

imply that (*i*) $u(\cdot)\in\mathcal{U}$ *iff* $\hat{u}(\cdot)\in\hat{\mathcal{U}}$, (*ii*) $\mathcal{P}(\cdot)$ *solves the* (*Cauchy*) *problem* $(2.4)^{c_{\eta,d}(\cdot)}\cup$ $\cup\,(2.5)_1$ *iff* $\hat{\mathcal{P}}(\cdot)$ *solves problem* $[(2.19)_{\eta,d}]_{1,2}$ *and* (*iii*) *remembering* $(2.3)^{c_{\eta,d}(\cdot)}$ *and* $(2.18)_{\eta,d}$

$$(2.22) \qquad \mathfrak{J}_{\eta,d}[\mathcal{P}(\cdot),u(\cdot)]=\hat{\mathfrak{J}}_{\eta,d}[\hat{\mathcal{P}}(\cdot),\hat{u}(\cdot)].$$

For $u(\cdot)\in\mathcal{U}$ and $\eta\in D$ we call $\mathcal{P}(\cdot,u(\cdot),\eta,d)$ the solution in $[s_0,s_1]$ of the problem $(2.4)^{c_{\eta,d}(\cdot)}\cup(2.5)_1$, while for $\hat{u}(\cdot)\in\hat{\mathcal{U}}$ and $\eta\in\overline{D_d}$ we call $\hat{\mathcal{P}}(\cdot,\hat{u}(\cdot),\eta,d)$ the solution in $[0, 2m+1]$ of problem $[(2.19)_{\eta,d}]_{1,2}$; and we set

$$(2.23) \qquad \xi_{\eta,d,u(\cdot)}\doteq(\mathcal{P}(\cdot,u(\cdot),\eta,d),u(\cdot)), \qquad \hat{\xi}_{\eta,d,\hat{u}(\cdot)}\doteq(\hat{\mathcal{P}}(\cdot,\hat{u}(\cdot),\eta,d),\hat{u}(\cdot)).$$

After Theorem 4.1 in [10, p. 44] the above impulsive problem $(2.1\text{-}5)^{c(\cdot)}$ can be reduced to the ordinary (auxiliary) optimization problem $(2.18\text{-}19)_{0,d}$, in the sense that

$$(2.24) \qquad J_d^*=\hat{J}_{0,d}^*\doteq\inf\{\hat{\mathfrak{J}}_{0,d}[\hat{\xi}_{0,d,\hat{u}}]:\hat{u}\in\hat{\mathcal{U}}\}.$$

3. Second step for the reduction of the nonmonotone case to the monotone one. A simple definition and some properties of the new weak infimum J^*

In the Second Step we let d_m tend to δ by keeping $\delta=d_1<\ldots<d_m$. Before studying this limit we consider $\beta_{\eta,d}(t)\doteq\beta_\eta(t)$, $\alpha_\eta[\beta_\eta(t)]$, and $\hat{c}(t)\doteq\hat{c}_d(t)$ given by (2.16) and (2.17), as functions of (η,d,t) defined for $t\in[0,2m+1]$ and $\eta\in\overline{D}$, of course in connection with a given admissible choice of $\tilde{\chi}(\cdot)$ – see (2.11-12).

P3.1. *Those functions are continuous on* $\overline{\Delta}\times[0,2m+1]$ *and their derivatives w.r.t. t are uniformly continuous on* $\Delta_j\doteq\overline{\Delta}\times(j,j+1)$ $(j=0,\ldots,2m)$, *where*

$$(3.1) \qquad \Delta\doteq\{(\eta,d):\delta=d_1<d_1+\eta_1<d_2<\ldots<d_m<d_m+\eta_m<\delta+\varepsilon_0\}.$$

Then, by $(2.20)_{\eta,d}$, our assumptions imply the following proposition.

P3.2. *For every* $R>0$ *the functions* $\hat{\varphi}_{\eta,d}(t,\mathcal{P},u)$ *and* $\hat{L}_{\eta,d}(t,\mathcal{P},u)$ *are meaningful and uniformly continuous on* $\Delta_j\times[-R,R]\times\hat{\mathcal{U}}$ $(j=0,\ldots,2m)$.

Now it is not difficult to see that, first, P3.3. *For some* $R > 0$ *the solution* $\widehat{\mathscr{P}}(\cdot, \hat{u}(\cdot), \eta, d)$ *of problem* $[(2.19)_{\eta, d}]_{1,2}$ *in* $[0, 2m+1]$ *exists and has a* C^0*-norm less than* R $\forall (\eta, d, \hat{u}) \in \bar{\Delta} \times \widehat{\mathcal{U}}$.

Second, by (2.18-19) and (2.23),

P3.4. *The functions* $\hat{\xi}_{\eta, d, \hat{u}}$ *and* $\widehat{\mathfrak{J}}_{\eta, d}[\hat{\xi}_{\eta, d, \hat{u}}]$ *of* $(\eta, d, \hat{u})$ *are continuous w.r.t.* $(\eta, d) \in \bar{\Delta}$ *uniformly w.r.t.* $\hat{u} \in \widehat{\mathcal{U}}$.

For d regarded as fixed at the outset, as it is done in [10], proposition P3.4 practically reduces to P3.9 in [10, p. 44]; and its proof, briefly sketched above in the general case, is in effect a straightforward generalization of the reasoning presented in [10, sect. 3] just to prove P3.9 there.

