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On topological degree and Poincaré duality

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Fisica matematica. — *On topological degree and Poincaré duality.* Nota di FRANCO CARDIN, presentata (*) dal Socio A. Bressan.

ABSTRACT. — In this *Note* we investigate about some relations between Poincaré dual and other topological objects, such as intersection index, topological degree, and Maslov index of Lagrangian submanifolds. A simple proof of the Poincaré-Hopf theorem is recalled. The Lagrangian submanifolds are the geometrical, multi-valued, solutions of physical problems of evolution governed by Hamilton-Jacobi equations: the computation of the algebraic number of the branches is showed to be performed by using Poincaré dual.

KEY WORDS: Topological degree; Poincaré duality; Maslov index; Lagrangian manifolds; Solutions of Hamilton-Jacobi equation.

RIASSUNTO. — *Sul grado topologico e sulla dualità di Poincaré.* Sono studiate le relazioni tra la dualità di Poincaré ed altri oggetti topologici quali l'indice d'intersezione, il grado topologico, l'indice di Maslov di sottovarietà Lagrangiane. Viene richiamata una semplice dimostrazione del teorema di Poincaré-Hopf. Le soluzioni multivoche dell'equazione di Hamilton-Jacobi, relativa a qualche problema fisico di evoluzione, sono geometricamente rappresentate da sottovarietà Lagrangiane: il calcolo del numero algebrico delle falde è realizzato mediante il duale di Poincaré.

1. INTRODUCTION

The intersection index of two transverse submanifolds U and V can be computed, via the Thom isomorphism, by means of the Poincaré dual of U and V. This *Note* explores some applications of this fact. An earlier application was given in [6], where it was remarked that the Maslov cohomological class of a Lagrangian submanifold Λ is precisely the Poincaré dual of the singular (Maslov) cycle of Λ, and this is recalled here, after some preliminary definitions given in Sections 2 and 3, in the Section 4. The Poincaré dual is useful to compute the topological degree of functions; indeed, we define the topological degree as a suitable intersection number, and we can directly prove the well known homotopic invariance of the topological degree in Section 5. Another topological application is presented in the Section 6: the proof of the Poincaré-Hopf theorem is quickly revisited, and in order to use the present framework, the Euler characteristic of $\chi(M)$ is here defined as the self-intersection index of the zero-section of the tangent bundle TM of a compact manifold M (see [9]). In the last Section 7 is presented a way to compute the algebraic number of branches of a Lagrangian submanifold L, thought of as a «multi-function», which is the geometrical solution of a Hamilton-Jacobi partial differential equation, related, for example, to some physical problems of propagation in the space-time. This computation, and the selection of the appropriate branches, are useful to build some new weak solutions (in the sense of «functions»), like *e.g.* the viscosity solutions, see [5].

(*) Nella seduta del 3 novembre 1994.

2. Poincaré duality

Let M be a n-dimensional oriented smooth manifold. Denote by $H^k(M)$ the k-dimensional space of the de Rham cohomology. The Poincaré duality, produced by the nondegenerate pairing

$$H^{n-k}(M) \times H_c^k(M) \ni ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \in \boldsymbol{R} \tag{2.1}$$

(c: compact support), establishes the existence of the following isomorphism:

$$H^{n-k}(M) \simeq (H_c^k(M))^* , \tag{2.2}$$

see [3, p. 44].

Let S be a k-dimensional oriented submanifold of M, and $j: S \hookrightarrow M$ the corresponding embedding map. By Stokes's theorem, the map

$$H_c^k(M) \ni [\omega^{(k)}] \mapsto \int_S j^* \omega^{(k)} \in \boldsymbol{R} \tag{2.3}$$

is a linear functional on $H_c^k(M)$; hence, by the above duality, a cohomological class $[\eta_S] \in H^{n-k}(M)$ is associated to S, the so-called Poincaré dual of S:

$$\int_S \omega^{(k)} \wedge \eta_S = \int_S j^* \omega^{(k)} , \qquad \forall \omega^{(k)} \in H_c^k(M) . \tag{2.4}$$

Localization Principle [3, p. 67]: the support of the Poincaré dual of a submanifold S can be shrunk into any given tubular neighborhood of S.

