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Fisica matematica. — *Equilibrium confocal textures in a smetic-A cell.* Nota (*) di
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ABSTRACT. — We study the textures of smectic-A liquid crystals consisting in curved, but strictly equidistant lamellae. Assuming translational symmetry, we can generate them from a single curve. The free energy is a non-trivial functional of it. We learn how to derive the equilibrium equation for this curve, when the texture is confined between two parallel plates, which exert a weak anchoring on the orientation of the lamellae, but do not interfere directly with their position. Finally, we describe an instability that may arise in the cell when the anchoring conditions on the two plates are antagonistic.

KEY WORDS: Smectic-A liquid crystals; Confocal textures; Free energy; Instability.

RIASSUNTO. — *Tessiture confocali di equilibrio in una cella di smettico A.* Si considerano le tessiture di uno smettico A che consistono in strati strettamente equidistanti, anche se generalmente curvi. Nell'ipotesi che gli strati siano invarianti per traslazione lungo una direzione prescritta, queste tessiture sono generate da una curva; il funzionale dell'energia libera dipende solo da essa. Deriviamo l'equazione di equilibrio per questa curva, che vale quando la tessitura si dispiega tra due piatti paralleli che esercitano un ancoraggio debole sull'orientamento degli strati, ma non interferiscono sulla loro posizione. La *Nota* si conclude con la descrizione di un'instabilità della tessitura, che può manifestarsi quando gli ancoraggi ai due piatti sono antagonisti.

1. INTRODUCTION

The molecules of smectic liquid crystals, which resemble rods, tend to be organized in parallel layers, nearly a molecular length apart, called *lamellae* (see *e.g.* [2, Ch.7]). The molecules of smectics-A tend to be perpendicular to the layers. They are completely free to move within one layer, but migrations towards other layers are highly impeded; in the ground state all layers are equidistant planes. Thus, smectics are at the same time one-dimensional crystals, in that the lamellae pile up with a long-range positional order, and two-dimensional liquids, in that the molecules in each lamella are not restrained to obey any positional order. Among all liquid crystals, smectics are the closest to very crystals, and retain a fair amount of their elasticity.

The elasticity of smectics has been widely studied. Theories allowing only for quasi-planar layers are described in [2, 3]; also non-linear theories, such as Leslie, Stewart and Nakagawa's [6], and covariant theories, such as Kléman and Parodi's [5], have appeared in the literature.

In many cases the dilation of the lamellae is negligible, and so the elastic description becomes much simpler [4]: the texture consists of lamellae all parallel to one another (as in the focal conics described by [1]). The bulk free energy density is a function of the total curvature $\bar{\sigma}$ (*i.e.*, twice the mean curvature) of the lamellae:

$$f_c := (1/2)K\bar{\sigma}^2 , \tag{1.1}$$

where K is a positive material modulus. In (1.1) terms reducible to surface energies by

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integration over the volume are omitted. Furthermore, $\bar{\sigma} = \operatorname{div} \boldsymbol{n}$, where $\boldsymbol{n}$ is the lamella normal.

The distortions that a smectic-A liquid crystal may undergo under the assumption that its lamellae suffer no dilation are described below. For each lamella, we call *focal surface* the locus of its centers of curvature; generally, it consists of two separate sheets, one for each center of curvature. Since the lamellae are all parallel, they possess the same focal surface: we say that they form a *confocal texture*. We call *generators* the straight lines orthogonal to the lamellae; they envelope the focal surface and are also the integral lines of the orientation field $\boldsymbol{n}$. Clearly, the focal surface of a confocal texture is a singular surface for the orientation field, since there the curvature $\bar{\sigma}$ diverges.

Let $\mathcal{B}$ be the region in space occupied by a smectic-A confocal texture. We assume that its focal surface lies all outside $\mathcal{B}$. This implies that there are no defects of $\boldsymbol{n}$ in $\mathcal{B}$. As a confocal texture is entirely determined by any of its lamellae, the curvature energy stored in $\mathcal{B}$ depends on the shape of one *reference* lamella. Thus, the curvature energy stored in $\mathcal{B}$,

$$\mathcal{F}_c := \int_{\mathcal{B}} f_c \, dv \,, \tag{1.2}$$

is a functional that depends indeed on a single surface.