It is natural to define the *weak infimum* J^* of the functional to minimize by

$$(3.2) \qquad J^* \doteq \liminf_{d_m \to \delta^+} J_d^* \quad (\text{keeping } \delta = d_1 < \ldots < d_m) \ - \ \text{see } (2.15).$$

We now call $\underline{\delta}$ the point $(\delta, \ldots, \delta) \in \boldsymbol{R}^m$, so that $(0, \underline{\delta})$ is the only point $(\eta, d) \in \bar{\Delta}$ with $d_m + \eta_m = \delta$; and we prove that

P3.5. J^* *has the following two properties*

$$(3.3) \qquad J^* = J^{\#} \doteq \liminf_{(\eta, d)(\in \Delta) \to (0, \underline{\delta})} J^*_{\eta, d}; \qquad J^* = J^*_{\eta, \underline{\delta}}$$

being

$$(3.4) \qquad \hat{J}^*_{\eta, d} \doteq \inf\{\widehat{\mathfrak{J}}_{\eta, d}[\hat{\xi}_{\eta, d, \hat{u}}] : \hat{u} \in \widehat{\mathcal{U}}\} \quad \forall (\eta, d) \in \bar{\Delta};$$

furthermore

$$(3.5) \qquad J^*_{\eta, d} = \hat{J}^*_{\eta, d} \quad \forall (\eta, d) \in \Delta.$$

Proof. Fix any $\varepsilon > 0$. Then by P3.4, for some $\rho_0 > 0$

$$(3.6) \qquad |\widehat{\mathfrak{J}}_{0, \underline{\delta}}[\hat{\xi}_{0, \underline{\delta}, \hat{u}}] - \widehat{\mathfrak{J}}_{\eta, d}[\hat{\xi}_{\eta, d, \hat{u}}]| < \varepsilon$$

$$\forall (\eta, d, \hat{u}) \in \bar{\Delta} \times \widehat{\mathcal{U}} \text{ with } |\eta| < \rho_0, \ |d - \underline{\delta}| < \rho_0.$$

Furthermore by definitions $(2.15)_{3,4}$ and (3.2), for some $\rho_1 \in (0, \rho_0)$ the first of the inequalities

$$(3.7) \qquad J^* \leqslant J_d^* + \varepsilon \leqslant \widehat{\mathfrak{J}}_{0, d}[\hat{\xi}_{0, d, \hat{u}}] + \varepsilon \qquad \forall (0, d, \hat{u}) \in \bar{\Delta} \times \widehat{\mathcal{U}} \text{ with } |d - \underline{\delta}| < \rho_1$$

holds; and the second follows from theorem (2.24).

The definition $(3.3)_2$ of $J^{\#}$ implies that, for some $\rho \in (0, \rho_1)$, some $(\bar{\eta}, \bar{d}, \bar{v}) \in \Delta \times \mathcal{U}$ satisfies the conditions $|\bar{\eta}| < \rho$, $|\bar{d} - \underline{\delta}| < \rho$, and the first one of

$$(3.8) \qquad J^{\#} + \varepsilon > \mathfrak{J}_{\bar{\eta}, \bar{d}}[\xi_{\bar{\eta}, \bar{d}, \bar{v}}] = \widehat{\mathfrak{J}}_{\bar{\eta}, \bar{d}}[\hat{\xi}_{\bar{\eta}, \bar{d}, \hat{v}}] \qquad \text{for } \hat{v} \doteq \bar{v} \circ \beta_{\bar{\eta}, \bar{d}} \in \widehat{\mathcal{U}}.$$

Then by definition $(3.8)_3$ and P2.1, $(3.8)_{2,4}$ also hold. The first of the inequalities

$$(3.9) \qquad J^* - J^{\#} \leqslant \widehat{\mathfrak{J}}_{0, \bar{d}}[\hat{\xi}_{0, \bar{d}, \hat{v}}] - \widehat{\mathfrak{J}}_{\bar{\eta}, \bar{d}}[\hat{\xi}_{\bar{\eta}, \bar{d}, \hat{v}}] + 2\varepsilon < 4\varepsilon$$

follows from (3.7-8). Since $\rho < \rho_0$, by using (3.6) first with $(\eta, d, \hat{u}) = (0, \bar{d}, \hat{v})$ and then with $(\eta, d, \hat{u}) = (\bar{\eta}, \bar{d}, \hat{v})$ we deduce $(3.9)_2$.

By ε's arbitrariness (3.9) yields the first of the inequalities

$$J^* \leqslant J^\#, \qquad J^\# \leqslant J^*. \tag{3.10}$$

By the definitions (3.2) of J^* and $(3.3)_2$ of $J^\#$, $(3.10)_2$ is an exercise on infimum limits. Thus $(3.3)_1$ has been proved.