Let U^h and V^k be two closed oriented submanifolds of M with transversal intersection, that is,

$$T_x U^h \oplus T_x V^k = T_x M , \qquad \forall x \in U^h \cap V^k (\dim U^h \cap V^k = h + k - n) . \tag{2.5}$$

In view of Thom's isomorphism theory (see [3,(6.31)]) we have

$$\eta_{U^h \cap V^k} = \eta_{U^h} \wedge \eta_{V^k} . \tag{2.6}$$

Here, with a little abuse of notation, we mean $U^h \cap V^k \neq V^k \cap U^h$, in the sense that $U^h \cap V^k$ inherits in a standard way the orientation of the pair (U^h, V^k).

3. Intersection index

When $h + k = n$, so $\dim U^h \cap V^k = 0$, the above intersection $U^h \cap V^k$ consists of a finite set of points. Each of these points is endowed with a (suitable) inherited orientation multiplicity, $+1$ or -1. The sum of these multiplicities is the so-called intersection index $U^h \bigcirc V^k$ (see [7, vol. 2, Sect. 15]). By remembering that the Poincaré dual of a point (with inherited orientation) can be represented by a bump function whose integral is 1 [3, p. 52, p. 68] we have that

$$U^h \bigcirc V^k = \int_M \eta_{U^h \cap V^k} , \tag{3.1}$$

and, by using the formula (2.6),

$$U^h \bigcirc V^k = \int_M \eta_{U^h} \wedge \eta_{V^k} \,. \tag{3.2}$$

Now, let Σ be a $(n-1)$-dimensional oriented submanifold of M, and let γ be a closed curve on M transversally crossing Σ. Let $[\eta_\Sigma] \in H^1(M)$ and $[\eta_\gamma] \in H^{n-1}(M)$ be the Poincaré duals of Σ and γ respectively. We compute the difference between the number n_+ of transition points from the negative side to the positive side and the number n_- of transition points in the reverse direction. This number is the intersection index $\Sigma \bigcirc \gamma$,

$$\Sigma \bigcirc \gamma = \int_M \eta_\Sigma \wedge \eta_\gamma = \oint_\gamma \eta_\Sigma \,. \tag{3.3}$$

4. Maslov index

Let Λ be a Lagrangian submanifold of the cotagent bundle $T^* M$,

$$\Lambda \overset{\iota}{\hookrightarrow} T^* M \xrightarrow{\pi_M} M \,, \tag{4.1}$$

that is, $\dim \Lambda = \dim M$ and the restriction of the symplectic 2-form $\omega = dp_i \wedge dq^i$ on Λ is vanishing identically, $\iota^* \omega = 0$. The graphs of the differentials of smooth functions $M \ni q \mapsto S(q) \in \boldsymbol{R}$ are Lagrangian manifolds, but the converse is not true, and a theorem by Maslov-Hörmander characterizes the local structure of the generic Lagrangian submanifolds by means of the so-called Morse families or generating functions (see [12]). Denote by Z the set of points λ on Λ where

$$\operatorname{rank}[\mathrm{D}(\pi_M \circ \iota)(\lambda)] < \max (= \dim M) \,. \tag{4.2}$$

Under some genericity assumptions, the singular locus Z is the union of a $(n-1)$-dimensional orientable submanifold of Λ and subsets of dimension $\leqslant n-3$. Thus Z naturally determines a cycle $[Z]$ in $H_{n-1}(\Lambda, \boldsymbol{Z})$ (singular homology with integer coefficients), called the Maslov cycle of Λ, see [8]. The Maslov index $m(\gamma)$ of a closed smooth curve γ in Λ is given by the intersection index $Z \bigcirc \gamma$. By means of the Poincaré dual class $[\eta_Z] \in H^1_c(\Lambda)$ related to Z we can compute $m(\gamma)$, see (3.3),

$$m(\gamma) = Z \bigcirc \gamma = \int_\Lambda \eta_{Z \cap \gamma} = \oint_\gamma \eta_Z = n_+ - n_- \,. \tag{4.3}$$

We remark that the cohomological class $[\eta_Z] \in H^1_c(\Lambda)$ is obtained, as a rule, by considering a rather different approach: given an atlas of Morse families (or generating functions) covering Λ (see *e.g.* [12]), one builds a suitable 1-cocycle in the Čech cohomology, and, by the isomorphism with de Rham cohomology, finally finds the Maslov class, which is, precisely, the above Poincaré dual class $[\eta_Z]$, see [6].