The boundary condition that we consider for these textures is the *weak anchoring*. It is a condition on $\boldsymbol{n}$, that is, on the orientation of the lamellae, not on their position in space: all orientations are allowed, but each interacts differently with the boundary. We assume that the interaction energy per unit surface of the boundary is a function γ of the angle between $\boldsymbol{n}$ and the normal $\boldsymbol{\nu}$ to the boundary. We further assume that the boundary does not interfere directly with the positions of the lamellae: they are regarded as free to intersect the boundary anywhere, without suffering any force.

In section 2 below we study the case of a cell confined between two parallel plates providing weak anchoring for $\boldsymbol{n}$. In it we consider only confocal textures with translational symmetry: it is sufficient to know one plane curve to determine completely the shape of a lamella. The total energy stored in the cell is a non-trivial functional of this curve.

We present in section 3 a method to calculate the first variation of the energy functional, which is primarily meant for free-boundary problems (see [8]).

In section 4 we deduce the equilibrium equation for the confocal textures envisaged in this paper; it is the same as that given without proof in [4]. Finally, we describe in section 5 an instability of the texture in the cell, which arises when the anchoring conditions on the two plates are antagonistic.

2. Energy functional

Let $(o, \boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ be a Cartesian frame with origin in the point o. A cell bounded by two parallel plates, d apart, is represented as the set $\mathcal{B} = \{(x, y, z) \,|\, x, y \in]0, L[, z \in]0, d[\}$. The bounding plates lie at $z = 0$ (plate 1) and $z = d$ (plate 2).

We consider confocal textures in $\mathcal{B}$ that are invariant under translations along the y-axis, and such that the focal surface lies outside $\mathcal{B}$.

Imagine a family of parallel curves in the plane (x, z). Any of these describes the whole texture; we choose one, that we call *reference curve*, and we represent it through a vector-valued mapping $\boldsymbol{p}(s) := p(s) - o$ of the arc-length s,

$$\boldsymbol{p}(s) = x(s)\,\boldsymbol{e}_x + z(s)\,\boldsymbol{e}_z\,. \tag{2.1}$$

The tangent unit vector to the reference curve is defined as usual:

$$\boldsymbol{t} := \boldsymbol{p}'\,; \tag{2.2}$$

here and in what follows a prime denotes differentiation with respect to s. We call ϑ the angle in $[-\pi/2, \pi/2]$ between $\boldsymbol{t}$ and $\boldsymbol{e}_x$; thus,

$$\boldsymbol{t} = \cos\vartheta\,\boldsymbol{e}_x + \sin\vartheta\,\boldsymbol{e}_z \quad \text{and} \quad x' = \cos\vartheta\,, \;\; z' = \sin\vartheta\,. \tag{2.3}$$

Let $\boldsymbol{n}$ be the unit vector defined by

$$\boldsymbol{n} := -\sin\vartheta\,\boldsymbol{e}_x + \cos\vartheta\,\boldsymbol{e}_z\,; \tag{2.4}$$

it is orthogonal to $\boldsymbol{t}$ and such that

$$\boldsymbol{t}' = \sigma\boldsymbol{n}\,, \tag{2.5}$$

where $\sigma := \vartheta'$ is the *curvature* of $\boldsymbol{p}$. It follows from (2.4) that

$$\boldsymbol{n}' = -\sigma\boldsymbol{t}\,. \tag{2.6}$$

Every point $p(s)$ is traversed by one generator of the texture, which is parallel to $\boldsymbol{n}$; we call ξ the co-ordinate along it, oriented like $\boldsymbol{n}$ and with origin in $p(s)$. Thus, for a given reference curve $\boldsymbol{p}$, the pairs (s, ξ) define a system of curvilinear co-ordinates, whose co-ordinate lines are the curves parallel to $\boldsymbol{p}$ and the generators orthogonal to them: these are the *confocal co-ordinates*. The point with confocal co-ordinates (s, ξ) is given by $p(s, \xi) = p(s) + \xi\boldsymbol{n}(s)$ and its Cartesian co-ordinates are

$$X = x(s) - \xi z'(s)\,, \qquad Z = z(s) + \xi x'(s)\,; \tag{2.7}$$

thus, by (2.6),

$$\nabla\boldsymbol{n} = -(\sigma/(1 - \xi\sigma))\,\boldsymbol{t}\otimes\boldsymbol{t}\,,$$

and so

$$\bar{\sigma} := \operatorname{div}\boldsymbol{n} = -\sigma/(1 - \xi\sigma)\,. \tag{2.8}$$

The Jacobian of the transformation defined in (2.7) is $J(s, \xi) = 1 - \xi\sigma(s)$. For a given point $p(s)$, we call $l_1(s)$ and $l_2(s)$ the ξ-co-ordinates of the points on the generator through it that belong to plate 1 and plate 2, respectively. Clearly,

$$l_1(s) = -z(s)/\cos\vartheta(s)\,, \qquad l_2(s) = (d - z(s))/\cos\vartheta(s)\,. \tag{2.9}$$

Likewise, we call $\sigma_1(s)$ and $\sigma_2(s)$ the curvature of the curves that intersect the two plates at the points with confocal co-ordinates $(s, l_1(s))$ and $(s, l_2(s))$.