Equality (3.5) follows from $(2.15)_{1,2}$, (3.4) and $(3.8)_2$. Then – remembering (3.6) – by $(3.3)_2$ and (3.4), for some $(\eta, d, \hat{v}) \in \Delta \times \hat{\mathcal{U}}$, we have the inequalities

$$|\eta| < \rho_0, \quad |d - \underline{\delta}| < \rho_0, \quad |J^\# - \hat{J}^*_{\eta,d}| < \varepsilon, \quad \hat{\mathfrak{J}}_{\eta,d}[\hat{\xi}_{\eta,d,\hat{v}}] < J^*_{\eta,d} + \varepsilon. \tag{3.11}$$

By (3.4) with $\eta = 0$ we have the first of the relations

$$\hat{J}^*_{0,\underline{\delta}} \leqslant \hat{\mathfrak{J}}_{0,\underline{\delta}}[\hat{\xi}_{0,\underline{\delta},\hat{v}}] \leqslant \hat{\mathfrak{J}}_{\eta,d}[\hat{\xi}_{\eta,d,\hat{v}}] + \varepsilon < J^\# + 3\varepsilon. \tag{3.12}$$

By $(3.11)_{1,2}$, $(3.6)_1$ holds for $\hat{u} \equiv \hat{v}$; this yields $(3.12)_2$. By $(3.11)_{3,4}$ we deduce $(3.12)_3$. The arbitrariness of $\varepsilon > 0$ yields the first of the relations

$$\hat{J}^*_{0,\underline{\delta}} \leqslant J^\#, \qquad J^\# \leqslant \hat{J}^*_{0,\underline{\delta}}. \tag{3.13}$$

Now we note that by (3.4) some further choice of $(\eta, d, \hat{v}) \in \Delta \times \hat{\mathcal{U}}$ satisfies $(3.11)_{1\text{-}3}$ and the first of the relations

$$\hat{J}^*_{0,\underline{\delta}} > \hat{\mathfrak{J}}_{0,\underline{\delta}}[\hat{\xi}_{0,\underline{\delta},\hat{v}}] - \varepsilon > \hat{\mathfrak{J}}_{\eta,d}[\hat{\xi}_{\eta,d,\hat{v}}] - 2\varepsilon \geqslant \hat{J}^*_{\eta,d} - 2\varepsilon > J^\# - 3\varepsilon. \tag{3.14}$$

The second follows from (3.6), valid for $\hat{u} \equiv \hat{v}$ again; by (3.4) $(3.14)_3$ holds, while $(3.11)_3$ yields $(3.14)_4$. By ε's arbitrariness $(3.13)_2$ also holds. We now deduce $(3.3)_3$ from $(3.3)_1$ and (3.13). ■

It is worth noting that

P3.6. (*a*) *For the weak infimum* J^* *– see* (3.2) *– we have that*

$$J^* = \lim_{(\eta,d)(\in\Delta)\to(0,\underline{\delta})} \hat{J}^*_{\eta,d} = \lim_{(\eta,d)(\in\Delta)\to(0,\underline{\delta})} J^*_{\eta,d}. \tag{3.15}$$

(*b*) *If* (*i*) $\{\hat{\xi}_{0,\underline{\delta},\hat{u}_s}\}_{s>0}$ *is any among the infinitely many minimizing sequences of problem* $(2.18\text{-}20)_{0,\underline{\delta}}$ *and* (*ii*) $\hat{u}_s = u_{s,\eta,d} \circ \beta_{\eta,d}$, *which determines* $u_{s,\eta,d}$ *for* $s > 0$ *and* $(\eta, d) \in \Delta$, *then*

$$\begin{aligned} J^* = \lim_{s\to\infty} \hat{\mathfrak{J}}_{0,\underline{\delta}}(\hat{\xi}_{0,\underline{\delta},\hat{u}_s}) &= \lim_{(s,\eta,d)(\in N\times\Delta)\to(+\infty;\,0,\underline{\delta})} \hat{\mathfrak{J}}_{\eta,d}(\hat{\xi}_{\eta,d,\hat{u}_s}) = \\ &= \lim_{(s,\eta,d)(\in N\times\Delta)\to(+\infty;\,0,\underline{\delta})} \mathfrak{J}_{\eta,d}(\xi_{\eta,d,u_s}). \end{aligned} \tag{3.16}$$

(*c*) *If* (*iii*) *problem* $(2.18\text{-}20)_{0,\underline{\delta}}$ *has a solution* $\hat{\xi}_{0,\underline{\delta},\hat{u}^*}$ *and* (*iv*) $\hat{u}^* = u_{\eta,d} \circ \beta_{\eta,d}$, *which determines* $u_{\eta,d}$ *for* $(\eta, d) \in \Delta$, *then* e.g. *the lower limit in* (3.3) *can be replaced by the limit for* $(\eta, d) \to (0, \underline{\delta})$, i.e. *for* $d_m + \eta_m \to \delta^+$, *with* $(\eta, d) \in \Delta$:

$$J^* = \hat{\mathfrak{J}}_{0,\underline{\delta}}[\hat{\xi}_{0,\underline{\delta},\hat{u}^*}] = \lim_{(\eta,d)(\in\bar{\Delta})\to(0,\underline{\delta})} \hat{\mathfrak{J}}_{\eta,d}[\hat{\xi}_{\eta,d,\hat{u}^*}] = \lim_{(\eta,d)(\in\Delta)\to(0,\underline{\delta})} \mathfrak{J}_{\eta,d}[\xi_{\eta,d,u_{\eta,d}}]. \tag{3.17}$$

Note, first, that by (*b*) *there is some sequence* $u_s \in \mathcal{U}$ *such that* J^* *equals the R.H.S. of* $(3.16)_3$.