5. Topological degree

Let $f: X \to Y$ be a smooth function, where $\dim X = \dim Y$, X is compact and Y is connected. We recall that the topological degree of f at a regular value $y' \in Y$ is (see *e.g.* [11]):

$$\deg(f, y') = \sum_{x_\alpha : f(x_\alpha) = y'} \operatorname{sgn}(\det Df(x_\alpha)) . \tag{5.1}$$

Let us define

$$\operatorname{graph} f := \{(x, y) \in X \times Y : y = f(x)\}, \quad X_{y'} := \{(x, y) \in X \times Y : y = y'\} . \tag{5.2}$$

We notice that, at the regular values $y' \in Y$, graph f and $X_{y'}$ have transversal intersection. We can recognise that the topological degree is the intersection index of graph $f \cap X_{y'}$,

$$\deg(f, y') = \operatorname{graph} f \bigcirc X_{y'} . \tag{5.3}$$

We quickly obtain the following main property of $\deg(f, y')$, that is, its independence from y'; by the above construction with the Poincaré dual, we can write

$$\deg(f, y') = \int_{X \times Y} \eta_{\operatorname{graph} f} \wedge \eta_{X_{y'}} = \int_{X_{y'}} \eta_{\operatorname{graph} f} = \int_{X_{y''}} \eta_{\operatorname{graph} f} = \deg(f, y'') \tag{5.4}$$

since $X_{y'}$ and $X_{y''}$, $\forall y', y'' \in Y$, are homotopically (hence homologically) trivially equivalent.

6. Poincaré-Hopf theorem

Let M be a compact oriented manifold, and let $T^0 M = \{(x, 0) : x \in M\} \subset TM$ be the zero section of TM. Following Hirsch [9], we define the characteristic of Euler as the self-intersection index of the zero section:

$$\chi(M) := T^0 M \bigcirc T^0 M . \tag{6.1}$$

Let ξ be a vector field with a finite number of zeros $\{x_i\}_{i=1,\dots,k}$ and $\operatorname{rank}(D\xi(x_i)) = \dim(M)$, $i = 1, \dots, k$. Every vector field, like the above ξ, produces a homotopic (and hence homologic) ε-deformation of $T^0 M$,

$$\operatorname{graph}(\varepsilon\xi) := \{(x, \varepsilon\xi) : x \in M\} \subset TM . \tag{6.2}$$

Such an ε-deformation is transverse to $T^0 M$ in TM, and this holds in view of the above rank condition on ξ (here, we drop the classical discussion about the relaxation of this generic condition); we may write

$$\chi(M) = T^0 M \bigcirc T^0 M = \operatorname{graph}(\varepsilon\xi) \bigcirc T^0 M , \tag{6.3}$$

and, in view of (5.3),

$$\chi(M) = \deg(\varepsilon\xi, 0) = \sum_i \operatorname{sgn}(\det(\varepsilon D\xi(x_i))) = \sum_i \operatorname{index}(\xi)|_{x_i} , \tag{6.4}$$

where for the last definition in (6.4) one can see *e.g.* [2, p. 288].

7. A (SORT OF) TOPOLOGICAL DEGREE FOR LAGRANGIAN SUBMANIFOLDS RELATED TO HAMILTON-JACOBI EQUATIONS

Let M be a (classical model of) space-time, $M = \boldsymbol{R}^{m+1} \ni (t, q^i)$, $i = 1, \ldots, n$. Let

$$\mathbf{H}: T^* \boldsymbol{R}^{n+1} \to \boldsymbol{R}, \qquad (t, q^i; p_0, p_j) \mapsto \mathbf{H}(t, q^i; p_0, p_j) = p_0 + \mathcal{H}(t, q^i, p_j) \tag{7.1}$$

be a possibly time-depending Hamiltonian function. By solving geometrically the Cauchy problems for the H-J equation related to $\mathbf{H}$ we mean looking for Lagrangian submanifolds Λ of $T^* \boldsymbol{R}^{n+1}$ contained into $\mathbf{H}^{-1}(0)$, and such that the elements of $\iota(\Lambda)$ of the form $(0, q^i; p_0, p_j)$ are given by

$$(0, q^i; p_0, p_j) = \left(0, q^i; -\mathcal{H}\left(0, q^i, \frac{\partial \sigma}{\partial q^j}(q^i)\right), \frac{\partial \sigma}{\partial q^j}(q^i)\right), \tag{7.2}$$

where the function

$$\sigma: \boldsymbol{R}^n \to \boldsymbol{R}, \qquad q^i \mapsto \sigma(q^i) \tag{7.3}$$

represents the initial data. When Λ is globally given by the graph of the differential of a function $S(t, q^i)$, then we say that $S(t, q^i)$ is a *classical* solution of the H-J equation,

$$\frac{\partial S}{\partial t}(t, q^i) + \mathcal{H}\left(t, q^i, \frac{\partial S}{\partial q^j}(t, q^i)\right) = 0, \qquad S(0, q^i) = \sigma(q^i). \tag{7.4}$$