We consider an arc $\mathcal{A}$ in the reference curve such that its points satisfy $x' \neq 0$ and the generators through them intersect both plates of the cell. Let a be the length of this arc. An equilibrium texture makes stationary also the curvature energy relative to $\mathcal{A}$ subject to the condition that both generators at $s = 0$ and $s = a$ be held fixed. The curvature energy relative to $\mathcal{A}$ is equal to $LF_c[\boldsymbol{p}]$, where

$$F_c[\boldsymbol{p}] = \frac{1}{2}K\int_0^a ds \int_{l_1(s)}^{l_2(s)} \bar{\sigma}^2(1-\xi\sigma(s))\,d\xi = -\frac{1}{2}K\int_0^a \sigma(s)\ln\left(\frac{1-l_2(s)\sigma(s)}{1-l_1(s)\sigma(s)}\right)ds\,.$$

We suppose that a weak anchoring acts on the generators of the texture at both plates of the cell; we take the anchoring energy per unit area to be given on plate 1 and plate 2 by functions γ_1 and γ_2 of the angle between the generator and the normal to the plates. As shown by (2.4), this angle is precisely ϑ. The total anchoring energy relative to $\mathcal{A}$ is $LF_a[\boldsymbol{p}]$, where

$$F_a[\boldsymbol{p}] := \int_{P_1}\gamma_1(\vartheta)\,dX + \int_{P_2}\gamma_2(\vartheta)\,dX\,, \tag{2.10}$$

and P_1 and P_2 are segments on the corresponding plates, which depend on $\mathcal{A}$. From $(2.3)_3$ and (2.9), we arrive at

$$l_i' = -(1-l_i\sigma)\,\mathrm{tg}\,\vartheta \qquad \text{for } i = 1, 2\,; \tag{2.11}$$

by using this formula and $(2.7)_1$ we see that along the line where $\xi = l_i(s)$ we have that $dX/ds = (1-l_i\sigma)/\cos\vartheta$, and so we give (2.10) the following form

$$F_a[\boldsymbol{p}] = \int_0^a \sum_{i=1}^{2}\{\varphi_i(\vartheta)(1-l_i\sigma)\}\,ds\,,$$

where $\varphi_i(\vartheta) := \gamma_i(\vartheta)/\cos\vartheta$ for $i = 1, 2$.

The functional of the reference curve that we study in the following is $F[\boldsymbol{p}] := F_c[\boldsymbol{p}] + F_a[\boldsymbol{p}]$.

3. First variation

In this section we compute the first variation of F.

Let $\varepsilon_0 > 0$ be given. For each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ we define the curve

$$\boldsymbol{p}_\varepsilon(s) := \boldsymbol{p}(s) + \varepsilon\boldsymbol{u}(s)\,, \tag{3.1}$$

where $\boldsymbol{u}$ is a vector-valued function that satisfies the conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}(a) = \boldsymbol{0}\,. \tag{3.2}$$

Clearly, s is no longer the arc-length of $\boldsymbol{p}_\varepsilon$; letting this latter be denoted by s_ε, we easily obtain

$$ds_\varepsilon/ds = |\boldsymbol{p}' + \varepsilon\boldsymbol{u}'| = 1 + \varepsilon\boldsymbol{t}\cdot\boldsymbol{u}' + o(\varepsilon)\,. \tag{3.3}$$

Furthermore, if $\boldsymbol{t}_\varepsilon$ denotes the unit vector tangent to $\boldsymbol{p}_\varepsilon$, we have that $\boldsymbol{t}_\varepsilon = d\boldsymbol{p}_\varepsilon/ds_\varepsilon = (ds_\varepsilon/ds)^{-1}\boldsymbol{p}_\varepsilon' = \boldsymbol{t} + \varepsilon(\boldsymbol{n}\cdot\boldsymbol{u}')\boldsymbol{n} + o(\varepsilon)$. Thus, the tangents at $s = 0$ and $s = a$ remain both uncharged, provided that

$$\boldsymbol{n}\cdot\boldsymbol{u}'\big|_{s\in\{0,a\}} = 0\,. \tag{3.4}$$

We call first variation of F at the curve $\boldsymbol{p}$ the linear functional of $\boldsymbol{u}$ defined by

$$\delta F(\boldsymbol{p})[\boldsymbol{u}] := \frac{d}{d\varepsilon}F[\boldsymbol{p}_\varepsilon]\big|_{\varepsilon=0}$$

and subject to (3.2) and (3.4). To guide the reader in computing δF, we collect below a few preliminary results.