Second, in all the limits considered in P3.6 we can restrict (η, d) by the conditions $(\eta, d) \in \Delta$ and $\eta_i = d_i - d_{i-1}$, which in effect eliminates the interval $[d_{i-1} + \eta_{i-1}, d_i]$

$(i = 2, \dots, m)$. This result is natural and it is expected to be reached directly if, instead of using our two-step treatment of the nonmonotone case, one extends directly the whole treatment of the nonmonotone case written in [10].

PROOF OF P3.6. Consider any $\varepsilon > 0$. Then for some $\rho_0 > 0$ (3.6) holds; and by (3.2), for some $\bar{d}$ with (d) $\delta = \bar{d}_1 < \dots < \bar{d}_m < \delta + \rho_0$ the first of the relations

$$J^* + 4\varepsilon \geq J_{\bar{d}} + 3\varepsilon \geq \widehat{\mathfrak{J}}_{0,\bar{d}}(\widehat{\xi}_{0,\bar{d},\hat{u}}) + 2\varepsilon \geq \widehat{\mathfrak{J}}_{\eta,d}(\widehat{\xi}_{\eta,d,\hat{u}}) \geq \widehat{\mathfrak{J}}^*_{\eta,d} \qquad \forall(\eta, d) \in \bar{\Delta} \cap [B(0, \rho_0) \times B(\underline{\delta}, \rho_0)] \tag{3.18}$$

holds. Furthermore by (2.24) some $\hat{u} \in \widehat{\mathfrak{U}}$ renders $(3.18)_2$ true. Now by (d) and (3.6) we easily deduce that

$$|\widehat{\mathfrak{J}}_{0,\bar{d}}(\widehat{\xi}_{0,\bar{d},\hat{u}}) - \widehat{\mathfrak{J}}_{\eta,d}(\widehat{\xi}_{\eta,d,\hat{u}})| < 2\varepsilon \quad \forall(\eta, d) \in \bar{\Delta} \cap [B(0, \rho_0) \times B(\underline{\delta}, \rho_0)]. \tag{3.19}$$

This implies $(3.18)_3$. Lastly (3.4) yields $(3.18)_4$. By ε's arbitrariness (3.18) implies that

$$J^* \geq \limsup_{(\eta, d)(\in \bar{\Delta}) \to (0, \underline{\delta})} \widehat{\mathfrak{J}}_{\eta,d}\,.$$

Then, by $(3.3)_{1,2}$, $(3.15)_1$ holds; and (3.5) yields $(3.15)_2$.

To prove part (b) we first deduce $(3.16)_1$ from $(3.3)_3$ and the assumptions in (b). Now we consider any $\varepsilon > 0$. By $(3.16)_1$, for some $S \in \mathbf{N}$

$$|J^* - \widehat{\mathfrak{J}}_{0,\underline{\delta}}(\widehat{\xi}_{0,\underline{\delta},\hat{u}_s})| < \varepsilon \qquad \forall s > S\,.$$

Furthermore, for some ρ_0, (3.6) holds. Then, by putting $\hat{u} = \hat{u}_s$ in $(3.6)_1$, one easily sees that

$$|J^* - \widehat{\mathfrak{J}}_{\eta,d}(\widehat{\xi}_{\eta,d,\hat{u}_s})| < 2\varepsilon \qquad \forall(\eta, d) \in \bar{\Delta} \cap [B(0, \rho_0) \times B(\underline{\delta}, \rho_0)], \qquad \forall s > S\,.$$

By ε's arbitrariness this yields $(3.16)_2$. Lastly assumption (ii) in (b) and P2.1 for $\hat{u} = \hat{u}_s$ – see (2.22-23) – yield $(3.16)_3$.

To prove part (c) we first deduce $(3.17)_1$ from $(3.3)_3$ and assumption (iii). Thus, setting $\hat{u}_s = \hat{u}^*$ $(s \in \mathbf{N}^*)$, $\{\widehat{\xi}_{0,\underline{\delta},\hat{u}_s}\}_{s>0}$ is a minimizing sequence of problem $(2.18\text{-}20)_{0,\underline{\delta}}$; and by assumptions (ii) and (iv), $u_{s,\eta,d} = u_{\eta,d}$ for all $s > 0$ and all $(\eta, d) \in \Delta$. Therefore, by part (b), (3.16) and $(3.17)_1$ imply $(3.17)_{2,3}$. ∎

We are now particularly interested in the afore-mentioned limit (η, d) $(\in \Delta) \to$ $\to (0, \underline{\delta})$ and hence in the case $(\eta, d) = (0, \underline{\delta})$. In this (2.16) yields

$$\beta_0(t) = \begin{cases} s_0 + (\delta - s_0)t\,, & t \in [0, 1]\,, \\ \delta\,, & t \in [1, 2m]\,, \\ \delta + (s_1 - \delta)(t - 2m)\,, & t \in [2m, 2m + 1]\,, \end{cases} \tag{3.20}$$

while (2.16) and (2.17) imply

$$(3.21)\quad \hat{c}(t) = \lim_{(\eta, d)(\in \Delta) \to (0, \hat{\delta})} c_\eta[\beta_\eta(t)] = \begin{cases} c[s_0 + (\hat{\delta} - s_0)t], & t \in [0, 1], \\ c_{i-1} + \sigma_i(t - 2i + 1), & t \in [2i - 1, 2i], \\ c_l, & t \in [2l, 2l + 1], \\ c[\hat{\delta} + (s_1 - \hat{\delta})(t - 2m)], & t \in [2m, 2m + 1], \end{cases}$$