We can interpret Λ as a sort of multi-function (just like a Riemann surface in complex analysis, see [12]). If the Hamiltonian $\mathcal{H}$ produces a global flux for the associated Hamiltonian system of o.d.e.'s (the characteristics), then, under a suitable non-characteristic condition on the initial function (see *e.g.* [10, 4], and the literature therein quoted), the geometrical Cauchy problem is globally solved by a Lagrangian submanifold, which however is not globally transverse to the base manifold $M = \boldsymbol{R}^{n+1}$; hence, as is well known, there need not exist global classical solutions like $S(t, q^i)$. Then Λ has a non trivial Maslov cycle Z. The intersection index of a (connected piece of) characteristic curve γ of the Hamiltonian system $\mathcal{H}$ on Λ with Z, $Z \bigcirc \gamma$, is just the Morse index of γ, and it is the Maslov index for a non-closed curve (see [1, appendix 11]).

In order to calculate the number of branches (with orientation) connected to a global geometrical solution Λ, we proceed as follow. If the solution of H-J is classical, $\Lambda = \operatorname{graph}(dS)$, the following topological degree is well defined at the regular points (p_0', p_j'):

$$\deg(dS; p_0', p_j') = \int_{T^* \boldsymbol{R}^{n+1}} \eta_{\operatorname{graph}(dS)} \wedge \eta_{\boldsymbol{R}^{n+1}_{(p_0', p_j')}}, \tag{7.5}$$

where

$$\boldsymbol{R}^{n+1}_{(p_0', p_j')} := \{(t, q^i; p_0, p_j) \in T^* \boldsymbol{R}^{n+1} : (p_0, p_j) = (p_0', p_j')\}, \tag{7.6}$$

and the supports of the Poincaré duals $\eta_{\operatorname{graph}(dS)}$ and $\eta_{\boldsymbol{R}^{n+1}_{(p_0', p_j')}}$ are chosen in small tubular neighborhoods of $\operatorname{graph}(dS)$ and $\boldsymbol{R}^{n+1}_{(p_0', p_j')}$ respectively.

We can make two generalizations: first, we extend the formula (7.5) to general Lagrangian submanifolds $\Lambda(\neq \text{graph}(dS))$; second, instead to consider $\boldsymbol{R}^{n+1}_{(p_0', p_j')}$, we consider

$$\boldsymbol{R}^{n+1}_{(t', q'^i)} := \{(t, q^i; p_0, p_j) \in T^* \boldsymbol{R}^{n+1} : (t, q^i) = (t', q'^i)\}. \tag{7.7}$$

The new formula is thus

$$\mathrm{DEG}(\Lambda; t', q'^i) := \int_{T^* \boldsymbol{R}^{n+1}} \eta_\Lambda \wedge \eta_{\boldsymbol{R}^{n+1}_{(t', q'^i)}}, \tag{7.8}$$

and it counts the number of (p_0, p_j) corresponding on Λ to (t', q'^i).

The integer $\mathrm{DEG}(\Lambda; t', q'^i)$ is well defined where Λ is transversal to $\boldsymbol{R}^{n+1}_{(t', q'^i)}$, and it is singular precisely on (the projection on the base $\boldsymbol{R}^{n+1} = \{(t, q^i)\}$ of) the singular Maslov cycle Z of Λ. Finally, we have

$$\mathrm{DEG}(\Lambda; t', q'^i) = \int_{\boldsymbol{R}^{n+1}_{(t', q'^i)}} \eta_\Lambda = \int_{\boldsymbol{R}^{n+1}_{(t'', q''^i)}} \eta_\Lambda = \mathrm{DEG}(\Lambda; t'', q''^i), \tag{7.9}$$

for every pair (t', q'^i), (t'', q''^i) homotopically connected by some curve not crossing the Maslov singular cycle Z.

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