Let $\boldsymbol{n}_\varepsilon$ be the perturbed orientation field; it is given by $\boldsymbol{n}_\varepsilon = \boldsymbol{t}_\varepsilon \times \boldsymbol{e}_y = \boldsymbol{n} - \varepsilon(\boldsymbol{n}\cdot\boldsymbol{u}')\boldsymbol{t} + o(\varepsilon)$. Let ϑ_ε be the angle between $\boldsymbol{t}_\varepsilon$ and $\boldsymbol{e}_x$; it satisfies the equation $\vartheta_\varepsilon = \vartheta + \varepsilon(\boldsymbol{n}\cdot\boldsymbol{u}') + o(\varepsilon)$. Let σ_ε denote the curvature of $\boldsymbol{p}_\varepsilon$; thus $d\boldsymbol{t}_\varepsilon/ds_\varepsilon = \sigma_\varepsilon \boldsymbol{n}_\varepsilon$, and so $\sigma_\varepsilon = (ds_\varepsilon/ds)^{-1} d\boldsymbol{t}_\varepsilon/ds\cdot\boldsymbol{n}_\varepsilon = \sigma + \varepsilon((\boldsymbol{n}\cdot\boldsymbol{u}')' - \sigma\boldsymbol{t}\cdot\boldsymbol{u}')$. Similarly, from (2.9) we arrive at the following formulae for the perturbed values of both l_1 and l_2: $l_{i\varepsilon} = l_i + \varepsilon(l_i \operatorname{tg}\vartheta\boldsymbol{n}\cdot\boldsymbol{u}' - \boldsymbol{e}_z\cdot\boldsymbol{u}/\cos\vartheta) + o(\varepsilon)$ for $i = 1, 2$.

An arc of length a is changed by the deformation in (3.1) into one of length a_ε, and so

$$F_c[\boldsymbol{p}_\varepsilon] = -\frac{1}{2}K\int_0^{a_\varepsilon} \sigma_\varepsilon \ln\left(\frac{1 - l_{2\varepsilon}\sigma_\varepsilon}{1 - l_{1\varepsilon}\sigma_\varepsilon}\right) ds_\varepsilon\,.$$

Changing the integration variable s_ε into s, by $(3.3)_1$ we give this integral the following form

$$F_\varepsilon[\boldsymbol{p}_\varepsilon] = -\frac{1}{2}K\int_0^{a} \sigma_\varepsilon \ln\left(\frac{1 - l_{2\varepsilon}\sigma_\varepsilon}{1 - l_{1\varepsilon}\sigma_\varepsilon}\right) |\boldsymbol{p}' + \varepsilon\boldsymbol{u}'|\, ds\,, \tag{3.5}$$

where the integrand is to be read as a function of s. From (3.5), through a tedious, but easy, computation we arrive at

$$\begin{aligned}\delta F_c(\boldsymbol{p})[\boldsymbol{u}] = -\frac{1}{2}K\int_0^a \Bigg\{&\left(\ln\left(\frac{1 - l_2\sigma}{1 - l_1\sigma}\right) - \frac{l\sigma}{(1 - l_1\sigma)(1 - l_2\sigma)}\right)(\boldsymbol{n}\cdot\boldsymbol{u}')' + \\ &+ \frac{l\sigma^2}{(1 - l_1\sigma)(1 - l_2\sigma)}(\boldsymbol{t}\cdot\boldsymbol{u}' - \operatorname{tg}\vartheta\boldsymbol{n}\cdot\boldsymbol{u}' + \sigma(\operatorname{tg}\vartheta\boldsymbol{t}\cdot\boldsymbol{u} + \boldsymbol{n}\cdot\boldsymbol{u}))\Bigg\} ds\,,\end{aligned} \tag{3.6}$$

where $l(s) := l_2(s) - l_1(s)$ is the total length of the generator through the point $p(s)$. Successive integrations by parts in (3.6) lead us to