for $i = 1$ to m and $l = 1$ to $m - 1$. Thus by (2.6) and (2.20)

$$(3.22)\quad \begin{cases} \hat{\varphi}_0(t, \mathcal{P}, u) = \\ \quad = \begin{cases} \varphi[s_0 + (\hat{\delta} - s_0)t, \mathcal{P}, u](\hat{\delta} - s_0), & t \in [0, 1], \\ \varphi^1(\hat{\delta}, \mathcal{P}, u, \Theta(\hat{\delta}, c(\cdot)), c_{i-1} + \sigma_i(t - 2i + 1))\sigma_i, & t \in [2i - 1, 2i], \\ 0, & t \in [2l, 2l + 1], \\ \varphi[\hat{\delta} + (s_1 - \hat{\delta})(t - 2m), \mathcal{P}, u](s_1 - \hat{\delta}), & t \in [2m, 2m + 1], \end{cases} \\ \hat{L}_0(t, \mathcal{P}, u) = \\ \quad = \begin{cases} L[s_0 + (\hat{\delta} - s_0)t, \mathcal{P}, u](\hat{\delta} - s_0), & t \in [0, 1], \\ L^1(\hat{\delta}, \mathcal{P}, u, \Theta(\hat{\delta}, c(\cdot)), c_{i-1} + \sigma_i(t - 2i + 1))\sigma_i, & t \in [2i - 1, 2i], \\ 0, & t \in [2l, 2l + 1], \\ L[\hat{\delta} + (s_1 - \hat{\delta})(t - 2m), \mathcal{P}, u](s_1 - \hat{\delta}), & t \in [2m, 2m + 1], \end{cases} \end{cases}$$

for $i = 1$ to m and $l = 1$ to $m - 1$. By $(3.22)_1$ it follows that

$$(3.23)\qquad \hat{\mathcal{P}}(t) = \text{const} \quad \forall t \in [2l, 2l + 1] \quad (l = 1, \dots, m - 1).$$

Remark that, by (2.10-11) and (3.20-23), $c(\cdot)$'s values in (d_l, d_{l+1}), *i.e.* those of $\tilde{\chi}(\cdot)$ there, are irrelevant, as far as the limits at $\hat{\delta}$ are concerned.

The results (3.22-23) allow us to simplify the auxiliary problem $(2.18\text{-}19)_{0, \hat{\delta}}$ by cancelling the intervals $(2l, 2l + 1)$ for $l = 1, \dots, m - 1$. Thus in the simplified problem, say (2.18-19)′, $\hat{c}(\cdot)\colon [0, m + 1] \to \boldsymbol{R}$ has the definition

$$(3.24)\quad \hat{c}_{0, \hat{\delta}}(t) = \hat{c}(t) = \begin{cases} c[s_0 + (\hat{\delta} - s_0)t], & t \in [0, 1], \\ c_{l-1} + \sigma_l(t - l + 1), & t \in [l - 1, l] \quad (l = 1, \dots, m), \\ c[\hat{\delta} + (s_1 - \hat{\delta})(t - m)], & t \in [m, m + 1]. \end{cases}$$

4. On the (second-step) impulsive problem $(\mathcal{P}_\nu; \sigma_{r,1}, \dots, \sigma_{r,m_r})$ in the nonmonotone case for $\nu = 1$. Extended functional to minimize; ordinary and impulsive parts of differential constraints, complementary equations, Pontrjagin's conditions; border and junction conditions

By means of the preceding limit $(\eta, d) \to (0, \hat{\delta})$ we have in effect associated to our original regular problem both (α) an impulsive problem $\mathcal{P}_1$, *i.e.* $(2.1\text{-}5)^{c(\cdot)}$ with the curvature $c(\cdot)$ defined on the whole set $[s_0, s_1]\backslash\{\delta\}$, in a way compatible with (2.11):

$$c(t) = \chi(t) \quad \forall t \in [s_0, \delta), \qquad c(t) = \tilde{\chi}(t) \quad \forall t \in (\delta, s_1]; \tag{4.1}$$

and (β) the monotonicity type $(\sigma_1, \ldots, \sigma_m)$.

REMARK 4.1. *Since in the limit problem* $d_m = \delta$

$$c(\delta^+) - c(\delta^-) = \sigma_1 + \ldots + \sigma_m, \tag{4.2}$$

for $m > 1$ *it suffices to know the «nonmonotonicity type»* $(\sigma_2, \ldots, \sigma_m)$; *consequently in the monotone case considered in* [10] *there is no need to know any monotonicity type.*

Because of the «nonmonotone» monotonicity type now associated to problem $\mathcal{P}_1$, this cannot be treated like in [10]; in fact above formula (3.24) we replaced the corresponding auxiliary problem $(2.18\text{-}19)_0$ with a simplified one, called $(2.18\text{-}19)'$. Thus it would be better to speak of the problem $(\mathcal{P}_1; \sigma_1, \ldots, \sigma_m)$.