$$\delta F_c(\boldsymbol{p})[\boldsymbol{u}] = \int_0^a \{\pi_c^{(t)}\boldsymbol{t}\cdot\boldsymbol{u} + \pi_c^{(n)}\boldsymbol{n}\cdot\boldsymbol{u}\}\, ds\,,$$

where

$$\pi_c^{(t)} := \frac{1}{2}K\left\{\left(\frac{l\sigma^2}{(1 - l_1\sigma)(1 - l_2\sigma)}\right)' - \left(\frac{l\sigma}{(1 - l_1\sigma)(1 - l_2\sigma)} - \ln\left(\frac{1 - l_2\sigma}{1 - l_1\sigma}\right)\right)'\sigma\right\},$$

$$\pi_c^{(n)} := \frac{1}{2}K\left\{\left(\frac{l\sigma}{(1 - l_1\sigma)(1 - l_2\sigma)} - \ln\left(\frac{1 - l_2\sigma}{1 - l_1\sigma}\right)\right)'' - \left(\frac{l\sigma^2\operatorname{tg}\vartheta}{(1 - l_1\sigma)(1 - l_2\sigma)}\right)'\right\}.$$

Now it follows from (2.11) that

$$\left(\ln\frac{1 - l_2\sigma}{1 - l_1\sigma}\right)' = -\frac{l\sigma'}{(1 - l_1\sigma)(1 - l_2\sigma)}\,.$$

Thus we see that $\pi_c^{(t)}$ vanishes identically. This was indeed to be expected, since when $\boldsymbol{u}$

is everywhere parallel to $\boldsymbol{t}$ the shape of the perturbed curve $\boldsymbol{p}_\varepsilon$ and that of $\boldsymbol{p}$ are the same up to the first order in ε.

We compute in a similar way the variation of F_a, and we arrive at

$$\delta F_a(\boldsymbol{p})[\boldsymbol{u}] = \int_0^a \sum_{i=1}^2 \{-l_i\varphi_i(\boldsymbol{n}\cdot\boldsymbol{u}')' + (\dot{\varphi}_i(1-l_i\sigma) - \varphi_i l_i \sigma \operatorname{tg}\vartheta)\boldsymbol{n}\cdot\boldsymbol{u}' + \\ + \varphi_i \boldsymbol{t}\cdot\boldsymbol{u}' + \sigma\varphi_i(\boldsymbol{n}\cdot\boldsymbol{u} + \operatorname{tg}\vartheta\boldsymbol{t}\cdot\boldsymbol{u})\}\,ds\,, \tag{3.7}$$

where a superimposed dot denotes differentiation with respect to ϑ. For future use the reader should note now that, by the clain rule, $\varphi' = \dot{\varphi}\sigma$. Integrating by parts in (3.7) we conclude that

$$\delta F_a(\boldsymbol{p})[\boldsymbol{u}] = \int_0^a \pi_a^{(n)}\boldsymbol{n}\cdot\boldsymbol{u}\,ds\,,$$

where

$$\pi_a^{(n)} = -((\dot{\gamma}_1 + \dot{\gamma}_2)/\cos\vartheta)'\,.$$

Thus, the first variation of F is

$$\delta F_a(\boldsymbol{p})[\boldsymbol{u}] = \int_0^a \pi^{(n)}\boldsymbol{n}\cdot\boldsymbol{u}\,ds \qquad \text{with } \pi^{(n)} := \pi_c^{(n)} + \pi_a^{(n)}\,. \tag{3.8}$$

4. Equilibrium textures

In the preceding section we have computed the first variation of F when the reference curve $\boldsymbol{p}$ is perturbed along an arc whose length can be chosen arbitrarily small. The equilibrium textures are generated by curves for which $\pi^{(n)}$ vanishes identically. Now we give $\pi_c^{(n)}$ a more expressive form (see [4]):

$$\pi_c^{(n)} = K\{l((\sigma_1\sigma_2\tilde{\sigma}/\sigma^3)\sigma' - \sigma_1\sigma_2\operatorname{tg}\vartheta)\}'\,,$$

where $\tilde{\sigma} := (\sigma_1 + \sigma_2)/2$, and then we write the equilibrium equation for $\boldsymbol{p}$ as

$$Kl((\sigma_1\sigma_2\tilde{\sigma}/\sigma^3)\sigma' - \sigma_1\sigma_2\operatorname{tg}\vartheta) = (\dot{\gamma}_1(\vartheta) + \dot{\gamma}_2(\vartheta))/\cos\vartheta + c\,, \tag{4.1}$$

where c is an arbitrary integration constant.