Let $s \mapsto \Theta(s) = \Theta(s, c(\cdot)) = (x(s), y(s), x'(s), y'(s))$ be the solution to the Cauchy problem $(2.1\text{-}2)^{c(\cdot)}$ in the case (4.1). Referring to this we construct $\varphi(s, \mathcal{P}, u)$ and $L(s, \mathcal{P}, u)$ by means of (2.6) and we set – see [10, (3.16)]

$$\begin{cases} \varphi_i(c, \mathcal{P}, u) \doteq \varphi^1[\delta, \mathcal{P}, u, \Theta(\delta), c] \\ L_i(c, \mathcal{P}, u) \doteq L^1[\delta, \mathcal{P}, u, \Theta(\delta), c] \end{cases} \qquad (i = 1, \ldots, m). \tag{4.3}$$

To introduce *extended solutions* – see [10, (6.9)] – in the present framework, we consider the process

$$\xi \doteq (\mathcal{P}(\cdot), u(\cdot), \{\mathcal{P}_i(\cdot), u_i(\cdot)\}_{i=1, \ldots, m}) \tag{4.4}$$

where the function $\mathcal{P}(\cdot)$: $[s_0, s_1]\backslash\{\delta\} \to \boldsymbol{R}$ is AC on both $[s_0, \delta)$ and $(\delta, s_1]$, and where, calling U the set of control values,

$$u \in \mathcal{B}([s_0, s_1], U), \qquad \mathcal{P}_i(\cdot) \in AC([c_{i-1}, c_i]), \qquad u_i \in \mathcal{B}([c_{i-1}, c_i], U) \qquad (i = 1, \ldots, m). \tag{4.5}$$

Furthermore we define the *extension* of the functional $\mathfrak{J}(\cdot)$ to minimize – see $(2.3)^{c(\cdot)}$ – by

$$\bar{\mathfrak{J}}[\xi] \doteq \mathfrak{J}[\mathcal{P}(\cdot), u(\cdot)] + \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} L_i[c, \mathcal{P}_i(c), u_i(c)]\, dc. \tag{4.6}$$

In our simple problem $(\mathcal{P}_1; \sigma_1, \ldots, \sigma_m)$ the differential constraints (in $\mathcal{P}(\cdot)$ and $\mathcal{P}_i(\cdot)$) and the complementary equations (in $\lambda(\cdot)$ and $\lambda_i(\cdot)$) have the *ordinary parts* in $[s_0, s_1]\backslash\{\delta\}$ – see [10, (6.17)]

$$d\mathcal{P}/ds = \varphi[s, \mathcal{P}, u(s)], \qquad d\lambda/ds = -\lambda\varphi_{,\mathcal{P}}[s, \mathcal{P}, u(s)] + L_{,\mathcal{P}}[s, \mathcal{P}, u(s)] \tag{4.7}$$

for a.e. $s \in [s_0, s_1]$, and the *impulsive parts*

$$d\mathcal{P}_i/dc = \varphi_i[s, \mathcal{P}_i, u_i(c)], \qquad d\lambda_i/dc = -\lambda_i\varphi_{i,\mathcal{P}}[c, \mathcal{P}_i, u_i(c)] + L_{i,\mathcal{P}}[c, \mathcal{P}_i, u_i(c)] \tag{4.8}$$

for a.e. $c \in [c_{i-1}, c_i]$ $(i = 1, \ldots, m)$. Furthermore we have the control constraints

$$u(s) \in U, \qquad u_i(c) \in U \quad (i = 1, \ldots, m), \tag{4.9}$$

the initial and junction conditions for $\mathcal{P}(\cdot)$ and $\mathcal{P}_i(\cdot)$

$$(4.10)\qquad \begin{cases} \mathcal{P}(s_0) = \mathcal{P}_0\,, & \mathcal{P}(\delta^-) = \mathcal{P}_1(c_0)\,, \quad \mathcal{P}_m(c_m) = \mathcal{P}(\delta^+)\,, \\ \mathcal{P}_i(c_i) = \mathcal{P}_{i+1}(c_i) & (i = 1, \ldots, m-1)\,, \end{cases}$$

and the terminal and junction conditions for $\lambda(\cdot)$ and $\lambda_i(\cdot)$

$$(4.11)\qquad \begin{cases} \lambda(s_1) = -\Psi'[\mathcal{P}(s_1)]\,, & \lambda(\delta^-) = \lambda_1(c_0)\,, \quad \lambda_m(c_m) = \lambda(\delta^+)\,, \\ \lambda_i(c_i) = \lambda_{i+1}(c_i) & (i = 1, \ldots, m-1)\,. \end{cases}$$

Note that $(4.10)_4$ and $(4.11)_4$ are lacking in the analogue [10, (6.18)] of (4.10-11).

Similarly, Pontrjagin's optimization condition has the *ordinary part*

$$(4.12)\qquad \lambda(s)\,\varphi[s, \mathcal{P}(s), u(s)] - L[s, \mathcal{P}(s), u(s)] = \\ = \max\{\lambda(s)\,\varphi[s, \mathcal{P}(s), u] - L[s, \mathcal{P}(s), u] \colon u \in U\}$$

for a.e. $s \in [s_0, s_1]$ – see [10, (6.19)] – as well as the *impulsive parts*

$$(4.13)\qquad \lambda_i(c)\,\varphi_i[c, \mathcal{P}_i(c), u_i(c)] - L_i[c, \mathcal{P}_i(c), u_i(c)] = \\ = {}^{\max}_{\min}\,\{\lambda_i(c)\,\varphi_i[c, \mathcal{P}_i(c), u] - L_i[c, \mathcal{P}_i(c), u] \colon u \in U\}$$

for a.e. $c \in [c_{i-1}, c_i]$ and $\sigma_i \gtrless 0$ $(i = 1, \ldots, m)$.

5. Extension of section 4 to the case of $\nu > 1$ discontinuity points with arbitrarily given nonmonotonicity types

Now we briefly consider our impulsive (or better structurally discontinuous) problem $(\mathcal{P}_\nu; \sigma_{r,1}, \ldots, \sigma_{r,m_r})$ (r being regarded to run on $\{1, \ldots, \nu\}$) in the general case $(\nu \geq 1)$, where the single discontinuity point δ (for $c(\cdot)$) is replaced by δ_1 to δ_ν. We assume that

(i) $s_0 < \delta_1 < \ldots < \delta_\nu < s_1$ and δ_r has the monotonicity type $(\sigma_{r,1}, \ldots, \sigma_{r,m_r})$ in the sense that δ_r corresponds to a short interval $[\tilde{a}_r, \tilde{b}_r]$ where the regular system $\tilde{S}$ (to be treated approximately) has an *impulsive character* of type $(\sigma_{r,1}, \ldots, \sigma_{r,m_r})$ $(r = 1, \ldots, \nu)$ – see (i) above (2.8) regarding $\tilde{a}_r = \delta$, $\tilde{b}_r = \delta + \varepsilon_0$, and $\sigma_{r,i} = \sigma_i$.

Now, *e.g.* expressions (4.3-4), (4.6), (4.8-9), and (4.13) simply become

$$(5.1)\qquad \begin{cases} \varphi_{r,i}(c, \mathcal{P}, u) \doteq \varphi^1[\delta_r, \mathcal{P}, u, \Theta(\delta_r), c]\,, \\ L_{r,i}(c, \mathcal{P}, u) \doteq L^1[\delta_r, \mathcal{P}, u, \Theta(\delta_r), c]\,, \end{cases}$$

$$(5.2)\qquad \xi \doteq (\mathcal{P}(\cdot), u(\cdot), \{\mathcal{P}_{r,i}(\cdot), u_{r,i}(\cdot)\}_{i=1,\ldots,m_r;\, r=1,\ldots,\nu})\,,$$

$$(5.3)\qquad \bar{\mathfrak{J}}[\xi] \doteq \mathfrak{J}[\mathcal{P}(\cdot), u(\cdot)] + \sum_{r=1}^{\nu} \sum_{i=1}^{m_r} \int_{c_{r,i-1}}^{c_{r,i}} L_i[c, \mathcal{P}_{r,i}(c), u_{r,i}(c)]\,dc\,,$$

$$(5.4)\qquad \begin{cases} d\mathcal{P}_{r,i}/dc = \varphi_{r,i}[c, \mathcal{P}_{r,i}, u_{r,i}(c)]\,, \\ d\lambda_{r,i}/dc = -\lambda_{r,i}\varphi_{r,i,\mathcal{P}}[c, \mathcal{P}_{r,i}, u_{r,i}(c)] + L_{r,i,\mathcal{P}}[c, \mathcal{P}_{r,i}, u_{r,i}(c)]\,, \end{cases}$$

for a.e. $c \in [c_{r,i-1}, c_{r,i}]$,

$$u(s) \in U\,, \qquad u_{r,i}(c) \in U\,, \tag{5.5}$$

and

$$\begin{aligned}\lambda_{r,i}(c)\,\varphi_{r,i}[c, \mathcal{P}_{r,i}(c), u_{r,i}(c)] - L_{r,i}[c, \mathcal{P}_{r,i}(c), u_{r,i}(c)] = \\ = {}^{\max}_{\min}\, \{\lambda_{r,i}(c)\,\varphi_{r,i}[c, \mathcal{P}_{r,i}(c), u] - L_{r,i}[c, \mathcal{P}_{r,i}(c), u]\colon u \in U\}\end{aligned} \tag{5.6}$$

for a.e. $c \in [c_{r,i-1}, c_{r,i}]$ and $\sigma_{r,i} \gtrless 0$ respectively, where

$$c_{r,1} \doteq c(\delta_r^-)\,, \qquad c_{r,i} \doteq c_{r,i-1} + \sigma_{r,i} \quad (c_{r,m_r} \doteq c(\delta_r^+)) \tag{5.7}$$

for $i = 1$ to m_r and $r = 1$ to ν. Furthermore $(4.10)_{2,4,5}$ and (4.11) become

$$\begin{cases} \mathcal{P}(\delta_r^-) = \mathcal{P}_{r,1}(c_{r,0}), \qquad \mathcal{P}_{r,m_r}(c_{r,m_r}) = \mathcal{P}(\delta_r^+)\,, \\ \mathcal{P}_{r,i}(c_{r,i}) = \mathcal{P}_{r,i+1}(c_{r,i}) \qquad (i = 1, \dots, m_r - 1; r = 1, \dots, \nu) \end{cases} \tag{5.8}$$

and

$$\begin{cases} \lambda(s_1) = -\Psi'[\mathcal{P}(s_1)]\,, \qquad \lambda(\delta_r^-) = \lambda_{r,1}(c_{r,0})\,, \qquad \lambda_{r,m_r}(c_{m_r}) = \lambda(\delta_r^+)\,, \\ \lambda_{r,i}(c_{r,i}) = \lambda_{r,i+1}(c_{r,i}) \qquad (i = 1, \dots, m_r - 1)\,. \end{cases} \tag{5.9}$$

Of course (4.7) and (4.12) need not to be changed.