This equation can be interpreted as representing the balance of the torques that act on a generator per unit area of the lamellae. According to this interpretation, $\dot{\gamma}_i/\cos\vartheta$ for $i = 1, 2$ are the torques exerted by the plates, $-Kl\,(\sigma_1\sigma_2\tilde{\sigma}/\sigma^3)\sigma'$ is the integral along the generator of the curvature torques acting on the single lamellae, and $Kl\sigma_1\sigma_2\operatorname{tg}\vartheta$ is the torque produced by the change in length of the generator, the constant c represents an external torque exterted by the lateral boundary of the cell and sustained by the lamellae.

Equation (4.1) is a third-order differential equation for the mapping $\boldsymbol{p}$. Since c is arbitrary, we would need four boundary conditions at one end-point of the reference curve to determine completely its equilibrium shape. Here, however, we immagine to fix both end-points of the curve and prescribe the tangent there, as the curve were clamped. The boundary-value problem that results for (4.1) may then exhibit a bifurcation, as is shown by the elementary example in the following section.

It should be noted that in the equilibrium equation obtained by Fournier [4] there was no constant torque like c in (4.1), because in that case the end-points of the reference curve were free on the lateral boundary. Finally, to compare better these two equilibrium equations the reader should heed that a misprint in both equations (8) and (9) of [4] changed the correct exponents 3 and 1/3 into 2 and 1/2.

5. Instability

Let the reference curve be so constrained that its end-points have the same height and tangents parallel to $\boldsymbol{e}_x$. Suppose that for the anchoring energy at both plate 1 and plate 2 the orientation of the generators with $\vartheta = 0$ is a stable equilibrium configuration for one plate, say plate 1, and unstable for the other. More precisely, we assume that

(5.1) $$\dot{\gamma}_1(0) = \dot{\gamma}_2(0) = 0$$

and

$$\ddot{\gamma}_1(0) < 0\,, \qquad \ddot{\gamma}_2(0) > 0\,.$$

Under assumption (5.1), we see from (4.1) that the texture in which the lamellae are all parallel to the bounding plates is an equilibrium texture; a reference curve would be $\boldsymbol{p}_0(s) = s\boldsymbol{e}_x$, $s \in [0, L]$. We now explore the stability of this texture, evaluating the functional F on all smooth curves close to $\boldsymbol{p}_0$. If ε is a small parameter, the small deformations of $\boldsymbol{p}_0$ are described by

(5.2) $$\boldsymbol{p}_\varepsilon(s) := s\boldsymbol{e}_x + \varepsilon v(s)\,\boldsymbol{e}_z \qquad \text{for } s \in [0, L]\,,$$

where the function v is of class C^2 and satisfies

(5.3) $$v(0) = v(L) = 0 \qquad \text{and} \qquad v'(0) = v'(L) = 0\,.$$

For $\boldsymbol{p}_\varepsilon$ as in (5.2), we arrive at $F[\boldsymbol{p}_\varepsilon] = F[\boldsymbol{p}_0] + (1/2)\,\varepsilon^2\delta^2 F[v'] + o(\varepsilon^2)$, where

(5.4) $$\delta^2 F[v'] := \int_0^L \{Kdv''^2 + (\ddot{\gamma}_1(0) + \ddot{\gamma}_2(0))v'^2\}\,ds\,.$$

$\delta^2 F[v']$ is the second variation of F at $\boldsymbol{p}_0$: it is a quadratic functional of v' subject to $(5.3)_2$. The linear stability of the unperturbed texture depends on the sign of this functional.

It is known (cf. *e.g.* [7, p.185]) that if w is a function of class C^1 on $[0, L]$ which vanishes at the end-points of this interval, then

$$\int_0^L w^2\,ds \leqslant (L/\pi)^2 \int_0^L w'^2\,ds\,,$$

where the equality is attained if and only if

(5.5) $$w = w_0 \sin(\pi s/L)$$

for any constant w_0. Applying this theorem to the function $w = v'$ in (5.4), we conclude that if $\omega := \ddot{\gamma}_1(0) + \ddot{\gamma}_2(0) + Kd\pi^2/L^2 > 0$ then $\delta^2 F$ is positive definite, and so the curve p_0 is linearly stable. On the other hand, if $\omega < 0$ we see that $\delta^2 F[w] < 0$ whenever w is as in (5.5), and so p_0 becomes unstable. We expect a bifurcation to occur at $\omega = 0$, though a nonlinear analysis is needed to bring it into the open.

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