6. Extended processes, Pontrjagin's maximum principle, and existence theorem for the general problem $(\mathcal{P}_\nu; \sigma_{r,1}, \dots, \sigma_{r,m_r})$

Definition 6.1. (*a*) In connection with the general optimization problem $(\mathcal{P}_\nu; \sigma_{r_1}, \dots, \sigma_{r_{m_r}})$ we say that the process ξ – see (5.2) – is an *admissible* (*extended*) *process*, if it solves the ODEs $(4.7)_1$, $(5.4)_1$ and it satisfies the conditions (5.5), (5.8) and (5.9).

(*b*) We say that the admissible process

$$\xi^* \doteq (\mathcal{P}^*(\cdot), u^*(\cdot), \{\mathcal{P}^*_{r,i}(\cdot), u^*_{r,i}(\cdot)\}_{i=1,\dots,m_r; r=1,\dots,\nu}) \tag{6.1}$$

is an *extended solution* to the above problem if

$$\bar{\mathfrak{J}}[\xi^*] \leqslant \bar{\mathfrak{J}}[\xi] \quad \text{for all admissible processes } \xi \text{ – see (5.2-3)}. \tag{6.2}$$

One can easily prove that

P6.1. *For the afore-mentioned solution ξ^*, we have that $\mathfrak{J}[\xi^*] = J^*$ where J^* is the analogue for $(\mathcal{P}_\nu; \sigma_{r,1}, \dots, \sigma_{r,m_r})$ of the weak infimum J^* defined by (2.15) for $(\mathcal{P}_1; \sigma_1, \dots, \sigma_m)$.*

Pontrjagin's maximum principle (PMP) – see [10, Theor. 6.2] – can be extended as follows.

Theorem 6.1 (PMP). *Let ξ^* be an extended solution to problem $(\mathcal{P}_\nu; \sigma_{r,1}, \dots, \sigma_{r,m_r})$. Then there are some functions $\lambda(\cdot)\colon [s_0, s_1] \to \boldsymbol{R}$ and $\lambda_{r,i}(\cdot) \in AC([c_{r,i-1}, c_{r,i}])$ – see (5.7) – $(i = 1, \dots, m_r; r = 1, \dots, \nu)$, such that (i) $\lambda(\cdot)$ is AC on $[s_0, \delta_1]$, $(\delta_{j-1}, \delta_j]$, $(j = 2, \dots, \nu)$, and $(\delta_\nu, s_1]$, (ii) the ODEs $(4.7)_2$ and $(5.4)_2$ are solved while* *conditions* (5.8) *and* (5.9) *are satisfied, and* (*iii*) *Pontrjagin's optimization conditions* (4.12) *and* (5.6) *hold in their general versions for* $(\mathcal{P}_\nu; \sigma_{r,1}, \ldots, \sigma_{r,m_r})$.

In order to state the existence theorem below we consider the following sets

(6.3) $$F^*(s, \mathcal{P}) \doteq \{(y_0, y): y_0 \geq L(s, \mathcal{P}, u), y = \varphi(s, \mathcal{P}, u), u \in U\} \qquad \forall (s, \mathcal{P}) \in ([s_0, s_1] \setminus \{\delta_1, \ldots, \delta_\nu\}) \times \boldsymbol{R},$$

and

(6.4) $$F^*_{r,i}(c, \mathcal{P}) \doteq \{(y_0, y): y_0 \geq L_{r,i}(c, \mathcal{P}, u)\,\sigma_{r,i}, y = \varphi_{r,i}(c, \mathcal{P}, u)\,\sigma_{r,i}, u \in U\} \qquad \forall (c, \mathcal{P}) \in [c_{r,i-1}, c_{r,i}] \times \boldsymbol{R} \quad (i = 1, \ldots, m_r; r = 1, \ldots, \nu).$$

THEOREM 6.2. *If the sets* (6.3-4) *are convex, then an extended solution* ξ^* *to problem* $(\mathcal{P}_\nu; \sigma_{r,1}, \ldots, \sigma_{r,m_r})$ *exists* (5).

In analogy with [10, Corollary 4.1] one can prove that by means of ξ^* some minimizing sequences can be constructed.

Let us note that in (6.1) $\mathcal{P}^*(\cdot)$ is allowed to take some negative values – see also [10, p. 51]. This is not possible in various physical applications that we have in mind (in fact some among the phase conditions [10, (6.20)] have to be satisfied in them) – see *e.g.* [6, 9-13]. In spite of this the present theory can be applied to several optimization problems – such as problems (*A*) to (*F*) in [6], added with some monotonicity types – in that every admissible process $(\mathcal{P}(\cdot), u(\cdot))$ for them satisfies some among the phase conditions [10, (6.20)].

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(5) This theorem is a counterpart in the nonmonotone case of [10, Corollary 4.1] which refers to the auxiliary problem [10, $(3.19\text{-}20)_0$].

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